Iterations of Local Averaging of a Function : a repeated Convolution

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"I did not, however, commit suicide;
for I wished to know more of Mathematics."

This paper will explore a "smoothing process" on a real-valued function on $\mathbb{R}^1$. We start with a function g_0 at time 0, and set $g_{t+1}(x)$ equal to the value of a local average of g_t near x. Thus we obtain a sequence of functions $\{g_t\}$ which is determined by g_0 as well as how large a neighborhood we take in the local average. We will come up with conditions for when $\{g_t\}$ converges and compute the limit. We will also look at several properties of functions that are preserved under the smoothing process.

Let a bounded interval [-r,r] and a measurable function $g_0(x) : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be given. We are also given a "weighting function" $f(s): [-r,r] \to \mathbb{R}$. Define $\|f\| = \int_{-r}^{r} |f(s)| ds$. Now let

$$g_{t+1}(x) = \frac{1}{\|f\|} \int_{-r}^{r} g_t(x+s) f(s) ds$$

In effect the value of $g_{t+1}(x)$ is a weighted average of g_t on an r-neighborood of x. We are interested in the properties of the sequence of functions $\{g_t\}$. From now on we will concentrate on the special case f(s)=1 (so $\|f\|=1/2r$). This is the familiar way of computing the average value of a function over an interval by integrating the function over the interval and dividing by the length of the interval. We will discuss the general case afterwards.

We must first address the question of whether $\{g_t\}$ converges. This is true in some but not all of the cases, as we can see from

Example 1 : $g_0(x)=k$ for some $k \in \mathbb{R}$. Then $g_t(x)=k$ $\forall t$ and $\lim g_t = k$.

Example 2 : $g_0(x)=x^2$. Then $g_t(x)=x^2+tr^2/3$ and $\lim g_t = \infty$.

There are conditions we can put on g_0 in order to show that g_t converges. Before starting we should note that the smoothing process can be viewed as an iteration of a convolution : define $\varphi=(1/2r)\chi_{|x|<r}$ and let $\varphi_n = \varphi * \varphi * \ldots * \varphi$ (n convolutions). Then we see

that $g_t = g_0 * \varphi_t$. We may now begin with a useful tool :

Lemma : For any $r>0$, $\|\varphi_n\|_2 \to 0$ as $n \to \infty$.

proof : We compute $\hat{\varphi}$ directly and find that $\hat{\varphi}(0)=1$ and $|\hat{\varphi}(y)| < 1$ for $y \neq 0$. Now $\hat{\varphi}_n = [\hat{\varphi}]^n$ (by definition of φ_n) and so for $y \neq 0$, $|\hat{\varphi}_n(y)|^2 \to 0$ with $|\hat{\varphi}_n(y)|^2 \leq |\hat{\varphi}(y)|^2$ $\forall y$. Since $\varphi \in \mathcal{L}^2$ we know from the Plancherel Theorem that $\hat{\varphi} \in \mathcal{L}^2$ and so $|\hat{\varphi}(y)|^2 \in \mathcal{L}^1$. Thus we have dominated convergence and $\int |\hat{\varphi}_n(y)|^2 \to 0$ (i.e. $\|\hat{\varphi}_n\|_2 \to 0$). By the Plancherel Theorem $\|\varphi_n\|_2 = \|\hat{\varphi}_n\|_2$, and so $\|\varphi_n\|_2 \to 0$ □

We now give sufficient conditions for when g_t converges and when the limit is 0 :

Theorem 1 : If $g_0 \in \mathcal{L}^p(\mathbb{R})$ for any p with $1 \leq p < \infty$ then $\{g_n\}$ converges uniformly to 0.

proof : We divide the proof into 4 cases :

$p=2$: $\|g_n\|_{sup} = \|g_0 * \varphi_n\|_{sup} \leq \|g_0\|_2 \|\varphi_n\|_2$ by the Schwartz inequality and the translation invariance of $\| \ \|_2$. So by the lemma $\|g_n\|_{sup} \to 0$ as $n \to \infty$.

$p=1$:
$$\begin{aligned}\|g_0 * \varphi_n\|_{sup} &= \|g_0 * \varphi * \varphi_{n-1}\|_{sup} \\ &\leq \|g_0 * \varphi\|_2 \|\varphi_{n-1}\|_2 \\ &\leq \|g_0\|_1 \|\varphi\|_2 \|\varphi_{n-1}\|_2\end{aligned}$$
So by the lemma $\|g_0 * \varphi_n\|_{sup} \to 0$ as $n \to \infty$.

$1<p<2$: Let $E = \{x: |g_0(x)| \geq 1\}$. We write $g_0 = g_0 \chi_E + g_0 \chi_{E^c}$.

Now $g_0 \chi_E \in \mathcal{L}^1$ and $g_0 \chi_{E^c} \in \mathcal{L}^2$ and so

$$\|g_0 * \varphi_n\|_{sup} \le \|g_0 \chi_E * \varphi_n\|_{sup} + \|g_0 \chi_{E^c} * \varphi_n\|_{sup}$$
$$\to 0 \text{ as } n \to \infty.$$

$2 < p < \infty$: We first need to show that $\|\varphi_n\|_1 = 1 \ \forall (n \ge 1)$:

(i) This is clearly true for n=1.
(ii) Assume $\|\varphi_n\|_1 = 1$.
(iii)

$$\begin{aligned}
\|\varphi_{n+1}\|_1 &= \int_R \varphi_n * \varphi \\
&= \int_R \int_R \varphi_n(x-y)\, \varphi(y)\, dy\, dx \\
&= \int_R \tfrac{1}{2} \int_{-1}^{1} \varphi_n(x-y)\, dy\, dx \\
&= \tfrac{1}{2} \int_{-1}^{1} \int_R \varphi_n(x-y)\, dx\, dy \\
&\qquad\qquad \text{by the Fubini Theorem} \\
&= \tfrac{1}{2} \int_{-1}^{1} 1\, dy \quad \text{by (ii)} \\
&= 1.
\end{aligned}$$

Thus $\|\varphi_n\|_1 = 1 \ \forall (n \ge 1)$ by induction. Now let p' satisfy $1/p' + 1/p = 1$ so that $1 < p' < 2$. We now show that, since $\|\varphi_n\|_1 = 1$ and $\|\varphi_n\|_2 \to 0$, we have $\|\varphi_n\|_{p'} \to 0$:

Let $E_k^n = \{x : \varphi_n(x) \ge 1/k\}$ and F_k^n be the complement of E_k^n. Since $\|\varphi_n\|_1 = 1$ and $\varphi_n \ge 0$ we have $m(E_k^n) \le k$, for otherwise we would have $\|\varphi_n\|_1 > 1$. Thus we have

$$\int_{E_k^n} \varphi_n^{p'} \le \left\{\int_{E_k^n} (\varphi_n^{p'})^{2/p'}\right\}^{p'/2} \left\{\int_{E_k^n} 1\right\}^{1-p'/2} \text{ by Holder}$$

$$= \{\int_{E_k^n} \varphi_n\} \quad \{m(E_k^n)\}\}$$

$$\le (\|\varphi_n\|_2)^{p'/4}\, k^{1-p'/2}$$

From the definition of F_k^n we have

$$\int_{F_k^n} \varphi_n^{p'} = \int_{F_k^n} \varphi_n\, \varphi_n^{p'-1}$$

$$\le \frac{1}{k^{p'-1}} \int_{F_k^n} \varphi_n$$

$$\le \frac{1}{k^{p'-1}}$$

So $$\int_{\mathbb{R}} \varphi_n^{p'} = \int_{E_k^n} \varphi_n^{p'} + \int_{F_k^n} \varphi_n^{p'}$$

$$\le \|\varphi_n\|_2^{p'/4}\, k^{1-p'/2} + k^{1-p'}$$

So $\|\varphi_n\|_{p'} \to 0$ as $n \to \infty$. Thus the Hölder inequality gives $\|g_0 * \varphi_n\|_{\sup} \le \|g_0\|_p \|\varphi_n\|_{p'}$
$\to 0$ as $n \to \infty$.

Q.E.D.

Notice that the above proof works for any $r>0$. So we see that $\lim g_t = 0$ regardless of how large or small a neighborhood we average over. When $g_0 \in \mathcal{L}^\infty(\mathbb{R})$ we may not have convergence at all. We may inquire about those functions that are in $\mathcal{L}^p(I)$ for every bounded interval I, i.e. those functions in $\mathcal{L}^p_{bc}(\mathbb{R})$. This question cannot be answered unless more information is given, average over. Example 1 shows us that when $g_0 \in \mathcal{L}^\infty(\mathbb{R})$ we may not have convergence to 0. It is true when $g_0 \in \mathcal{L}^\infty(\mathbb{R})$, however, that we do have convergence to a constant :

Theorem 2 : If $g_0 \in \mathcal{L}^\infty(\mathbb{R})$ then there is a $k \in \mathbb{R}$ such that $\lim g_n = k$.

proof :

step 1 : Let $r>0$ be given. Then $\exists (g \in \mathcal{L}^\infty(\mathbb{R}))$ such that $g_n \to g$.

proof : Let $\varepsilon>0$ be given. Let $c=1/2r$. So $|\varphi_n(x)| \le c$ $\forall x,n$. Now

$$
\begin{aligned}
|g_n - g_m| &= |g_0 * \varphi_n - g_0 * \varphi_m| \\
&= |g_0 [\varphi_n - \varphi_m]| \\
&\le \|g_0\|_\infty \|\varphi_n - \varphi_m\|_1 \qquad \text{by the Hölder Inequality} \\
&\le \|g_0\|_\infty \|(\varphi_n - \varphi_m)(\varphi_n + \varphi_m)/c\|_1 \qquad \text{since } |\varphi_n(x)| \le c \ \forall x,n \\
&= (1/c)\|g_0\|_\infty \|\varphi_n^2 - \varphi_m^2\|_1 \\
&\le (1/c)\|g_0\|_\infty [\|\varphi_n^2\|_1 + \|\varphi_m^2\|_1] \qquad \text{by the Minkowski inequality} \\
&= (1/c)\|g_0\|_\infty [\|\varphi_n^2\|_2 + \|\varphi_m^2\|_2]
\end{aligned}
$$

Now by the lemma there is an N such that for $m,n \ge N$ we have $[\|\varphi_n^2\|_2 + \|\varphi_m^2\|_2] < c\varepsilon/\|g_0\|_\infty$. So

$|g_n - g_m| < \varepsilon$ for $m,n \ge N$. Thus $\{g_n\}$ is cauchy, and since $\mathcal{L}^{\infty}(\mathbb{R})$ is complete, $\exists (g \in \mathcal{L}^{\infty})$ such that $g_n \to g$.

step 2 : We now show that g is constant. Suppose $g_n(x) \to g(x)$ and that $g_n(y) \to g(y)$. Let $\varepsilon > 0$ be given. We may then pick an N_1 such that $|g_n(x)-g(x)| < \varepsilon/3$ and $|g_n(y)-g(y)| < \varepsilon/3$ for $n \ge N_1$. Also $|g_n(x)-g_n(y)| = |g_0 * [\varphi_n(x)-\varphi_n(y)]|$

$$\le \|g_0\|_{\infty} \|\varphi_n(x)-\varphi_n(y)\|_1$$

$$< \varepsilon/3 \text{ for } n \ge N_2 \text{ by the same}$$

type argument as in the first part of the proof. Now let $N = \max(N_1, N_2)$. Thus

$$|g(x)-g(y)| \le |g(x)-g_n(x)| + |g_n(x)-g_n(y)| + |g_n(y)-g(y)|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \text{ for } n \ge N$$

Since ε was arbitrary we see that $|g(x)-g(y)| = 0$, and so g is constant.

Q.E.D.

Note : It is clear that if $g_0 \in \mathcal{L}^{\infty}(\mathbb{R})$ and $g_n \to k \in \mathbb{R}$ then $g_n \to k$ regardless of how we choose $r > 0$ (i.e. how large a neighborhood we integrate over).

We have just adressed the question of convergence for functions in $\mathcal{L}^p(\mathbb{R})$ for $1 \le p \le \infty$, but what about those functions that are in $\mathcal{L}^p(I)$ for every bounded interval I, i.e. those functions in $\mathcal{L}^p_{bc}(\mathbb{R})$? This question cannot be answered unless more information is given, as we can see by example 2 (where a function in $\mathcal{L}^{1}_{bc}(\mathbb{R})$ diverges) and the following example :

<u>Example 3</u> : $g_0(x)=\sin x$. Then $g_t(x)=(\sin r/r)^t \sin x$ and $\lim g_t=0$.

The author has not found general conditions under which functions not in $\mathcal{L}^P(\mathbb{R})$ converge, though such functions clearly exist.

We now look at properties of functions that are preserved under the averaging process, and come up with a class of functions not necessarily in $\mathcal{L}^P(\mathbb{R})$ but that do converge.

We first note a few properties of g_t that are immediate consequences of the definitions. For $t\geq 1$ $g_t(x)$ is a uniformly continuous function since this is a property of convolutions. Also if $g_0(x)$ is monotone increasing (decreasing) then $g_t(x)$ is monotone increasing (decreasing) since if $a\geq b$ then $g_{t+1}(a)=(1/2r)\int g_t(a+s)ds \geq (1/2r)\int g_t(b+s)ds = g_{t+1}(b)$. Also if $g_0(x)$ is bounded below and above by m and M respectively, then $g_t(x)$ has the same bounds $\forall(t>0)$.

We say that g_0 **smooths evenly** if $g_t(x)=C(t)g_0(x)$ $\forall x,t$ where $C(t) : \mathbb{Z}^+ \rightarrow \mathbb{R}$. The functions that smooth evenly are those functions that look like expanded or contracted versions of themselves when undergoing the local averaging process. An example of a function that smooths evenly and converges was given in example 3. Here is a function that smooths evenly but does not converge :

<u>Example 4</u> : $g_0(x)=e^x$. Then $g_t(x)=(\sinh r/r)^t e^x$ and $\lim g_t(x)=\infty$ $\forall x$.

It turns out that in order to determine whether a function smooths evenly it is sufficient to see that it does after one step, i.e. $g_1(x)=Cg_t(x)$. This follows from the following theorem :

<u>Theorem 3</u> : Suppose $g_t(x)=C(t)g_0(x) + D(t)$. Then $C(t)=[C(1)]^t$ and $D(t)=([C(1)]^{t-1}+\ldots+[C(1)]+1)D(1)$.

proof : (i) The statement is trivially true for $t=1$.

(ii) Assume $g_t(x)=C(t)g_0(x)+D(t)$ where $C(t)$ and $D(t)$ are as above.

(iii) Now $g_{t+1}(x)=(1/2r)\int g_t(x+s)ds$

$=(1/2r)\int[C(t)g_0(x+s)+D(t)]ds$

$=(1/2r)\int[C(t)g_0(x+s)ds +D(t)$

$=C(t)g_1(x) + D(t)$

$=[C(1)]^t(C(1)g_0(x)+D(1)) + D(t)$

$=[C(1)]^{t+1}g_0(x) + ([C(1)]^t+...+[C(1)]+1)D(1)$

Thus the theorem is true by induction.

<u>**Corallary 3.1**</u> : $g_0(x)$ smooths evenly if and only if for some $k\in\mathbb{R}$, $g_t(x)=k^t g_0(x)$.

<u>**Corallary 3.2**</u> : Suppose g_0 smooths evenly with smoothing constant k and that $|g_0(x)|<\infty$ for all x. Then $k\neq 1$ and $g_0(x)\neq 0$ imply $\{g_t(x)\}$ is a strictly monotone sequence. Also :

$\lim g_t(x)=0$ if $k<1$

$\lim g_t(x)=g_0(x)$ if $k=1$

$\lim g_t(x)=0$ when $g_0(x)=0$

$\pm\infty$ when $g_0(x)\neq 0$ if $k>1$.

So we have extended the class of functions that converge to include those functions that smooth evenly with smoothing constant ≤ 1. The question of which functions smooth evenly is the same as which functions g satisfy $Cg=g*\varphi$ for some $C\in\mathbb{R}$. Appealing to a theorem about mean periodic functions (see Meyer, <u>Algebraic Numbers and Harmonic Analysis</u>, Thm.9.7) we find that the only candidates for functions that smooth evenly are functions of the form $(at+b)e^{\lambda t}$ with $a,b\in\mathbb{R}$ and $\lambda\in\mathbb{C}$, finite sums of these functions, and limits of the sequences of these (convergent) partial sums. Examples 1,3, and 4 are of this form. The only other thing we can say is that no function $h\in\mathcal{L}^1(\mathbb{R})$ smooths evenly (except for the trivial case when $h\equiv 0$ a.e.); for then there is a $C\in\mathbb{R}$ with $Ch=h*\varphi \Rightarrow C\hat{h}=\hat{h}\hat{\varphi} \Rightarrow C=(\sin rx/rx)$ for all x, which is a contradiction since $r>0$.

Our previous discussions have all concerned the special case $f(s)=1$, the simplest case of weighted averaging. If, in general, we take f to be bounded, then what we have proved should also hold. For general f the situation must be handled more delicately, and we wisely save this case for a later time.

We end our discussion with a few questions :

(1) We saw that for most functions g_0 that smooth evenly, $\{g_t(x)\}$ forms a monotone sequence at each x. However there are other functions that also have this property, such as $g_0(x)=x^3$. Here $g_t(x)=x^3+tr^2x$. Can we classify, perhaps with a certain condition, those functions g_0 with $\{g_t(x)\}$ forming a monotone sequence at each point?

(2) Does every continuous function h(x) come from a "smoothing process" that started with some arbitrary function?

(3) We call x a **fixed point** if $g_t(x)=g_0(x)$ for all $t\geq 0$. Given a function is there any way to determine it's fixed points under our process? Are there any functions besides those of the form $ax+b$ $(a,b\in\mathbb{R})$ with every point fixed?

(4) What properties of a function are preserved under the averaging process?

(5) Are there functions with $\lim g_t=g$ with g something other than 0, ∞, or $ax+b$?