James E. Georges
Department of Mathematics
California Polytechnic State University
San Luis Obispo, CA 93407

Annette M. Matthews
Department of Mathematics
Portland State University
Portland, OR 97207

Maximal Polygons for Equitransitive Periodic Tilings

Abstract --- It has been shown (Danzer, Grünbaum, and Shepherd: 1987) that in any periodic equitransitive tiling by convex polygonal tiles, the maximum number of sides of any tile is 66. This maximum is achieved in the periodic symmetry group p6m. We extend this result by determining the maximum number of sides in each of the remaining 16 periodic symmetry groups.

1. Introduction

A convex **tiling** is a set of closed convex polygonal regions, known as **tiles**, which cover the plane without gap or overlap. If the vertices of adjacent tiles meet, the tiling is **edge-to-edge.** A tiling is **periodic** if its group of symmetries is one of the seventeen periodic groups, often known as wallpaper groups. (See [3] for a full derivation of the seventeen periodic groups.) A periodic tiling is characterized by the fact that it has two translative symmetries in nonparallel directions. Suppose we represent these translations by the vectors x and y. The **period parallelogram** of a periodic tiling is the parallelogram with sides x and y having the minimal positive area. This parallelogram has the property that by replicating the region inside the parallelogram along the translation vectors, the entire tiling may be reconstructed.

A tiling is **equitransitive,** if for each k, all polygons having k sides are in the same transitivity class. i.e., all polygons having the same number of sides can be mapped to each other by a symmetry of the tiling.

In this paper we will study periodic equitransitive tilings with convex polygons. In any such tiling, the maximum number of sides on any tile is 66. (See [1].) This maximum is achieved in the periodic symmetry group p6m. We will consider bounds for the other sixteen periodic groups. To do this, we will prove the following theorem.

Theorem:: In any equitransitive tiling with one of the seventeen periodic symmetry groups, the maximum number of sides on any tile is given in Table I.

* Work on this paper was done while the authors were participants in the Research Experiences for Undergraduates program at Oregon State University. Their work was partially supported by NSF Grant DMS-8712402. The authors would like to thank Branko Grünbaum, Robby Robson, and Paul Cull for their kind support.

Table I Maximal polygons

	symmetry group	p_k max	maximal polygon
I	cmm	4	30
	p6	6	42
	p31m	6	42
	p3	3	24
	pgg	2	18
	p2	2	18
	pm	2	18
	cm	2	18
	p1	1	12
II	pmg	4	18
	pg	2	12
III	p4	4	28
	p4g	4	28
IV	p4m	4	24
	pmm	6	36
	p3m1	8	48
	p6m	12	66

Here p_k max is the maximum number of centroids possible in the period parallelogram, as explained below.

The proof of this theorem proceeds in stages. In section 2, we pose a lemma which gives an initial upper bound for the maximal polygon. We will call this initial upper bound m_g, where g is the symmetry group under consideration. For the groups labeled I, above, construction proves that m_g equals the number of sides on the maximal polygon. For the remaining cases, we must revise our initial estimates. This is done in sections 3 through 5.

2. The Initial Upper Bound

To get the initial bound on the maximal polygon for a particular symmetry group, we use the following lemma which is stated without proof. A proof can be found in [1].

Lemma: In any periodic tiling, if the period parallelogram contains the centroids of p_k k-gons, where k is an integer, then

$$3p_3 + 2p_4 + p_5 \geq \sum_{k=7}^{m_g}(k - 6)\, p_k.$$

From the lemma it is possible to get an estimate for m_g, the maximum number of sides on a polygon in a particular symmetry group. As exemplified below, m_g depends entirely upon the maximum value of p_k. Because we require our figures to be equitransitive, the maximum value of p_k will equal the maximum number of k-gon centroids in the period parallelogram. The maximum number of centroids depends on the symmetries present in the tiling group, as illustrated by the dots in right half of Figures 1 through 16 [2]. A key to these group diagrams is given in Table 2. Substituting this maximum value of p_k into lemma inequality yields an estimate for m_g, the maximal polygon.

As an example, we work through this process for the symmetry group cmm. An examination of Figure 1, the group diagram for cmm, shows that the maximum number of centroid images is achieved when a centroid is placed in "general position," off all lines of symmetry. In cmm, this maximum is four, which implies that there are at most four k-gons (for each k) in the period parallelogram. With p_k at most four, the lemma yields the following result.

$$\begin{aligned} 24 &= (3\cdot4) + (2\cdot4) + (1\cdot4), \\ &\geq 3p_3 + 2p_4 + p_5, \\ &\geq p_7 + 2p_8 + \ldots + (m_{cmm}-6)p_{m_{cmm}}. \end{aligned}$$

So, to maintain the inequality, $m_{cmm} = 30$.

To verify that this estimate does in fact correspond to a tiling, we must find a periodic equitransitive tiling with convex polygonal tiles in symmetry group cmm which contains 30-sided polygons. Figure 1 shows an example of such a tiling.

Using the lemma, similar estimates can be made for the groups p6, p31m, p3, pgg, p2, pm, cm, and p1. Figures illustrating the maximal p_k for these groups and the corresponding tilings are shown in Figures 2-9. Thus, for these first nine symmetry groups, the estimate for the maximal polygon given by the lemma produces an actual tiling.

3. The Second Set of Groups

The second set of groups are those in which *two* images of a polygon must always appear in the period parallelogram. There are exactly two symmetry groups in which this occurs: pmg and pg.

In pmg, for example, each center of symmetry occurs twice in the period parallelogram. This means that the period parallelogram must contain at least two of every polygon type, so $p_k \geq 2$. Additionally, the group symmetries shown in Figure 10 require that $p_k \leq 4$.

Table 2 Key to symmetry group diagrams

Symbol	Meaning
———	Line of reflection.
- - - - - - -	Line of glide reflection
◇ ◆	Center of 2-fold rotation. Black figure indicates that rotation lies on a line of reflection.
△ ▲	Center of 3-fold rotation.
□ ■	Center of 4-fold rotation.
⬡ ⬢	Center of 6-fold rotation.

With these constraints, the lemma gives

$$
\begin{aligned}
24 &= (3\cdot4) + (2\cdot4) + (1\cdot4), \\
&\geq 3p_3 + 2p_4 + p_5, \\
&\geq p_7 + 2p_8 + \ldots + (m_{pmg}-6)p_{m_{pmg}}, \\
24 &\geq (m_{pmg} - 6)\, 2, \\
18 &\geq m_{pmg}.
\end{aligned}
$$

So 18-gons are the maximum polygons possible for the symmetry group pmg. Figure 10 shows the corresponding tiling with 18-gons. A similar argument for the symmetry group pg yields a tiling with 12-gons. (See Figure 11).

4. The Third Set of Groups

In the third set of groups we find that m_g must be divisible by four. Two symmetry groups for which this occurs are p4 and p4g. For p4 and p4g, the maximum value of p_k is four. (See Figures 12 and 13.) So by the lemma, $m_g \leq 30$. We now show that in both of these cases $m_g = 28$.

Suppose 30-gons are possible in p4. Since the four-fold center of rotation is the only center which occurs once in the period parallelogram and since $p_{30} = 1$, the 30-gon must be centered on this four-center. But since 30 is not divisible by four, this is impossible. Placing the center of the 30-gon anywhere else in the period parallelogram would require that $p_{30} > 1$, so 30-gons are not possible in p4. For similar reasons 29-gons are not possible. Thus, we must take $m_{p4} = 28$. Figure 12 shows an example of an equitransitive p4 tiling with convex polygons using 28-gons.

Next, suppose $m_{p4g} = 30$ and 30-gons are possible in p4g. For $p_{30} = 1$, the 30-gons must be centered on either a two or a four center. The 30-gons cannot be on the four-centers since 30 is not divisible by four. Suppose that the 30-gons sit on two-centers. By inspection of the group diagram, one finds that each 30-gon can touch other 30-gons either four or zero times. Assume the 30-gons each touch four other 30-gons. Then the remaining 26 sides form a closed concave figure centered on the four-fold rotation. Now, the number of sides of any polygonal figure centered on the four-fold rotation must, of course, be divisible by four. Since 26 is not divisible by four, we have a contradiction.

The remaining possibility is that the 30-gons touch zero times. This possibility is ruled out as follows. The lemma dictates that, with a 30-gon present, only four other types of polygons can exist in the tiling: 3, 4, 5, and 6-gons. By the group symmetries, these four polygons can each compose at most eight sides of the 30-gon. Three of these polygons contributing eight sides each leaves six sides for the remaining polygon. The group symmetries, however, prohibit a polygon from contributing six sides. So again we have contradiction.

The symmetry group will not permit 29-gons since 29 is an odd number. Thus, $m_{p4g} \leq 28$. Figure 13 shows an example of an equitransitive tiling with convex 28-gons for the symmetry group p4g.

5. The Fourth Set of Groups

The remaining four symmetry groups, pmm, p3m1, p4m, and p6m, have the property that all centers of rotation lie on lines of reflection. Because of this symmetry, we can determine the number of distinct tiles which must be in the period parallelogram. By application of the lemma we are then able to reduce the initial estimate for the maximal polygon.

To illustrate this procedure, we consider the group p4m. Examining Figure 14, we see that for any k, there are at most eight k-gons in the period parallelogram. Application of the lemma yields $m_{p4m} \leq 54$. This upper bound assumes that for $k = m_{p4m}$, $p_k = 1$. In p4m, this is true only if the maximal polygon lies on a four-fold center of rotation. This requirement forces m_{p4m} to be divisible by four. Hence, $m_{p4m} \leq 52$ and the other possible values are also multiples of four .

A maximal polygon on a four-center can touch an identical polygon at most four times. This leaves $(m_{p4m} - 4)$ edges to be adjacent to other polygons. The lines of reflection passing through the four-centers allow these other polygons to contribute at most eight edges to the maximal polygon. From this we determine that the period parallelogram must include at least $(m_{p4m} - 4) / 8$ other polygons besides the one on the four-center. With this fact we show $m_{p4m} \neq 52$.

Suppose $m_{p4m} = 52$. Then there are *at least* $(52 - 4) / 8 = 6$ polygons other than 52-gon. But from the lemma, when $p_{52} = 1$, $p_k = 0$ for all $k \geq 9$. This leaves the inequality

$$p_7 + 2p_8 \leq 2.$$

The inequality shows that in addition to the 3, 4, 5, and 6-gons, we can have either two 7-gons or one 8-gon. So, with a 52-gon in the tiling, only five other polygon types are possible. This is not enough. Therefore, we have shown that p4m does not admit 52-gons. Stepping down by four, the next possibility is $m_{p4m} = 48$. The construction in Figure 14 shows an equitransitive p4m tiling with convex 48-gons.

Using similar techniques, it is possible to reduce initial estimates for maximal polygon in the groups pmm, p3m1, and p6m. The resulting tilings are illustrated in Figures 15-17. The above methods were used in [1] to get an estimate for the maximal polygons possible in the symmetry group p6m.

References

1. Ludwig Danzer, Branko Grünbaum, and G. C. Shephard, "Equitransitive Tilings or How to Discover New Mathematics", Math. Mag. , 60 (1987) 67-89.

2. Branko Grünbaum and G. C. Shephard, Tilings and Patterns, W. H. Freeman, New York, 1987.

3. George E. Martin, Transformation Geometry: An Introduction to Symmetry, Springer-Verlag, New York-Heidelberg-Berlin, 1982.

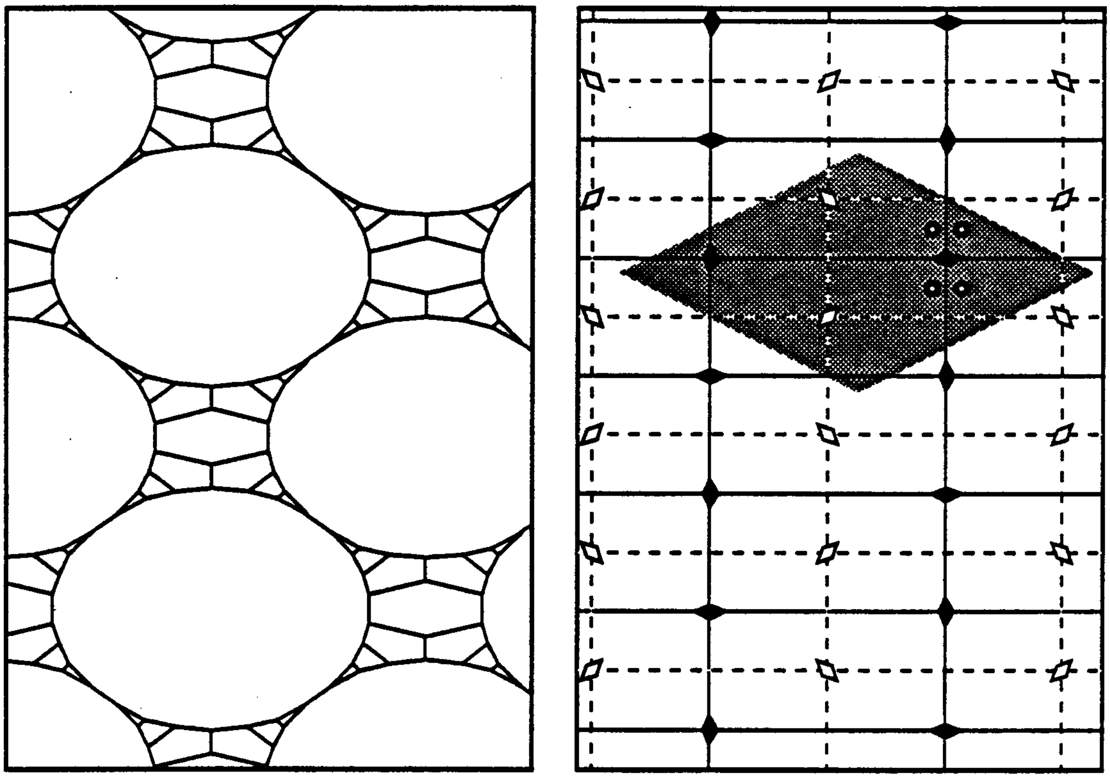


Figure 1. (left) cmm with 30-gons. In addition to the 30-gons there are 3, 4, 5, 6, and 7-gons present in the figure. (right) Group diagram for cmm demonstrating that the maximal value for p_k in cmm is 4.

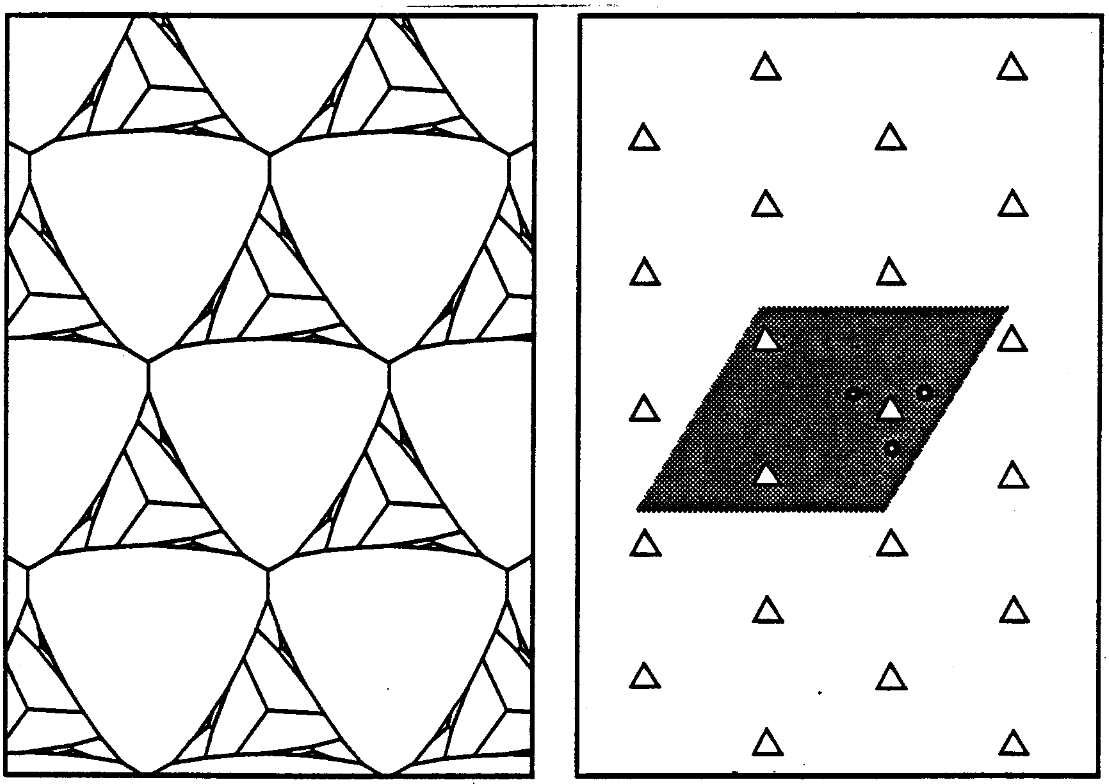


Figure 2. (left) p6 with 42-gons. In addition to the 42-gons there are 3, 4, 5, and 6-gons present in the figure. (right) Group diagram for p6 demonstrating that the maximal value for p_k in p6 is 6.

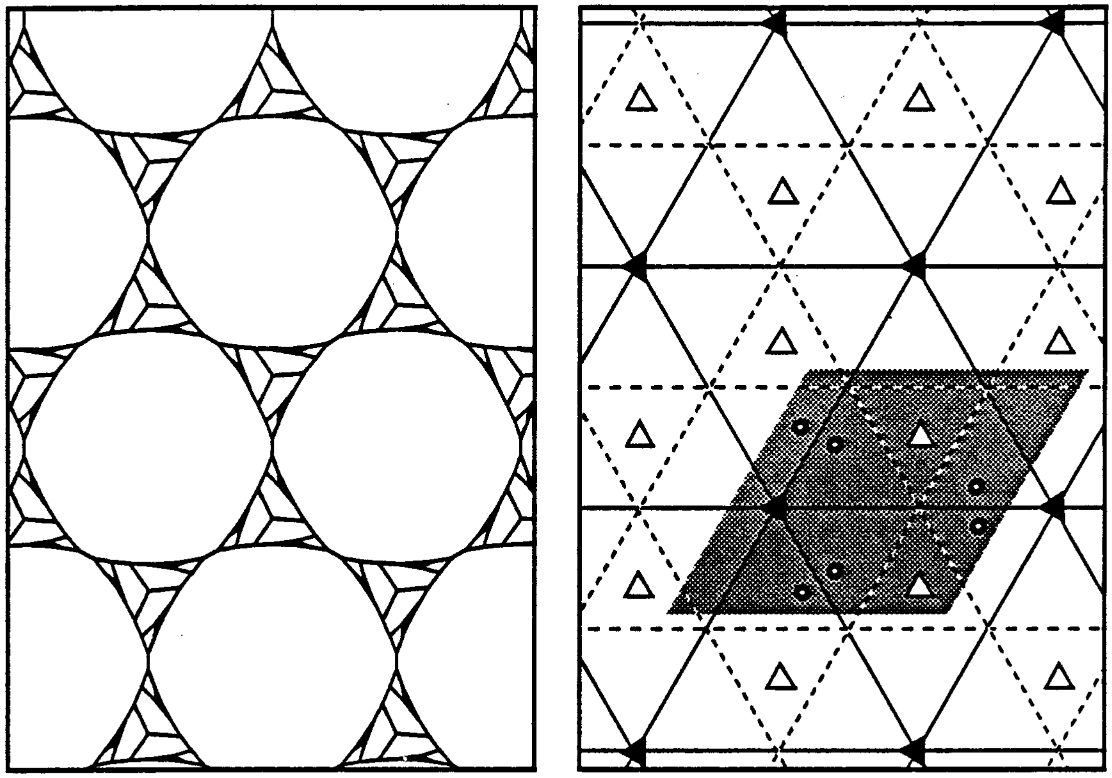


Figure 3. (left) p31m with 42-gons. In addition to the 42-gons there are 3, 4, 5, and 6-gons present in the figure. (right) Group diagram for p31m demonstrating that the maximal value for p_k in p31m is 6.

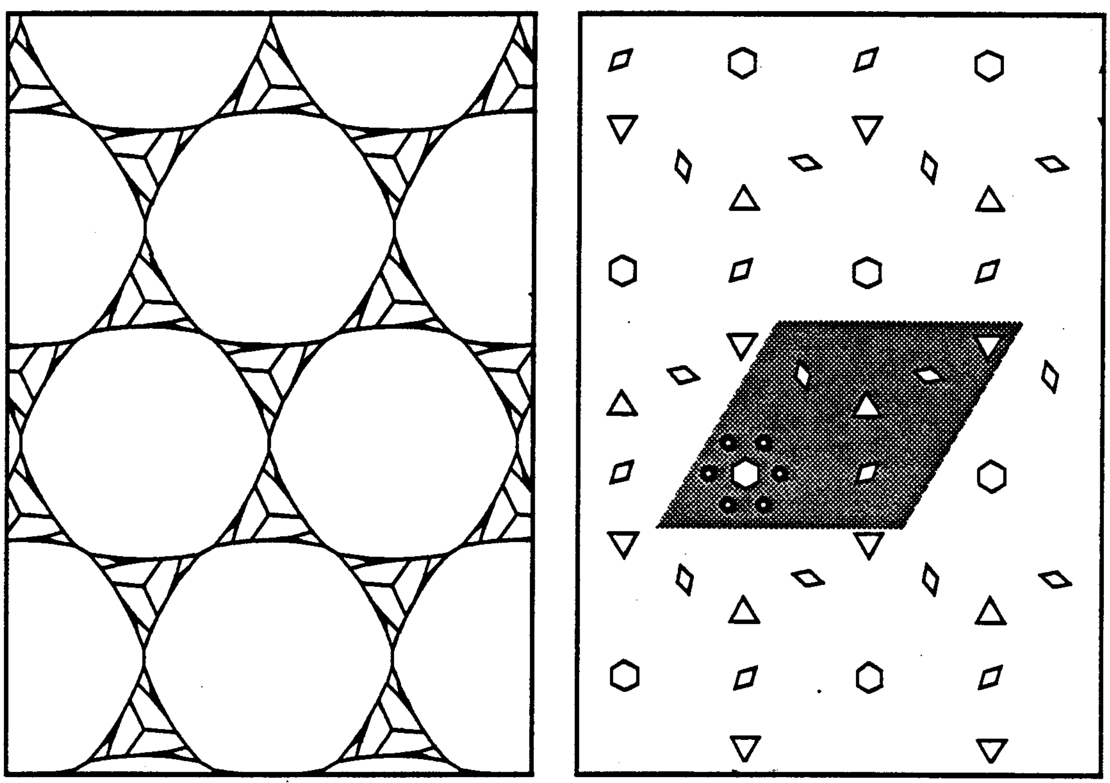


Figure 4. (left) p3 with 24-gons. In addition to the 24-gons there are 3, 4, 5, and 6-gons present in the figure. (right) Group diagram for p3 demonstrating that the maximal value for p_k in p31m is 6.

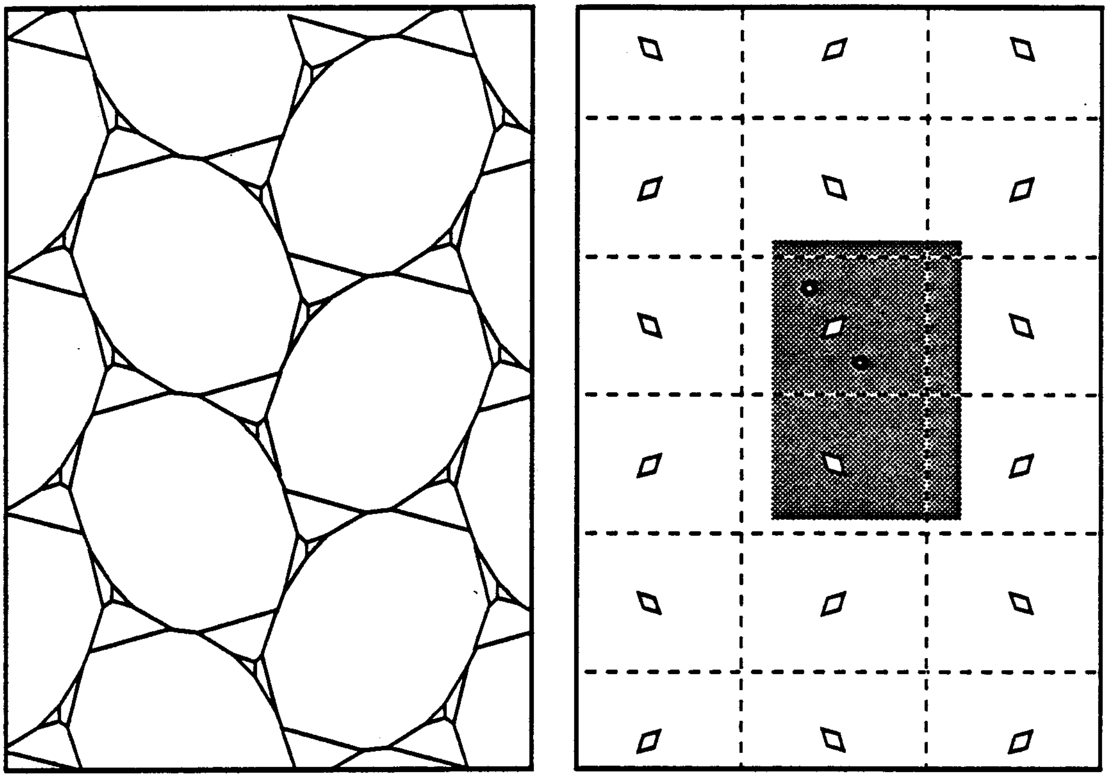


Figure 5. (left) pgg with 18-gons. In addition to the 18-gons there are 3, 4, and 5-gons present in the figure. (right) Group diagram for pgg demonstrating that the maximum value for p_k in pgg is 2.

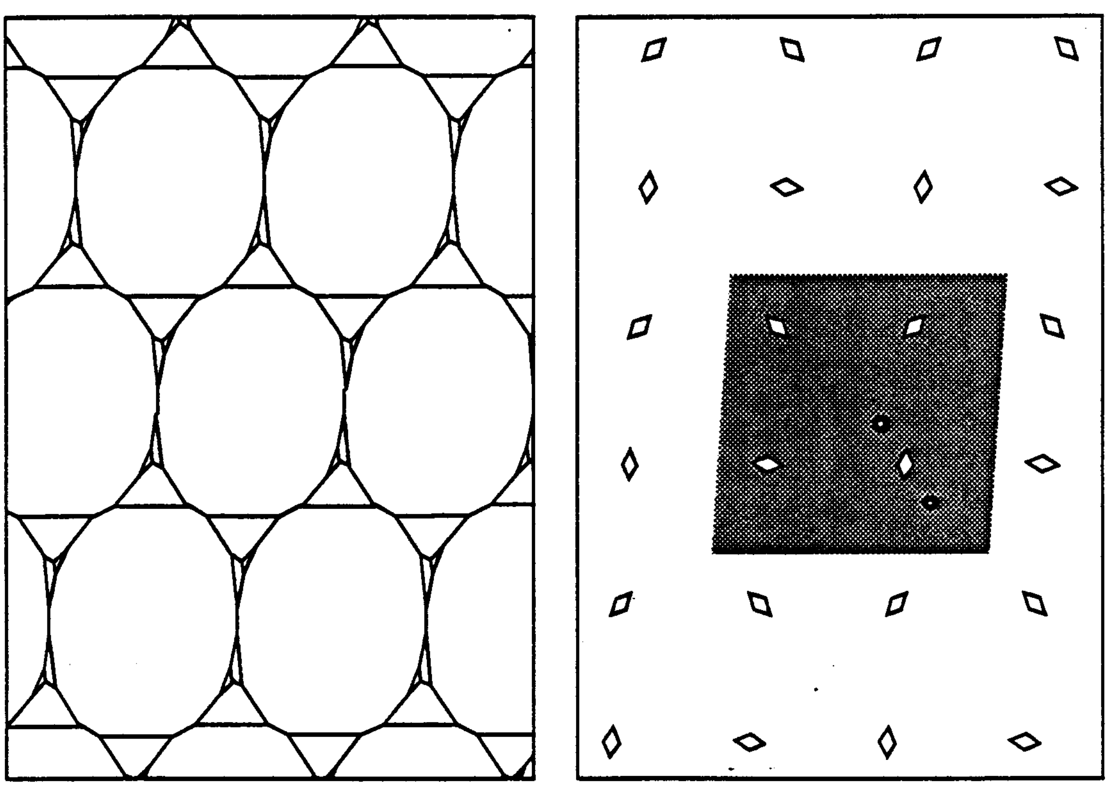


Figure 6. (left) p2 with 18-gons. In addition to the 18-gons there are 3, 4, and 5-gons present in the figure. (right) Group diagram for p2 demonstrating that the maximum value for p_k in p2 is 2.

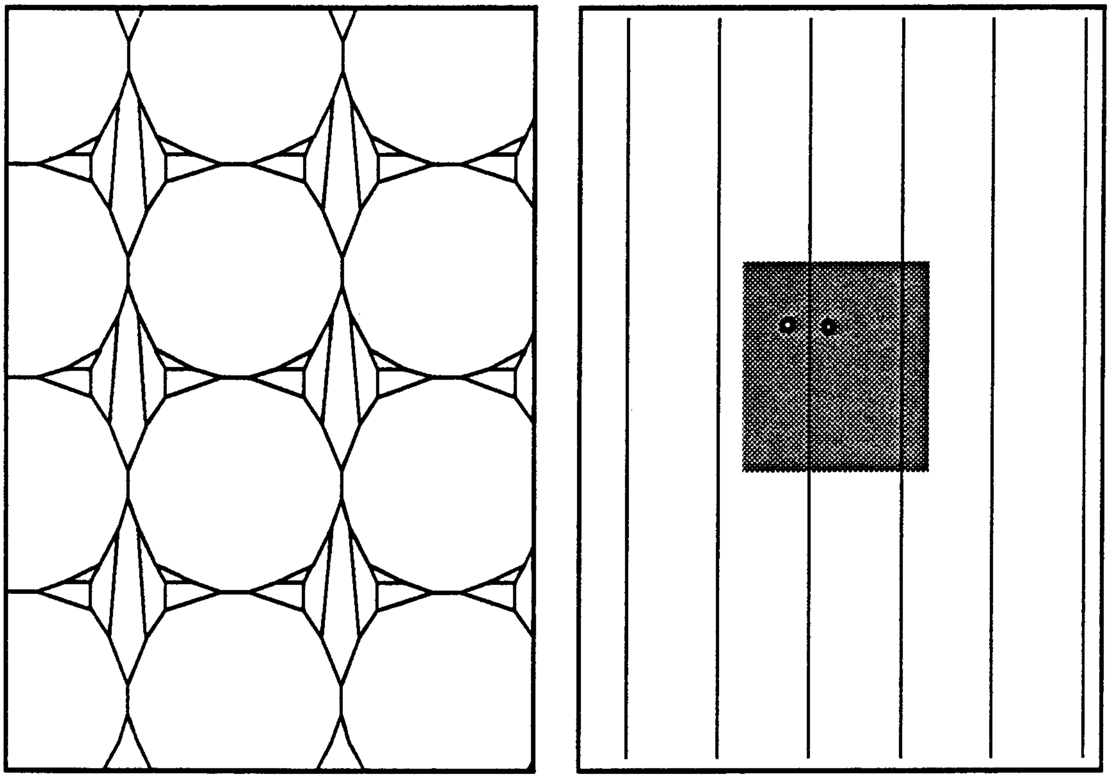


Figure 7. (left) pm with 18-gons. In addition to the 18-gons there are 3, 4, 5, and 6-gons present in the figure. (right) Group diagram for pm demonstrating that the maximum value for p_k in pm is 2.

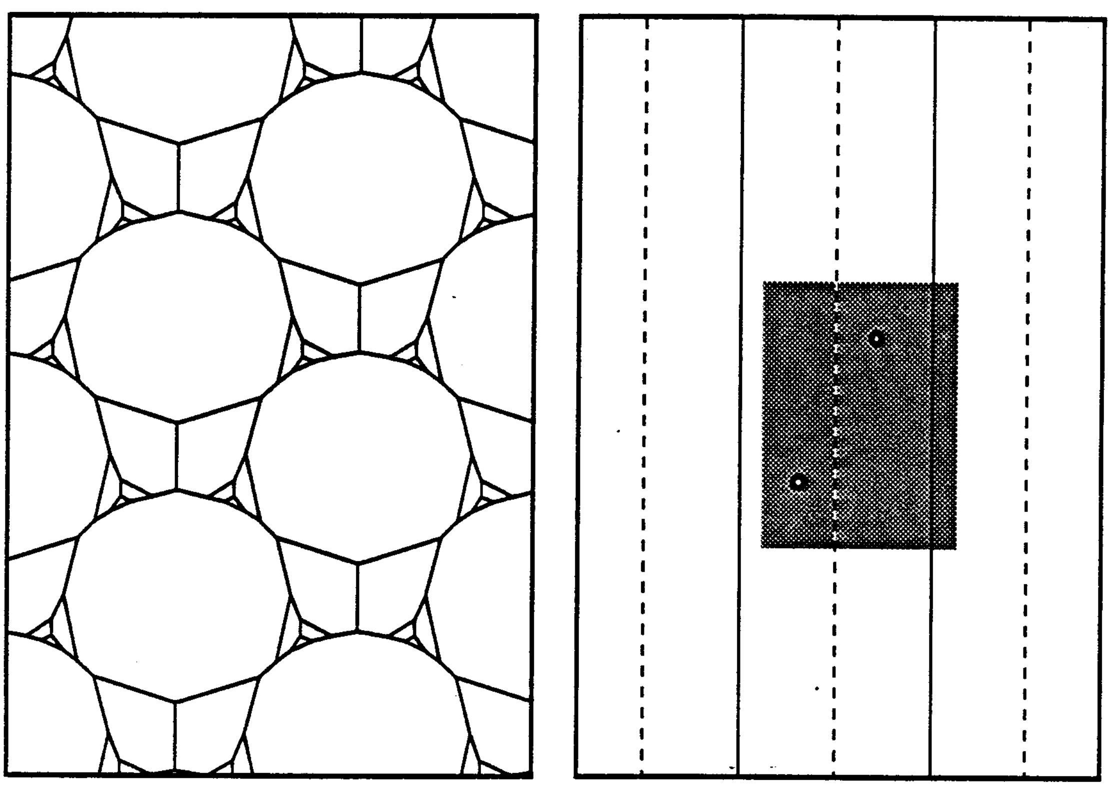


Figure 8. (left) cm with 18-gons. In addition to the 18-gons there are 3, 4, 5, and 6-gons present in the figure. (right) Group diagram for cm demonstrating that the maximum value for p_k in cm is 2.

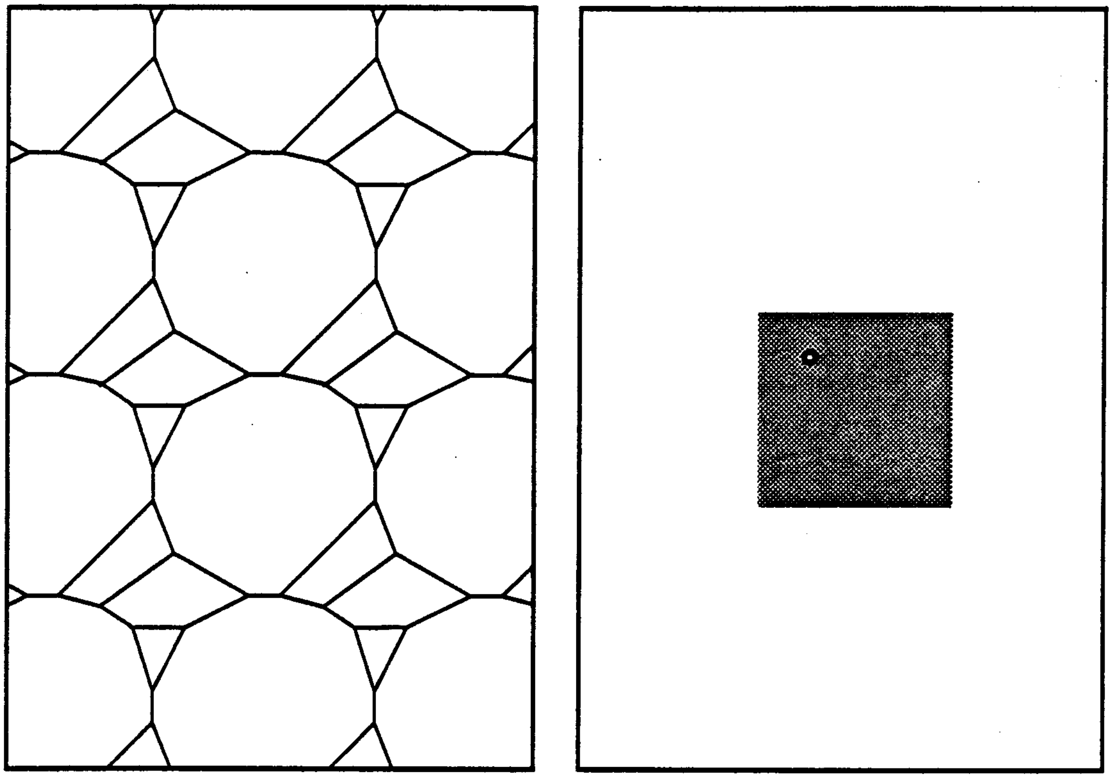


Figure 9. (left) p1 with 12-gons. In addition to the 12-gons there are 3, 4, and 5-gons present in the figure. (right) Group diagram for p1 demonstrating that the maximum value for p_k in p1 is 1.

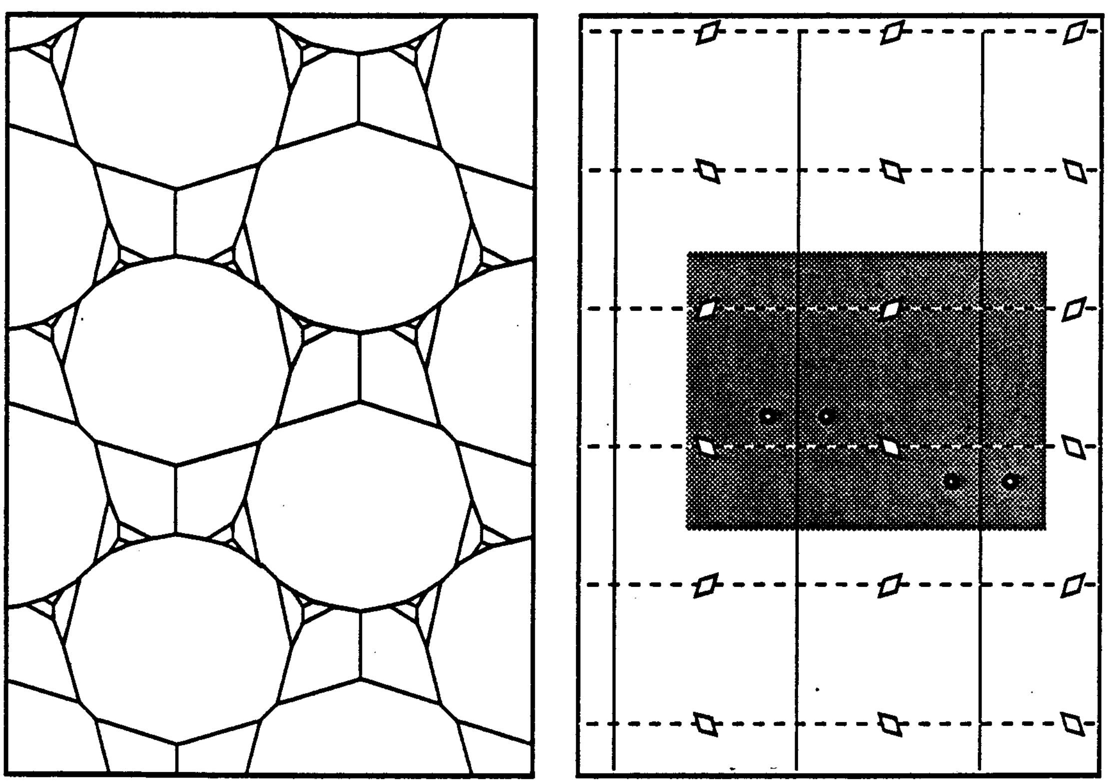
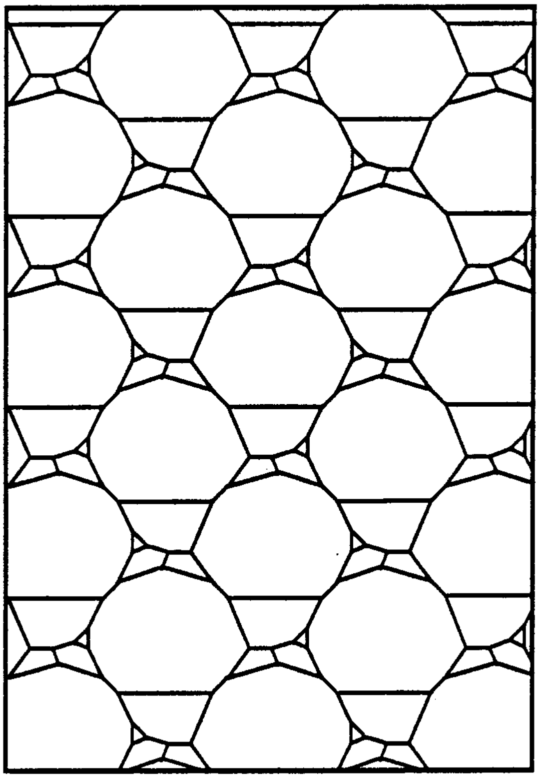


Figure 10. (left) pmg with 18-gons. In addition to the 18-gons there are 3, 4, 5, and 6-gons present in the figure. (right) Group diagram for pmg demonstrating that the maximum value for p_k in p1 is 4.



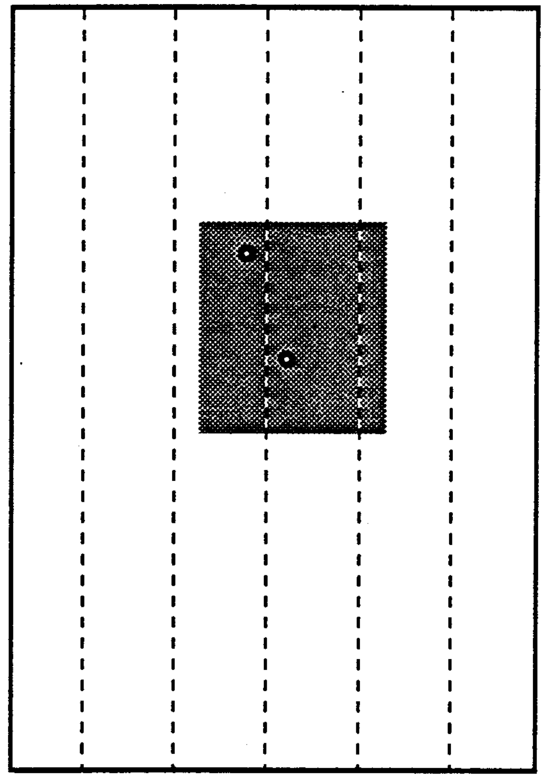
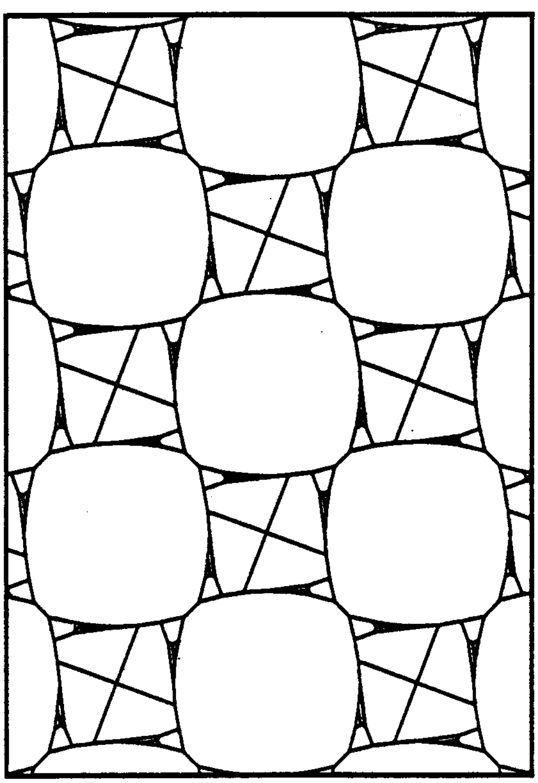


Figure 11. (left) pg with 12-gons. In addition to the 12-gons there are 3, 4, 5, and 6-gons present in the figure. (right) Group diagram for pg demonstrating that the maximum value for p_k in pg is 2.



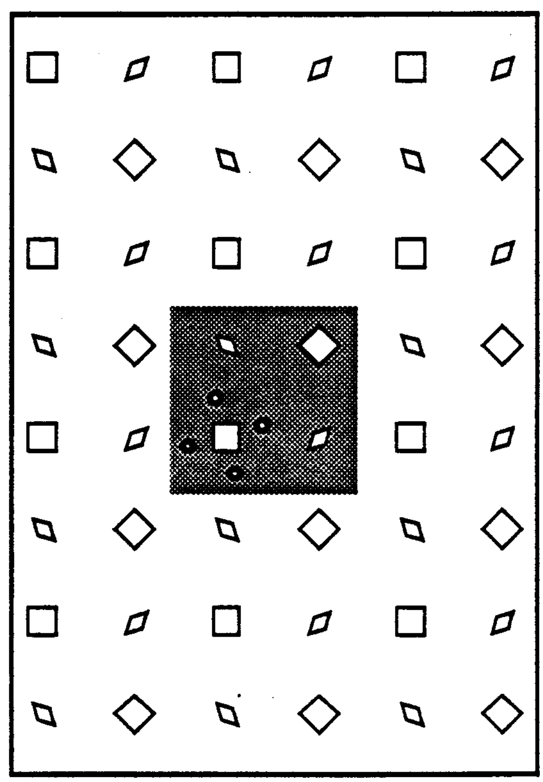


Figure 12. (left) p4 with 28-gons. In addition to the 28-gons there are 3, 4, 5, and 6-gons present in the figure. (right) Group diagram for p4 demonstrating that the maximum value for p_k in p4 is 4.

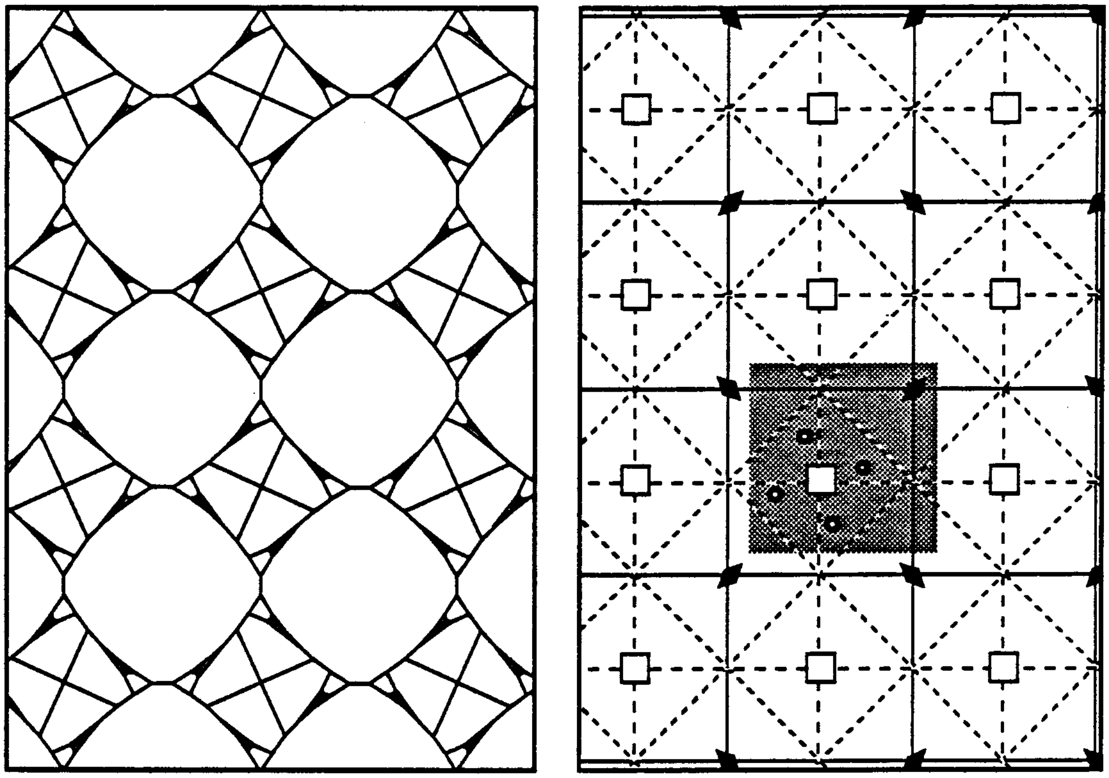


Figure 13. (left) p4g with 28-gons. In addition to the 28-gons there are 3, 4, 5, and 6-gons present in the figure. (right) Group diagram for p4g demonstrating that the maximum value for p_k in p4g is 4.

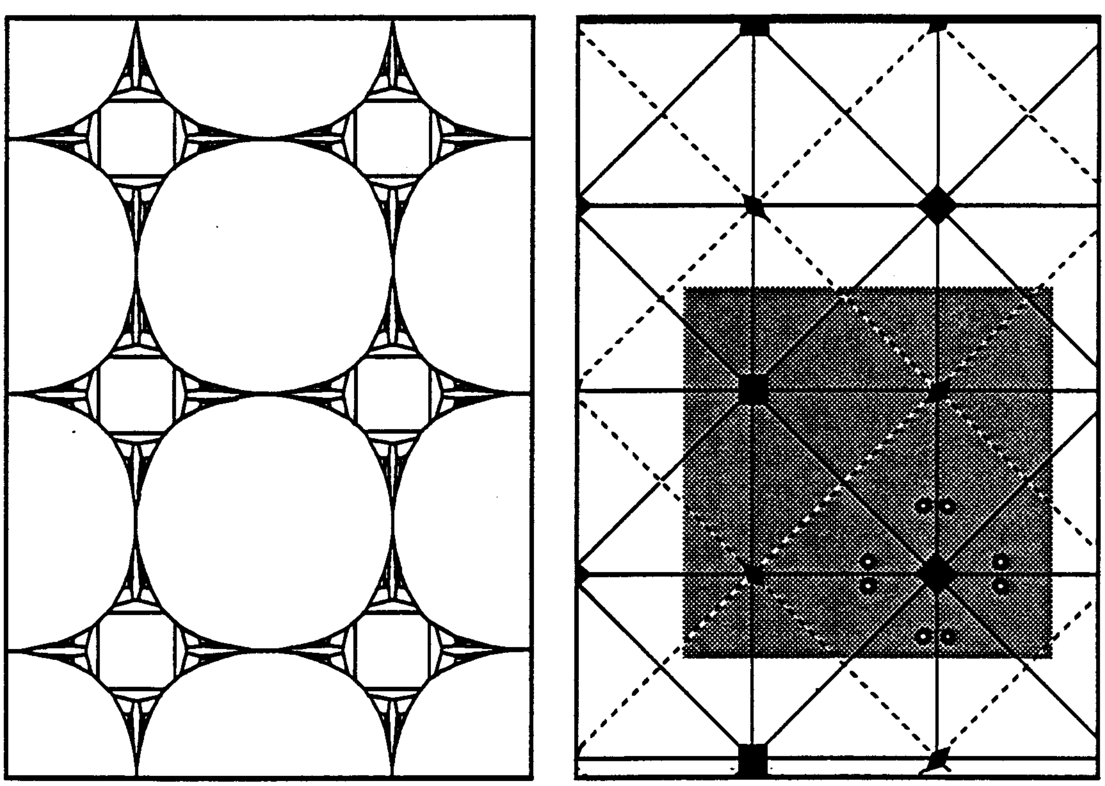


Figure 14. (left) p4m with 48-gons. In addition to the 48-gons there are 3, 4, 5, 6, 7, and 8-gons present in the figure. (right) Group diagram for p4m demonstrating that the maximum value for p_k in p4g is 4.

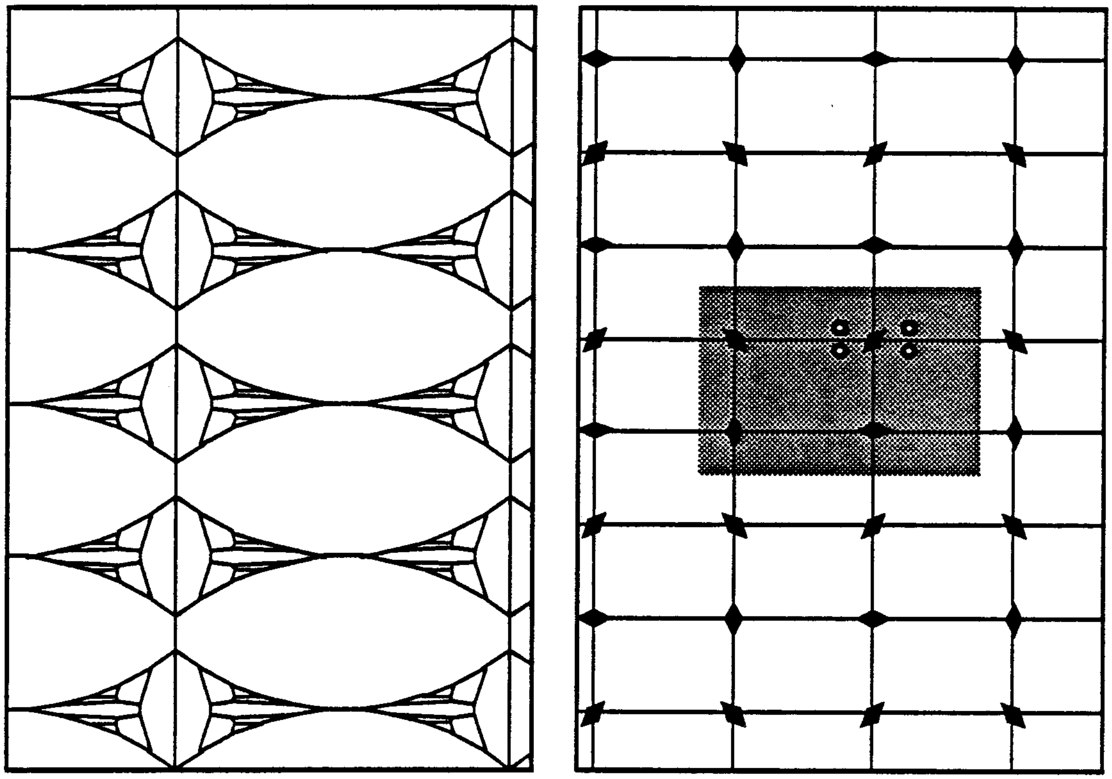


Figure 15. (left) pmm with 24-gons. In addition to the 24-gons there are 3, 4, 5, 6, and 7-gons present in the figure. (right) Group diagram for pmm demonstrating that the maximum value for p_k in pmm is 4.

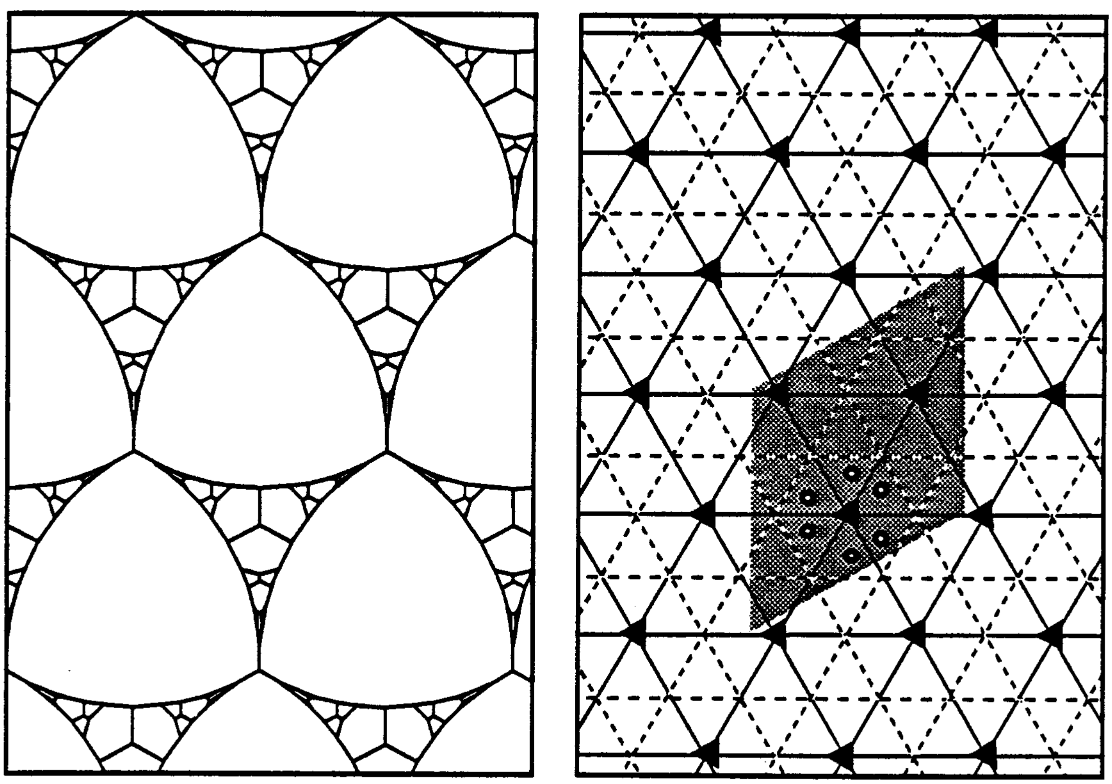


Figure 16. (left) p3m1 with 36-gons. In addition to the 36-gons there are 3, 4, 5, 6, and 7-gons present in the figure. (right) Group diagram for p3m1 demonstrating that the maximum value for p_k in p3m1 is 6.

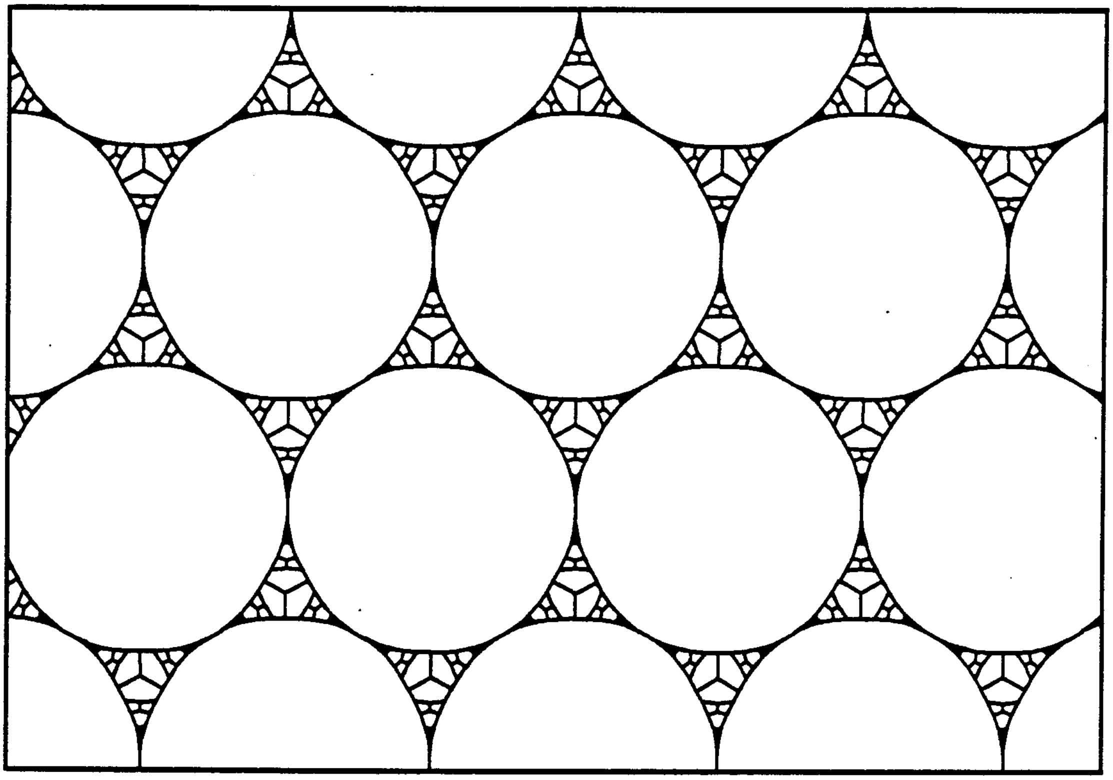


Figure 17. (left) p6m with 66-gons. In addition to the 66-gons there are 3, 4, 5, 6, and 7-gons present in the figure. A full explanation of this figure's derivation can be found in [1].