The Structure of Potentially Infinite Cardinal Numbers

Mike Nakamaye

August 20, 1987

We will develop a theory of potentially infinite cardinal numbers, during the process of which we will encounter several interesting and open-ended problems. Although, we only mention the concept of potential infinity once, it is in light of this concept that this paper should be read.

Part One

Introductory Definitions:

We define a set S of **primitive functions** recursively as follows:

1) x^n is in S $\forall n \in \mathbb{Z}^+$;

2) if $g(x) \in S$, then $h(x) = c + g(x) \in S$, $\forall c \in \mathbb{Z}^+$;

3) if $g(x) \in S$, then $h(x) = c \cdot g(x) \in S$, $\forall c \in \mathbb{Z}^+$;

4) if $g(x) \in S$, then $h(x) = 2^{g(x)} \in S$.

Note: each function in S is understood to be from $\mathbb{R}^+$ into $\mathbb{R}^+$, or , with restricted domain, from $\mathbb{Z}^+$ into $\mathbb{Z}^+$.

A function f is a primitive function if and only if it may be derived from rules 1-4 where each rule is applied at most a finite number of times.

We make the following observations:

Remark (1) Rules 2,3, and 4 produce 'larger' functions, by which we mean that in the definitions of rules 1,2, and 3, $h(x) > g(x)$ for all x in $\mathbb{R}^+$. This is clear in the case of rules 2 and 3 and it can be proved in the case of rule 4 simply by observing that $2^x > x$ if $x \geq 1$ and that if $x < 1$ we have $0 < x < 1 \leq 2^x$.

Remark (2) As a result of (1) we know that the function $f(x) = x$ is the smallest function is S, i.e., if $g(x) \in S$, then $g(x) \geq x$ for all x in $\mathbb{R}^+$. Furthermore, the set T_1 of functions in S which are derived using rule 4 once and only once has as its minimal member the function $f(x) = 2^x$. And, in general, the set T_n consisting of all functions in S which are derived using n and only n applications of rule 4 has as its minimal member:

$f(x) = 2^{2^{2^{\cdot^{\cdot^{2^x}}}}}$ where there are n 2's. For brevity we will give this function the following simpler notation: $f(x) = \Phi_n(x)$. Thus, $\Phi_0(x) = x$, $\Phi_1(x) = 2^x$, etc.

Motivated by the fact that we are interested in the number of applications of rule 4 required to form a function f in S, we make the following definition:

definition: let f be a function in S, then $\mathfrak{D}(f) = n$ where n is the number of applications of rule 4 required to derive f. We also define the following sets related to the function $\mathfrak{D}$: $\mathfrak{D}_0 = \{f \in S: \mathfrak{D}(f) = 0\}$, $\mathfrak{D}_1 = \{f \in S: \mathfrak{D}(f) = 1\}$, ... , $\mathfrak{D}_n = \{f \in S: \mathfrak{D}(f) = n\}$,....

Remark (3): There is no maximal function in $\mathfrak{D}_n$. To see this, it suffices to observe that $x^n \in \mathfrak{D}_0$ for all n in $\mathbb{Z}^+$ and that these functions increase as x increases. The cases for $\mathfrak{D}_1$, $\mathfrak{D}_2$, ... follow immediately.

Finally, in proving many of the theorems which follow, we make use of l'Hôpital's rule and hence we make:

Remark (4): all functions in S are differentiable on $\mathbb{R}^+$. This follows from the fact that rules 1, 2, and, 3 produce differentiable functions on $\mathbb{R}^+$ and from the fact that if g(x) is differentiable on $\mathbb{R}^+$, then $h(x) = 2^{g(x)}$ is differentiable on $\mathbb{R}^+$ $(h'(x) = \ln(2)\cdot g'(x)\cdot 2^{g(x)})$.

We now define the relation f is equivalent to g (written $f \sim g$) as follows:
Let f and g be primitive functions. Then $f \sim g$ if at least one of the following three conditions hold:

1) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = k$ where $k \in \mathbb{R}^+$;

2) $\lim_{x\to\infty} \frac{f(x)}{g(x)} > 1$ and $\lim_{x\to\infty} \frac{f(x)}{[g(x)]^n} < 1$ for some $n \in \mathbb{Z}^+$;

3) $\lim_{x \to +\infty} \frac{f(x)}{g(x)} < 1$ and $\lim_{x \to +\infty} \frac{[f(x)]^n}{g(x)} > 1$ for some $n \in \mathbb{Z}^+$.

Note: in each of these limit statements, we consider the symbol $+\infty$ to be greater than 1. In addition, since every limit we consider in this paper is a limit as x tends to infinity, we will abbreviate the symbol '$\lim_{x \to +\infty}$' by the simpler 'lim.'

We observe that by the definition of the relation $\sim$, $f \sim f$, $f \sim g \leftrightarrow g \sim f$, and lastly, $f \sim g$ & $g \sim h \rightarrow f \sim h$. Thus $\sim$ is an equivalence relation on S.

We make one more definition to facilitate the expression of certain theorems; **definition:** Let f be a primitive function. Then we define the series of functions $f_1, f_2, f_3, \ldots, f_n$ as the individual steps in the construction of the function f from rules 1 through 4; f_{i+1} is obtained from f_i by an application of rule 2, 3, or 4.. For, example, if $g(x) = 4 + 2^{(x+3)}$, then we have the following: $g_1(x) = x$, $g_2(x) = x + 3$, $g_3(x) = 2^{(x+3)}$, and $g_4(x) = 4 + 2^{(x+3)}$. We notice, however, that we have not established that this series of functions is well-defined. And, in fact, it is not. For instance, if $f(x) = 12 + 6x$, we have $f(x) = 3(4 + 2x) = 2(6 + 3x) = 6(2 + x)$. But, we can see that this indeterminacy only concerns the constant terms and that (1) $f_1(x)$ is **always** uniquely determined [It will simply be the power of x occurring in f(x).] and (2) the number of applications of rule 4 is also uniquely determined. These two invariants are sufficient for establishing the desired results.

§1

If $f, g \in \mathcal{D}_0$, then $f \sim g$.

Proof: It suffices to observe that $\lim \frac{f(x)}{g(x)} = k \cdot \lim \frac{f_1(x)}{g_1(x)}$ where $k \in \mathbb{R}^+$ and then just apply l'Hôpital's rule.

One implication of this theorem is that rules 2 and 3 preserve the property of similarity. The general result that $f \sim g \rightarrow f_n \sim g_n$ where f_n is derived form f by using rules 2 and 3 and g_n is derived from g using rules 2 and 3 may be proved from the definition of similarity by straightforward algebraic means.

§2

If $f \in \mathfrak{D}_m$, $(m \geq 1)$ then there exist unique natural numbers $n_1, n_2, \ldots n_m$ such that:

$$(*) \qquad \lim \frac{f(x)}{2^{2n_1^{2n_2^{\cdot^{\cdot^{x^{n_m}}}}}}} = k \in \mathbb{R}^+.$$

Since this is too awkward to write out, we introduce the following abbreviation for the denominator: $2_{(n_1,n_2, \ldots, n_m)}$.

The theorem is stated for $m \geq 1$ but can be extended to $m = 0$. For this special case, we say that if $f \in \mathfrak{D}_0$ then there exists a unique natural number n_0 such that $\lim \frac{f(x)}{x^{n_0}} = k \in \mathbb{R}^+$.

Proof: The case where $m = 0$ is trivial for we simply let n_0 be the exponent of $f_1(x)$. Uniqueness follows from the comments which proceed the definition of $f_1(x)$. We prove the general result by mathematical induction.

Suppose that the theorem holds for all functions f in $\mathfrak{D}_m$. Then we need to show that this implies the theorem for $\mathfrak{D}_{m+1}$. Let f be functions in $\mathfrak{D}_{m+1}$. We prove the result by examining the series of functions $f_1, f_2, \ldots, f_n$. By the definition of $\mathfrak{D}_{m+1}$, we know that rule 4 is applied exactly $m + 1$ times during the construction. Let k be the unique natural number with the following property: f_{k+1} is obtained from f_k by applying rule 4 for the $(m + 1)$st time during the construction $(f_{k+1} = 2^{f_k})$. We know, by the definition of $\mathfrak{D}_m$ that $f_k \in \mathfrak{D}_m$. Thus, there exist unique natural numbers $n_1, n_2, \ldots, n_m$ such that $2_{(n_1,n_2, \ldots, n_m)}$ satisfies (*).

Let $\lim \dfrac{f_k(x)}{2_{(n_1,n_2, \ldots, n_m)}} = k_1 \in \mathbb{R}^+$.

Then, we have (1) $\lim \frac{f_{k+1}(x)}{2_{(k_1,n_1,n_2,\ \ldots,\ n_m)}} = 1 \in \mathbb{R}^+$. This statement is not clearly true; in fact, for general functions it would be false. For example, let $f(x) = x + \sqrt{x}$ and $g(x) = x$. Then, we have $\lim \frac{f(x)}{g(x)} = 1$, but $\lim \frac{2^{f(x)}}{2^{1 \cdot g(x)}} = +\infty$. The reason why it holds for primitive functions is that there is only one term in $f_k(x)$ which tends to infinity as $x \rightarrow +\infty$, namely, the term which contains the x. All other terms are merely constants. Hence, we have:

$$\lim \frac{f_{k+1}(x)}{2_{(k_1,n_1,n_2,\ \ldots\ ,\ n_m)}} = 2^{\lim[\log_2(f_{k+1}(x)) - \log_2(2_{(k_1,n_1,n_2,\ \ldots\ ,\ n_m)})]}$$

$$= 2^{\log_2(2)\cdot\lim[f_k(x) - k_1 \cdot 2_{(n_1,n_2,\ \ldots\ ,\ n_m)}]};$$

But, from what has been said earlier it is clear that the above limit is equal to some constant $c \in \mathbb{R}^+$. This constant will be equal to the sum c of the constant terms in f_k. Thus, the entire limit is equal to 2^c which is clearly in $\mathbb{R}^+$. From here, the rest of the proof is straight forward. The functions f_n, f_{n-1}, ... f_{k+2} are each obtained from their predecessor by an application of either rule 3 or rule two. But rule 2 will not effect the validity of (1) while rule 3 merely multiplies the limit by a natural number. Therefore, we have the desired result that $\lim \frac{f(x)}{2_{(k_1,n_1,n_2,\ \ldots,\ n_m)}} = p \in \mathbb{R}^+$ where k_1 and n_i are uniquely determined.

§3

Theorem I: $\mathfrak{D}(f) > \mathfrak{D}(g) \rightarrow \lim \frac{f(x)}{g(x)} = +\infty$.

We prove this result, like the last, by mathematical induction.

Suppose $\mathfrak{D}(f) = 1$ and $\mathfrak{D}(g) = 0$. Then $\lim \frac{f(x)}{2^x} \geq 1$ by remark (2). Also, there exists a unique natural number n such that $\lim \frac{x^n}{g(x)} = k \in \mathbb{R}^+$. Thus we have the following:

$$\lim \frac{f(x)}{g(x)} \geq \lim \frac{2^x}{g(x)} = k \cdot \lim \frac{2^x}{x^n} = +\infty \text{ by l'Hôpital's rule.}$$

Next, suppose $\mathfrak{D}(f) = m + 1$ and $\mathfrak{D}(g) = m$. Then, by §2, we know that there exist natural numbers $n_1, n_2, \ldots, n_{m+1}, k_1, k_2, \ldots, k_m$ such that:

$$\lim \frac{f(x)}{2_{(n_1, n_2, \ldots, n_{m+1})}} = p \in \mathbb{R}^+ \quad \text{and} \quad \lim \frac{g(x)}{2_{(k_1, k_2, \ldots, k_m)}} = q \in \mathbb{R}^+.$$

We further have:

$$\lim \frac{f(x)}{g(x)} = \frac{p}{q} \cdot \lim \frac{2_{(n_1, n_2, \ldots, n_{m+1})}}{2_{(k_1, k_2, \ldots, k_m)}} =$$

$$(**) \quad \frac{p}{q} \cdot 2^{\log_2(2) \cdot \lim\left[n_1 \cdot 2_{(n_2, n_3, \ldots, n_{m+1})} - k_1 \cdot 2_{(k_2, k_3, \ldots, k_m)}\right]}.$$

But, we know that $2_{(n_2, n_3, \ldots n_{m+1})} \in \mathfrak{D}_m$ and that $2_{(k_2, k_3, \ldots, k_m)} \in \mathfrak{D}_{m-1}$ and hence

$\lim \dfrac{2_{(n_2, n_3, \ldots, n_{m+1})}}{2_{(k_2, k_3, \ldots, k_m)}} = +\infty$. But this implies that $(**) \to +\infty$. This establishes,

by mathematical induction, the fact that if $f \in \mathfrak{D}_n$ and $g \in \mathfrak{D}_{n-1}$ then $\lim \frac{f(x)}{g(x)} = +\infty$. The general case, where $g \in \mathfrak{D}_n$ and $f \in \mathfrak{D}_{n+k}$ follows by choosing functions $h_1 \in \mathfrak{D}_{m+1}$, $h_2 \in \mathfrak{D}_{m+2}$, ..., $h_{m+k-1} \in \mathfrak{D}_{m+k-1}$ and observing that $\lim \frac{f(x)}{h_{m+k-1}(x)} = +\infty$ and that the h_i's decrease as i decreases until we reach h_{m+1} which has the property that $\lim \frac{h_{m+1}(x)}{g(x)} = +\infty$. □

§4

Theorem I tells us that there might be a relationship between the equivalence classes of S under $\sim$ and the sets $\mathfrak{D}_1, \mathfrak{D}_2, \ldots$ since, if $f \in \mathfrak{D}_m$ and $g \in \mathfrak{D}_{m+k}$, then it is not the case that $f \sim g$ (This follows from the definition of $\sim$ and from the fact that if $f \in \mathfrak{D}_m$ then $(f)^n \in \mathfrak{D}_m$.). It would be nice if the equivalence classes of S were precisely $\mathfrak{D}_0$, $\mathfrak{D}_1$, $\mathfrak{D}_2$, ... Unfortunately, this is not the case as can be seen in following example:

Let $f(x) = 2^{x^2}$ and $g(x) = 2^x$. Then clearly $f, g \in \mathfrak{D}_1$. But, $f \not\sim g$ since

$\lim \frac{2^{x^2}}{[2^x]^n} = \lim \frac{2^{x^2}}{2^{n \cdot x}} = +\infty \quad \forall n \in \mathbf{Z}^+.$

Thus, if we want the equivalence classes of S to be $\mathfrak{D}_0, \mathfrak{D}_1, \mathfrak{D}_2, \ldots$, we have to change the definition of our equivalence relation. One way to do this would be to allow the insertion of arbitrary natural numbers at places other than the exponent of the entire function. For example, if $f(x) = 2^{2^{2^x}}$, we could allow the following sort of change: $f'(x) = 2^{3 \cdot 2^{5 \cdot 2^{4 \cdot x}}}$. This would eliminate the problem alluded to in the above example. Thus, we formulate the following revised definition of similarity:

Definition II: $f \sim g$ iff one of the following three conditions holds:

(1) $f \sim g$ under the first definition definition of similarity;

(2) $\lim \frac{g(x)}{f(x)} = +\infty$ and $\lim \frac{g(x)}{f^1(x)} = 0$ where $f^1(x)$ is defined as follows:

Let $2_{(n_1, n_2, \ldots, \ldots n_m)}$ be such that $\lim \frac{f(x)}{2_{(n_1, n_2, \ldots, n_m)}} = k \in \mathbb{R}^+$. Clearly, this is only possible when $\mathfrak{D}(f) \geq 1$, but the case where $\mathfrak{D}(f) = 0$ is not a problem since the in this case there is nowhere else to insert natural numbers other than as exponents of the functions. We define $f^1(x)$ as follows: $f^1(x) = 2_{(k_1 \cdot n_1, k_2 \cdot n_2, \ldots, k_m \cdot n_m)}$ where k_i are arbitrary natural numbers;

(3) $\lim \frac{f(x)}{g(x)} = +\infty$ and $\lim \frac{f(x)}{g^1(x)} = 0$ where $g^1(x)$ is defined in a similar manner.

We can now state:

Theorem II: The equivalence relation $\sim$, using its *second* definition, partitions S into the following equivalence classes: $\mathfrak{D}_0, \mathfrak{D}_1, \mathfrak{D}_2, \ldots$.

Proof: The proof has two steps. We prove (1) $f \sim g \rightarrow \mathfrak{D}(f) = \mathfrak{D}(g)$ and (2) $\mathfrak{D}(f) = \mathfrak{D}(g) \rightarrow f \sim g$.

We prove the contrapositive of (1), namely that $\mathfrak{D}(f) \neq \mathfrak{D}(g) \rightarrow f \nsim g$. Suppose that $\mathfrak{D}(f) \neq \mathfrak{D}(g)$. Assume, for the sake of argument, that $\mathfrak{D}(f) > \mathfrak{D}(g)$. Then $\mathfrak{D}(g) = m$ and $\mathfrak{D}(f) = m + k$ for some $m,k \in \mathbf{Z}^+$. Thus, by Theorem I, $\lim \frac{f(x)}{g(x)} = +\infty$. Furthermore, $g^1(x)$, by its definition, belongs to $\mathfrak{D}_m$ and hence:

$\lim \frac{f(x)}{g^1(x)} = +\infty$. Finally, $[g(x)]^n$ $(n \in \mathbb{Z}^+)$ is also in $\mathfrak{D}_m$ and hence $\lim \frac{f(x)}{[g(x)]^n} = +\infty$. Thus, $f \not\sim g$.

For (2), suppose $\mathfrak{D}(f) = \mathfrak{D}(g) = m$. Then, by §2 we know that there exist natural numbers $k_1, k_2, \ldots, k_m, c_1, c_2, \ldots, c_m$ such that:

(a) $\lim \frac{f(x)}{2_{(k_1, k_2, \ldots, k_m)}} = p \in \mathbb{R}^+$ and (b) $\lim \frac{g(x)}{2_{(c_1, c_2, \ldots c_m)}} = q \in \mathbb{R}^+$.

Suppose that $\lim \frac{f(x)}{g(x)} = j \in \mathbb{R}^+$. Then $f \sim g$. Suppose $\lim \frac{f(x)}{g(x)} = +\infty$. Then we have the following: let $g^1(x) = 2_{(c_1 \cdot k_1, c_2 \cdot k_2, \ldots, 2 \cdot c_m \cdot k_m)}$. Then $\lim \frac{f(x)}{g_1(x)} = o$ and hence, by the second definition of $\sim$, $f \sim g$. The case where $\lim \frac{f(x)}{g(x)} = 0$ is analogous to this last case. Hence $\mathfrak{D}(f) = \mathfrak{D}(g) \rightarrow f \sim g$ and the theorem is proved. □

§5

In this section we will make a few generalizations of the set S of primitive functions. The first and most natural such extension is to the function space S_1 which is defined as follows:

(1) if $f \in S$, then $f \in S_1$;

(2) if $f, g \in S_1$, then $f + g$ and $f \cdot g$ are also in S_1.

We make the observation that if $f \sim g$, then $(f + g) \sim (f \cdot g) \sim f \sim g$. The proof of this fact is straight forward: it involves simple algebraic manipulations in each of the four case for which similarity is defined. We demonstrate one of them for clarity: suppose $\lim \frac{f(x)}{g(x)} = +\infty$ and $\lim \frac{f(x)}{[g(x)]^n} = 0$ for some $n \in \mathbb{R}^+$. Then,

$\lim \frac{f(x)}{f(x) + g(x)} = +\infty$ and $\lim \frac{f(x)}{[f(x) + g(x)]^n} \leq \lim \frac{f(x)}{g(x)} = 0$. Therefore, $f \sim (f + g)$. The case for $f \cdot g$ and the other remaining categories of similarity use a closely analogous method.

Since we want Theorem II to hold in S_1, we make the following definition for the $\mathfrak{D}_i$'s:

definition: If $f,g \in \mathfrak{D}_n$ then $(f + g)$ and $(f \cdot g)$ also belong to $\mathfrak{D}_n$.

Another simple observation is that if $f \not\sim g$, then both $(f + g)$ and $(f \cdot g)$ are similar to the larger of the two functions. We make the analogous definition for the $\mathfrak{D}_i$'s:

definition: If $f \in \mathfrak{D}_m$ and $g \in \mathfrak{D}_n$ where $m < n$, then $(f + g) \in \mathfrak{D}_n$ and $(f \cdot g) \in \mathfrak{D}_n$.

Another extension of S which proves interesting is one which we call S_2 and it is defined as follows:

(1) if $f \in S$, then $f \in S_2$;

(2) the following group of functions belongs to S_2: x^x, x^{x^x}, $x^{x^{x^x}}$, ...;

(3) if $f,g \in S_2$, then both $(f + g)$ and $(f \cdot g)$ are likewise in S_2.

The reason why this function space is interesting is that it contains a model of transfinite cardinal arithmetic (TCA). This, incidentally, was the motivating force for this work. We pursue it in the following section.

§6

To see why S_2 is closely related to TCA, we make a few observations. First, we notice that the equivalence relation $\sim$ (under its second definition) imposes a natural on the sets S, S_1, and S_2. This order corresponds to at least two distinct notions: (1) the number of applications of rule 4 in generating our functions and (2) the sets Z^+, $\wp(Z^+)$, $\wp[\wp(Z^+)]$, ... where $\wp$ means 'power set of.' The intuitive justification for (2) consists in forming a correspondence between the functions in S_2 (or S or S_1) and certain sets which may be constructed in Zermelo-Fraenkel set theory. This correspondence is as follows:

(1) $f(x) = x^n \approx Z^n$;

(2) $f(x) = x^{x^{\cdot^{\cdot^{\cdot^x}}}} \approx Z^{Z^{\cdot^{\cdot^{\cdot^Z}}}}$;

(3) $h(x) = [c + f(x)] \approx c \cup f(x)$ where 'c' represents a set of cardinality c;

(4) $h(x) = c \cdot f(x) \approx c \times f(x)$;

(5) $h(x) = [f(x) + g(x)] \approx f(x) \cup g(x)$;

(6) $h(x) = f(x)\cdot g(x) \approx f(x) \times g(x)$;

(7) $h(x) = 2^{f(x)} \approx \wp[f(x)]$.

The equivalence classes $\mathfrak{D}_0$, $\mathfrak{D}_1$, ... correspond nicely with sets of cardinality $\aleph_0$, $\aleph_1$, ... In light of this fact, we define the cardinality of a function $f \in S_2$:

definition: the cardinality of f, written as $\tilde{f}$ and card(f), is equal to the subscript of the equivalence class to which f belongs, i.e. $f \in \mathfrak{D}_0 \rightarrow \tilde{f} = 0$, $f \in \mathfrak{D}_1 \rightarrow \tilde{f} = 1$, ...

Using the remarks in §5, we can see how closely related this concept of cardinality is to TCA. In particular, we know that if $\tilde{f} \geq \tilde{g}$, then card(f+g) = card(f·g) = card(f): in TCA, we have the similar statement, namely, if $\alpha \geq \beta$, then $(\alpha + \beta) = (\alpha\cdot\beta) = \alpha$, where α and β are any two transfinite cardinal numbers. The laws which govern TCA, such as commutativity of multiplication and addition, distributive property of both addition and multiplication, etc. are all valid for the cardinalities defined in S_2. Thus S_2 is a model of TCA.

§7

Employing this close relationship between TCA and the $\mathfrak{D}_i$'s, we prove the following:

Theorem III: Let $f,g \in S_2$. Then, if $\frac{f(\aleph_0)}{g(\aleph_0)} > 0$, $\lim \frac{f(x)}{g(x)} = +\infty$ where $\frac{f(\aleph_0)}{g(\aleph_0)}$ is defined as follows :

$$\frac{f(\aleph_0)}{g(\aleph_0)} = \begin{cases} 0, & \text{if } g(\aleph_0) = \wp[f(\aleph_0)] \text{ or } \wp\{\wp[f(\aleph_0)]\} \text{ or } \ldots \\ f(\aleph_0), & \text{if } f(\aleph_0) = \wp[g(\aleph_0)] \text{ or } \wp\{\wp[g(\aleph_0)]\} \text{ or } \ldots \end{cases}$$

The fraction is undefined when $f(\aleph_0) = g(\aleph_0)$.

Proof: The proof is an immediate consequence of Theorem I and the observation that the $\mathfrak{D}_i$'s form a model of TCA.

§8

We have defined cardinality on the sets S, S_1, and S_2 and we know when two functions have the same cardinality and when one is less than another. We have not, however, formally defined the latter. Thus we have:

definition: Let f,g $\in$ S (or S_1 or S_2). Then we say that $\tilde{f} < \tilde{g}$ iff $\lim \frac{f(x)}{[g(x)]^n} = +\infty$ $\forall n \in \mathbb{Z}^+$ *and* $\lim \frac{f(x)}{g^1(x)} = +\infty$. We say that $\tilde{f} \leq \tilde{g}$ iff $\lim \frac{f(x)}{g(x)} = +\infty$.

Pursuing this definition, we produce the following sufficient and necessary conditions for $\tilde{f} > \tilde{g}$: $\tilde{f} > \tilde{g}$ iff there exist functions $f_1 \in \mathfrak{D}_n$ and $g_1 \in \mathfrak{D}_m$ where $n > m$ such that $\lim \frac{f_1(x)}{f(x)} = 0$ and $\lim \frac{g_1(x)}{g(x)} = +\infty$.

This reformulation of $\tilde{f} < \tilde{g}$ points in the direction of another extension of the set S. We state this formally as follows:

(1) if f $\in S_1$ then f $\in S_3$;

(2) if $\exists$ two functions f_1 and f_2 in S_3 which both belong to $\mathfrak{D}_n$ for some n with the property that $\lim \frac{f_1(x)}{g(x)} \leq 1$ and $\lim \frac{f_2(x)}{g(x)} \geq 1$, then we say that (a) $g(x) \in S_3$ and (b) $g(x) \in \mathfrak{D}_n$.

It is clear that Theorem I as well as most of the comments in §5 and §7 still hold in S_3. But, it is not at all clear that, for instance, theorem II holds in S_3. In fact, it does not. To see this, we produce a counter-example.

Let $g(x) = x^2 - \ln(x)$. The functions $f_1(x) = x$ and $f_2(x) = x^3$ are both in S_3 and are both in $\mathfrak{D}_0$. Furthermore, they satisfy the requirements of (2) in the definition above. Therefore $g(x) \in S_3$ and $g(x) \in \mathfrak{D}_0$. But, there is no integral power m of x such that $\lim \frac{x^m}{g(x)} = k \in \mathbb{R}^+$. It is easy to produce similar examples for the other orders of functions.

In addition, the nice correspondence between functions and sets is lost because such functions as $g(x) = x^2 - \ln(x)$ cannot be set in any reasonable correspondence with sets. The reason for this is that in our function spaces 'cardinality' is defined more or less in terms of the rate of growth of functions, and hence functions like ln(x) which grow at slower rate than powers of x but which nevertheless tend to infinity as x tends to infinity have a cardinatlity between finite and the 'smallest' infinite cardinality, namely 0. With sets, it makes no difference how sparse a set is in the integers: as long as it is not finite, it will be equivalent to the entire set of integers-- sets are considered as wholes whereas our functions are considered only in relation to their growth. Our functions represent potential infinity while the sets of Zermelo-Fraenkel set theory are actually infinite. The second half of this paper will be concerned with the behavior of these potentially infinite cardinalities; but before concluding this first part, we mention a few more properties which hold in S, S_1, and S_2, but which fail to hold in S_3.

§10

All functions in S, S_1, and S_2 are bounded below by 0. This is not the case in S_3 since the function $h(x) = f(x) - n$ is in S_3, $\forall$ f $\in S_2$ and $\forall n \in \mathbf{Z}^+$. Hence, if we let f_1, f_2, f_3,... be an infinite sequence of functions in S_3 such that $\lim \frac{f_n(x)}{f_m(x)} = +\infty$, when $n > m$, then it is not clear that there exists a well-defined function g(x) such that $\lim g(x) = +\infty$ and $\lim \frac{f_n(x)}{g(x)} = +\infty$, $\forall n \in \mathbf{Z}^+$. (Any function space in which this is true we will call pseudo-complete. The property itself will be termed 'pseudo-completeness;' the justification for this name will emerge later.) When we say well-defined, we mean a closed general form which will be valid for all infinite sequences of functions which satisfy the hypotheses of the theorem, or if this is not possible, a general procedure for producing the function g(x).

An example of a pseudo-complete set of functions is the following: let S_{11} be the same as S_1 except that we allow all real powers of x greater than one rather than only integral powers. Let f_1, f_2, ... be an infinite sequence of functions in S_{11}

which satisfies the property delineated above. Then, we can define $g(x)$ as follows: Let $A_{x_0} = \text{lub}\{y \mid y = f_i(x_0) \text{ for some } i \in \mathbb{Z}^+\}$; we know this exists because the functions in S_{11} are bounded below by $h(x) = x$. If we let $g(x_1) = A_{x_1}$, $\forall x_1 \in \mathbb{R}^+$, then $g(x) \geq x$ and hence $\lim g(x) = +\infty$. Also, we know that $\lim \frac{f_n(x)}{f_{n+1}(x)} = +\infty$ and that $g(x) \leq f_{n+1}(x)$, $\forall x \in \mathbb{R}^+$; therefore $\lim \frac{f_n(x)}{g(x)} = +\infty$ $\forall n \in \mathbb{Z}^+$. Thus S_{11} is pseudo-complete. It is clear that the method used above to show that S_{11} is pseudo-complete is inapplicable in the case of S_3. We leave it as an open problem whether of not S_3 is pseudo-complete.

We observe that the functions whose cardinality is between n and n + 1 are excluded by the definition of S_3 (We haven't yet shown that there are such cardinalities and ask the reader to take it on faith until reading Part Two.). The reason why we selected the name pseudo-completeness is that the function spaces which we know to be pseudo-complete lack certain cardinalities-- exactly which cardinalities must be excluded is not entirely clear. It would be nice to prove a general theorem about the cardinalities of a function space and pseudo-completeness, but, it is not clear that such a general relationship exists; we leave this as another open problem.

PART TWO

§11

We will now attempt expand upon the comments in §9 and §10. We start this process by introducing another extension of S. Let S_4 be defined as follows:

(1) if $f \in S_2$, then $f \in S_4$;

(2) $\log_2(x), \log_2[\log_2(x)], \log_2\{\log_2[\log_2(x)]\}, \ldots \in S_4$ (We will write $\log^{(n)}(x)$ for $\log_2\{\log_2 \ldots \log_2(x)\}$ where there are n log's; this is *not* to be confused with $[\log(x)]^n$);

(3) if $f \in S_2$, then $(f)^{\log^{(n)}(x)} \in S_4$, $\forall n \in \mathbb{Z}^+$;

(4) if $f \in S_4$, then $(c + f)$ and $(c \cdot f)$ are also in S_4, $\forall c \in \mathbb{Z}^+$;

(5) $f, g \in S_4 \rightarrow (f \cdot g)$ and $(f + g) \in S_4$.

This, in practice, is extremely difficult to sort out and to avoid the problem of finding the general form of a function in S_4, we prove the following important theorem:

Theorem IV: Part A: $card(f) = card(g) \leftrightarrow card(f^h) = card(g^h)$ where f,g,h are arbitrary functions in S_4.

Proof:

Suppose card(f) = card(g). There are four ways in which this can happen:

(a) $\lim \frac{f(x)}{g(x)} = k \in \mathbb{R}^+$. In this case $\lim \frac{[f(x)]^{h(x)}}{[g(x)]^{h(x)}} = \lim \left(\frac{f(x)}{g(x)}\right)^{h(x)} = \begin{cases} 0, \text{ if } k < 1 \\ k \in \mathbb{R}^+ \text{ or } +\infty, \text{ if } k = 1 \\ +\infty, \text{ if } k > 1 \end{cases}$

Suppose $k < 1$. Then $\lim\left(\frac{f(x)}{g(x)}\right)^{h(x)} = 0$ and $\lim \frac{[f(x)]^{2 \cdot h(x)}}{[g(x)]^{h(x)}} = \lim \left(\frac{f^2(x)}{g(x)}\right)^{h(x)} = +\infty$; thus, if $k < 1$, $card(f^h) = card(g^{h(x)})$.

Suppose that $k \geq 1$ (when $k = 1$ we are only concerned with the case where the limit is $+\infty$). Then we have :

$\lim \left(\frac{f(x)}{g(x)}\right)^{h(x)} = +\infty$ and $\lim \frac{[f(x)]^{h(x)}}{[g(x)]^{h(x) \cdot 2}} = \lim \frac{[f(x)]^{h(x)}}{[g^2(x)]^{h(x)}} = 0$; and thus $card(f^h) = card(g^h)$.

(b) $\lim \frac{f(x)}{g(x)} = +\infty$ and $\lim \frac{f(x)}{[g(x)]^n} = 0$. In this case, we have $\lim \frac{[f(x)]^{h(x)}}{[g(x)]^{h(x)}} = \lim\left(\frac{f(x)}{g(x)}\right)^{h(x)} = +\infty$ and $\lim \frac{[f(x)]^{h(x)}}{[g^n(x)]^{h(x)}} = 0$; hence, we again have $card(f^h) = card(g^h)$.

The cases where (c) $\lim \frac{f(x)}{g(x)} = 0$ and $\lim \frac{f^1(x)}{g(x)} = +\infty$ and (d) $\lim \frac{g(x)}{f(x)} = +\infty$ while $\lim \frac{g(x)}{[f(x)]^n} = 0$ may be proved in the same manner as (b). Thus, we have proven that $card(f) = card(g) \rightarrow card(f^h) = card(g^h)$ and we now prove the reverse implication.

Suppose $card(f^h) = card(g^h)$. Again, there are for ways in which this can be true and we will give the direct proof of two of these.

(a) $\lim \left(\frac{f(x)}{g(x)}\right)^{h(x)} = k \in \mathbb{R}^+$. In this case, since $\lim h(x) = +\infty$, we know that $0 < \lim \frac{f(x)}{g(x)} \leq 1$ and hence card(f) = card(g).

(b) $\lim \left(\frac{f(x)}{g(x)}\right)^{h(x)} = +\infty$ and $\lim \left(\frac{f(x)}{g^n(x)}\right)^{h(x)} = 0$. The first of these two limit statements says that $\lim \frac{f(x)}{g(x)} > 1$, while the second tells us that $\lim \frac{f(x)}{g^n(x)} < 1$; hence card(f) = card(g). Again, cases (c) and (d) follow the method applied in (b). Therefore, part A of the theorem is proved and we now proceed to:

Part B: card(f) < card(g) $\leftrightarrow$ card(f^h) < card(g^h).

First, suppose that card(f) < card(g). From part A we know that it cannot be the case that card(f^h) = card(g^h). Suppose, then, that card(f^h) > card(g^h). This says that $\lim \left(\frac{f(x)}{g(x)}\right)^{h(x)} = +\infty$ and hence that $\lim \frac{f(x)}{g(x)} \geq 1$. But, since card(f) < card(g), $\lim \frac{f(x)}{g(x)} = 0$. This is a contradiction and hence it is not the case that card(f^h) > card(g^h). But the only possibility left is that card(f^h) < card(g^h). Strictly speaking of course, we have not proved anything since we are *assuming* that one of the three following hold:

(1) card(F) < card (G);

(2) card(F) = card(G);

(3) card(F) > card(G), where F and G are arbitrary functions in S_4. Equivalently, we have not shown that the limit statements occurring in the formulation of $\sim$ always exist. This is left as another open problem. One way around the difficulty would be to use a constructive proof rather than the *reductio ad absurdum* employed here. Again, we leave this as an open problem. At any rate, what we *have* established is that *if* the specified limits exist, then card(f) < card(g) $\rightarrow$ card(f^h) < card(g^h).

Now suppose that card(f^h) < card(g^h). Then $\lim \left(\frac{f(x)}{g(x)}\right)^{h(x)} = 0$ and thus $\lim \frac{f(x)}{g(x)} < 1$. Suppose that card(f) > card(g). Then $\lim \frac{f(x)}{g(x)} = +\infty$ and we have a contradiction. By part A, we know that it cannot be the case that card(f) = card(g). Hence, subject to the same restrictions indicated earlier, we may conclude that card(f) < card(g).

The purpose for this theorem which almost necessarily appears to be a digression is to avoid considering separately the cardinal numbers between 0 and 1, between 1 and 2, between 2 and 3, etc. This theorem tells us, in an indirect way, that the structure of the set of cardinalities between n and n + 1 is the same, for an arbitrary natural number n. More specifically, the structure of the set of cardinalities *less than* 0 is equivalent to the set of cardinalities between 0 and 1. We will try to elucidate this idea as we develop the cardinal numbers less than 0.

Closely related to this theorem is the following: $\text{card}(f^g) < \text{card}(f^h)$ iff $\text{card}(g) < \text{card}(h)$. This can be proved constructively in a straight forward manner and hence we will not prove it here.

§12

We begin this section by showing that the cardinality of $f(x) = \log(x)$ is less than 0. To do this, we select a function of cardinality 0, the simplest choice being $g(x) = x$, and use the definition of cardinality.

$\lim \frac{x}{\log(x)} = +\infty$ and thus $\tilde{f} \le 0$. We also have $\lim \frac{x}{[\log(x)]^n} = \lim \frac{1}{n[\log(x)]^{n+1} \cdot 1/x} = \lim \frac{x}{n[\log(x)]^{n+1}} = \lim \frac{x}{n \cdot (n+1) \cdot [\log(x)]^{n-2}} = \ldots. = \lim \frac{x}{n!} = +\infty$. Therefore, by definition, $\text{card}\left[\log(x)\right] < \text{card}(x) = 0$.

Next, we show that $\text{card}\left[\log^{(n)}(x)\right] < \text{card}\left[\log^{(n-1)}(x)\right]$. We have $\lim \frac{\log^{(n)}(x)}{\log^{(n+1)}(x)} = \lim \frac{1/\ x \cdot \prod_{i=1}^{n-1} \log^{(i)}(x)}{1/x \cdot \prod_{i=1}^{n} \log^{(i)}(x)} = \lim \log^n(x) = +\infty$ by l'Hôpital's rule. L'Hôpital's rule applied m times will also give us: $\lim \frac{\log^{(n)}(x)}{[\log^{(n+1)}(x)]^m} = \lim \frac{\log^{(n)}(x)}{m!} = +\infty$.

Thus, we know that $\text{card}\left[\log^{(n)}(x)\right] < \text{card}\left[\log^{(m)}(x)\right]$ iff $n > m$.

Using the fact that $\text{card}(x) > \text{card}\left[\log(x)\right] > \text{card}(n)$ for $n \in \mathbb{Z}^+$, and using the result related theorem IV we see that $\text{card}(x^n) = 0 < \text{card}(x^{\log(x)}) < \text{card}(x^x) = 1$. We can also see, by a similar argument, that $0 < \ldots. < \text{card}\left[x^{\log^{(m)}}(x)\right] <$

$\text{card}\left[x^{\log^{(m-1)}}(x)\right] < \ldots < \text{card}\left[x^{\log(x)}(x)\right] < \text{card}(x^x) = 1$. And it is in this way that we understand the statement that the structure of the set to cardinalities less than 0 is equivalent to the set of cardinalities between 0 and 1.

In concluding this section, we write in closed form the cardinalities of S_4 which are less than 0: $0 > \text{card}\left[\log(x)\right] > \text{card}\left[\log[\log(x)]\right] > \text{card}\left[\log^{(3)}(x)\right] > \ldots$.

§13

Before extending S_4, we make the comment that it is not pseudo-complete. Let $f_1(x) = \log(x)$, $f_2(x) = \log^{(2)}(x)$, $f_3(x) = \log^{(3)}(x)$, ... This infinite sequence of functions clearly satisfies the hypotheses for pseudo-completeness. There exists no function $g(x) \in S_4$ such that $\lim g(x) = +\infty$ and $\lim \frac{f_n(x)}{g(x)} = +\infty \ \forall \in \mathbb{Z}^+$. We also show that there is indeed such a function g(x).

Let g(x) be defined as follows:

$$g(x) = \begin{cases} \log(x), \text{ if } 0 \leq x < 2^4 \\ \log[\log(x)], \text{ if } 2^4 \leq x < 2^{2^5} \\ \log\{\log[\log(x)]\}, \text{ if } 2^{2^5} \leq x < 2^{2^{2^6}} \\ \vdots \\ \log^{(n)}(x), \text{ if } 2^{2^{\cdot^{\cdot^{n+2}}}} \leq x < 2^{2^{\cdot^{\cdot^{n+3}}}} \end{cases}$$ where there are n 2's.

We can see that $\lim g(x) = +\infty$ since, on any interval of the form $[2^{2^{\cdot^{\cdot^{n}}}}, 2^{2^{\cdot^{\cdot^{n+1}}}}]$, we have $(n-2) \leq x \leq (n-1)$. Thus as $n \to +\infty$, $g(x) \to +\infty$. It is also clear that $\lim \frac{f_i(x)}{g(x)} = +\infty$, since $\forall x > 2^{2^{\cdot^{\cdot^{i+3}}}}$ $g(x) \leq f_{i+1}(x)$. Thus, the reason why S_4 is pseudo complete is that it does not contain this limit cardinal. This limit cardinal is in some sense similar to the ideal number $-\infty$; if we say that $\text{card}(\log(x)) = -1$, $\text{card}\left[\log^{(2)}(x)\right] = -2$, ... this statement becomes a bit more believable. Of course, there is a slight problem with this modification, namely that we do not know whether or not $\lim h(x) = +\infty$ implies that $\text{card}\left[h(x)\right] \geq \text{card}\left[g(x)\right]$ for an arbitrary function h. We leave this as an open problem. One other difficulty with this introduction of a limit cardinal is that it excludes finite cardinal numbers from our system; indeed, any finite cardinal will have the property that $\text{card}(\alpha) < \text{card}\left[g(x)\right]$. If the situation is viewed in a certain light, however, this latter difficulty may be avoided. We are considering the cardinalities of potential *infinities* and hence the finite case is excluded from our consideration from the start. The concept of a minimal potential infinity is rather intriguing, but we leave this for the moment and continue to build our set of cardinal numbers.

§13

We now extend S_4 to $S_{4'}$, which is defined as follows:

(1) If $f \in S_4$, then $f \in S_{4'}$;

(2) If $f \in S_4$ then $(f)^{\log^{(n)}(x)} \in S_{4'}$, $\forall n \in \mathbb{Z}^+$.

As in §12 we will only consider those cardinalities which are less than 0. Thus, we will be considering functions of the form $[\log^{(n)}(x)]^{\log^{(m)}(x)}$ for some m, n $\in \mathbb{Z}^+$. We will abbreviate this as $f_{m,n}$. Before introducing the order of the new cardinals thus introduced, we make a few simple observations which will help to clarify the somewhat complicated situation.

In the first place, it is clear that $\text{card}(f_{m,n}) > \text{card}(f_{m+1,n}) > \text{card}(f_{m+2,n}) > \ldots.$ Equally clear is the fact that $\text{card}(f_{m,n}) > \text{card}(f_{m,n+1}) > \text{card}(f_{m,n+2}) > \ldots.$ But it is not clear, at first sight, what the order of these two sequences will be when they are combined. In fact, the actual result is somewhat counter-intuitive. We state the result and then proceed to justify it: $f_{m,n} > f_{m+1,n} > f_{m+2,n} > \ldots. > f_{m,n+1} > f_{m,n+2} > \ldots$, if $m \geq n$ and $f_{m,n} > f_{m,\,n+1} > f_{m,n+2} > \ldots > f_{m+1,n} > f_{m+2,n} > \ldots.$, if $m < n$. To see why this is true, we present a specific example whose generalization is sufficiently clear.

We have: $f_{2,5} > f_{3,5} > f_{4,5} \ldots.$ and similarly $f_{2,5} > f_{2,6} > f_{2,7} > \ldots$ Now, putting these together, we show that $\text{card}(f_{2,n}) > \text{card}(f_{m,5})$ where $m \geq 3$.

$\lim \frac{f_{2,n}}{[f_{m,5}]^n} \doteq {}_2\lim\left[\log^{(3)}(x)\cdot\log^{(n)}(x) \;-\; n\cdot\log^{(m+1)}(x)\cdot\log^{(5)}(x)\right].$ But since $m \geq 3$ it is clear that $\lim\left[\log^{(3)}(x)\cdot\log^{(n)}(x) \quad - \quad n\cdot\log^{(m+1)}(x)\cdot\log^{(5)}(x)\right] = +\infty$; the $\log^{(3)}(x)$ term dominates the others (To be strictly precise, we need to show that $f_{2,n}$ dominates $[\log^{(m)}(x)]^{n\cdot[\log^{(5)}(x)]^m}$, but the extra natural number m does not effect the limit). Thus we have the following ordering: $f_{2,5} > f_{2,6} > f_{2,7} > \ldots. > f_{3,5} > f_{4,5} > f_{5,5} >$ ....

When we step back to see what exactly is happening here, the behavior becomes somewhat simpler. Given two ordered pairs of natural numbers (a,b) and (α,β), we make the following definition: let $k_1 = \min(a+1,b)$ and $k_2 = \min(\alpha+1,\beta)$. Then, we can say that $\text{card}(f_{a,b}) > \text{card}(f_{\alpha,\beta})$ if $k_1 < k_2$: this follows from the method used above of looking at $2^{\lim[\log(f_{a,b} - f_{\alpha,\beta})]}$, in fact, this method is used in the remainder of this paper. If $k_1 = k_2$, then we look at $l_1 = \max(a+1,b)$ and $l_2 = \max(\alpha+1,\beta)$. If $l_1 < l_2$ then again we have $\text{card}(f_{a,b}) > \text{card}(f_{\alpha,\beta})$. If we have $l_1 = l_2$ and $k_1 = k_2$, then clearly $f_{a,b} = f_{\alpha,\beta}$. Thus we have a general procedure for determining the order of the cardinalities of S_4, which have the form $f_{m,n}$ with $m,n \in \mathbb{Z}^+$ and deriving the order given above is merely an exercise in the application of this procedure.

Implicit in this argument is the following: it is not the case that $f_{a,b} = f_{\alpha,\beta} \leftrightarrow a = \alpha \;\&\; b = \beta$. We demonstrate this as follows: $f_{a,b} = f_{\alpha,\beta} \leftrightarrow l_1 = l_2 \;\&\; k_1 = k_2$. There are numerous ways in which the latter half of the implication can be valid. We list them:

(a) $k_1 = (a + 1) = (\alpha + 1)$ and $l_1 = b = \beta$ [case where $a = \alpha$ and $b = \beta$];

(b) $k_1 = (a + 1) = \beta$ and $l_1 = b = (\alpha + 1)$;

(c) $k_1 = b = \beta$ and $l_1 = (a + 1) = (\alpha + 1)$ [same as case (a)];

(d) $k_1 = b = (\alpha + 1)$ and $l_1 = (a + 1) = \beta$ [same as case (b)].

Hence we have $f_{a,b} = f_{\alpha,\beta} \leftrightarrow (a = \alpha$ and $b = \beta)$ or $(a = \beta - 1$ and $b = \alpha + 1)$. And, as may be anticipated from the fact that there are no other cases when $f_{a,b} = f_{\alpha,\beta}$, applying (b) twice gives us $a = \alpha$ and $b = \beta$. An example of this transformation is the following: $[\log^{(3)}(x)]^{\log^{(3)}(x)} = [\log^{(2)}(x)]^{\log^{(4)}(x)}$. Although perhaps not at first apparent, this equality is clarified by taking the log of both sides. One apparent consequence of this transformation is that $\mathrm{card}(f_{a,b}) = \mathrm{card}(f_{\alpha,\beta}) \leftrightarrow f_{a,b}(x) = f_{\alpha,\beta}(x)$.

§14

Although we have a procedure for generating the order of the functions $f_{m,n}$, we do not know where the functions $g_n(x) = \log^{(n)}(x)$ occur in this ordering, nor have we given the general structure in a closed form. These two subjects will occupy us in this section.

We begin by showing that $g_{k_1} < f_{m,n} < g_{(k_1-2)}$ (To make this statement general one may define $g_0 = x$ and $g_{-1} = x^{\log(x)}$). The first inequality is self-explanatory while, for the second, if is sufficient to show that $f_{k_1,k_1} < g_{k_1-2}$. We know that lim $\frac{f_{k_1,k_1}}{g_{k_1-2}} = 2^{\lim(g_{k_1+1} \cdot g_k - g_{k_1-1})} = +\infty$. This gives us a general orientation of the relationship between the $f_{m,n}$ and the g_k. This information is not, however, sufficient to give the entire ordering because it does not pinpoint the location of the g_k. The derivation of this location is somewhat tedious and for this reason we simply state, without proof, the order of the cardinals less than or equal to $f_{1,1}$ in S_4':

$\mathrm{card}(f_{1,1}) > \mathrm{card}(f_{2,1}) > \mathrm{card}(f_{3,1}) > \ldots > \mathrm{card}(f_{1,2}) > \mathrm{card}(f_{1,3}) > \ldots. > \mathrm{card}(g_1) > \mathrm{card}(f_{2,3}) > \mathrm{card}(f_{3,3}) > \mathrm{card}(f_{4,3}) > \ldots > \mathrm{card}(g_2) > \mathrm{card}(f_{3,4}) > \mathrm{card}(f_{4,4}) > \ldots > \mathrm{card}(g_3) > \mathrm{card}(f_{4,5}) > \ldots\ldots\ldots > \mathrm{card}(f_{n+1,n+1}) > \mathrm{card}(f_{n+2,n+1}) > \mathrm{card}(f_{n+3,n+1}) > \ldots > \mathrm{card}(g_n) > \mathrm{card}(f_{n+1,n+2}) > \ldots\ldots\ldots\ldots$ It takes some time before one's intuition can become a trustworthy guide in these orderings and if this order is not clear to the reader, we suggest that he or she experiment with the limits involved until it does become clear.

We note that it appears as if all ordered pairs (m,n) do not occur on the list, for example, (2,2) is not on the list. But, (2,2) = (1,3) which *does* appear on the ordering. In fact, it is possible to show that if (m,n) does not occur on the list, then (n-1,m+1) does.

One important thing to notice in the ordering is that each g_i becomes a limit point in the sense that to the *left* side there is no adjacent cardinality. In other words, there is no minimal cardinal α with the property that $\alpha > \text{card}(g_i)$. Contrarily, we observe that there is a maximal cardinal α, namely $f_{i+1,i+2}$, such that $\alpha < \text{card}(g_i)$. This property will carry over when we further extend S_4' in the following section.

§15

We extend S_4' to S_4'' as follows:

(1) if $f \in S_4'$, then $f \in S_4''$;

(2) if $f \in S_4'$, then $(f)^{\log^{(n)}(x)} \in S_4''$, $\forall n \in \mathbb{Z}^+$.

We make the observation that $\left([\log^{(n)}(x)]^{\log^{(m)}(x)}\right)^{\log^{(k)}(x)} = [\log^{(n)}(x)]^{\log^{(m)}(x)\cdot\log^{(k)}(x)}$, which we will write as $h_{n,m,k}$. Observing that $h_{n,m,k} = h_{n,k,m}$, we will write both of these as $h_{n(m,k)}$ where (m,k) is understood as an *unordered* pair. Thus, we are interested in ordering the $h_{n(m,k)}$ together with the $f_{m,n}$ and the g_k (the symbols m,n, and k are used independently in the different sets of functions). Trying to discover this order is much more tedious and difficult than in the case where we only considered the g_k and the $f_{m,n}$. As in the last case we will not show the derivation. Rather we introduce a function which will tell us which of two functions has a larger cardinality. The purpose of this is that this function will provide us with a decision procedure for determining the relative size of any two functions the further extensions of S_4''.

Before defining the function, we first define the extensions of S_4'' which are of interest. S_4''' written as $S_{4^{(3)}}$ is defined as follows:

(1) if $f \in S_{4''}$, then $f \in S_{4^{(3)}}$;

(2) if $f \in S_{4''}$, then $(f)^{\log^{(n)}(x)} \in S_{4(3)}$, $\forall n \in \mathbb{Z}^+$.

We can now define $S_{4(n)}$ for all $n \geq 4$. The definition is recursive:

(1) if $f \in S_{4(n-1)}$ then $f \in S_{4(n)}$;

(2) if $f \in S_{4(n-1)}$ then $(f)^{\log^{(m)}(x)} \in S_{4(n)}$, $\forall m \in \mathbb{Z}^+$.

With this information we can now define a function $F\colon \mathbb{Z} \to \{-1, 0, 1\}$.

If n is the least integer for which $f \in S_{4(n)}$, then generalizing the comments made earlier in this section, $f = G_{a_1(a_2,a_3,\ldots,a_n)} = \left[\log^{(a_1)}(x)\right]^{\log^{(a_2)}(x)\cdot\log^{(a_2)}(x)\cdot\ \ldots\ \cdot\log^{(a_n)}(x)}$ for some set $\{a_i\}$ of natural numbers where $(a_2, a_3, \ldots, a_n)$ is an unordered (n-1)-tuple.

Let $(f_1, f_2, \ldots f_n)$ be $(a_1 + 1, a_2, a_3, \ldots a_n)$ in *increasing* order and let $(g_1, g_2, \ldots g_n)$ be defined similarly.

Then $F_{f,g}(x) = \begin{cases} +1 \text{ if } f_x > g_x \\ 0, \text{ if } f_x = g_x \\ -1, \text{ if } f_x < g_x \end{cases}$

The one stipulation needed is that $f_x = 0 \quad \forall x > n$ where in is the least integer such that $f \in S_{4(n)}$. Let $k_{f,g}$ be the *least* integer for which F is not equal to zero. Then we have the following important theorem:

Theorem V: $\text{card}(f) < \text{card}(g)$ iff $F(k) = 1$, $\text{card}(f) = \text{card}(g)$ iff $F(x) = 0$ $\forall x \in \mathbb{Z}^+$, and $\text{card}(f) > \text{card}(g)$ iff $F(k) = -1$.

We will not prove this since it is essentially a mechanical exercise using the same methods which have been applied throughout part II of this essay. We make two observations. First, the definition of F implies that $F_{f,g}(x) = -F_{g,f}(x)$ $\forall x \in \mathbb{Z}^+$. Secondly, we note that the theorem implies that two functions can have the same cardinality if and only if they appear in $S_{4(n)}$ for the same set of n's. We can, in fact, list the set of functions which are equivalent to a given function f. The transformations discussed in §13 carry over nicely for the general case of $S_{4(n)}$. The following set of n functions are equivalent to $f_{a_1(a_2,a_3,\ldots,a_n)}$:

(1) $f_{a_1(a_2,a_3,\ldots,a_n)}$

(2) $f_{a_2-1(a_1+1,a_3,a_4,\ \dots,\ a_n)}$

(3) $f_{a_3-1(a_1+1,a_2,a_4,\ \dots,\ a_n)}$

$\vdots$

(n-1) $f_{a_{n-1}-1(a_1+1,a_2,a_3,\ \dots,a_{n-2},a_n)}$

(n) $f_{a_n(a_1+1,a_2,a_3,\ \dots,\ a_{n-1})}$.

The structure of $S_{4^{(3)}}$ is fairly complicated and we will give the ordering of its cardinalities without proof (we omit the 'card' notation as it is implicit that we are discussing the functions' cardinalities):

$h_{1(1,1)} > h_{2(1,1)} > h_{3(1,1)} > \dots > h_{1(2,1)} > h_{2(2,1)} > \dots > f_{1,1} > h_{2(3,1)} > h_{3(3,1)} > h_{4(3,1)} > \dots > f_{2,1} > h_{3(4,1)} > h_{4(4,1)} > h_{5(4,1)} > \dots\dots\dots > h_{n(n+1,1)} > h_{n+1(n+1,1)} > h_{n+2(n+1,1)} > \dots > f_{n,1} > h_{n+1(n+2,1)} > \dots\dots\dots > f_{1,2} > h_{2(3,2)} > h_{3(3,2)} > h_{4(3,2)} > \dots > f_{1,3} > h_{3(4,2)} > h_{4(4,2)} > \dots\dots\dots > h_{n+1(n+2,2)} > h_{n+2(n+2,2)} > \dots > f_{1,n+2} > h_{n+2(n+3,2)} > \dots\dots\dots > g_1 > h_{2(3,3)} > h_{3(3,3)} > \dots > f_{2,3} > h_{3(4,3)} > h_{4(4,3)} > \dots > f_{3,3} > h_{4(5,3)} > h_{5(5,3)} > \dots\dots\dots > g_2 > h_{3(4,4)} > h_{4(4,4)} > \dots\dots\dots\dots\dots\dots\dots > g_n > h_{n+1(n+2,n+2)}$.

As in the last case, this is at first extremely opaque but it becomes much clearer with familiarity. We make the observation that, as in the last extension, each cardinality of $S_{4''}$ becomes a limit point to the left of elements in $S_{4^{(3)}}$ but that it has an immediate successor to the right. This behavior is in fact general and we prove the general result in the following theorem:

Theorem VI: If n is the minimal integer for which $f \in S_{4^{(n)}}$, then there is a maximal function $g \in S_{4^{(n+1)}}$ such that $f > g$ but there is no minimal function $h \in S_{4^{(n+1)}}$ such that $h > f$.

Proof: f is of the form $f_{a_1(a_2,a_3,\ \dots,\ a_n)}$; we look at the infinite series of functions $t_m = t_{a_1(a_2,a_3,\ \dots,\ a_n,m)}$. It is clear that these functions are all larger than f and that $t_1 > t_2 > \dots$.. Thus we need to show that there is no function $p \in S_{4^{(n+1)}}$ such that $t_m > p > f$, $\forall m \in \mathbf{Z}^+$. We show this using the function F(x) defined earlier. If there is such a function p(x) then we have that $F_{p,f}(k_{p,f}) = 1$ and $F_{p,t_m}(k_{p,t_m}) = -1$

$\forall m \in \mathbb{Z}^+$. But, by the definition of the t_m, we know that $F_{p,f}(x) = F_{p,t_m}(x)$ for $x = 1,2,3, \ldots n$. Since $p \in S_{4^{(n+1)}}$ we also know that $F_{p,f}(x) = F_{p,t_m}(x)$ $\forall x > n$. From this it follows that the only integer for which $f_{p,f}$ and F_{p,t_m} differ is $n+1$. But since the (n+1)st term of t_m grows arbitrarily large, it eventually will become larger than the (n+1)st term of p and thus we will have, contrary to our assumption, $F_{t_m,p}(k_{t_m,p}) = +1$. Hence, no such function p exists and f is a limit point on its left side.

The second part of the theorem we show by a simple construction. We know that f is of the form $f_{a_1(a_2, \ldots a_n)}$. Let $(b_1,b_2, \ldots b_n)$ be the a_i in *increasing* order. Then, let $g(x) \in S_{4^{(n+1)}} = h_{b_n(b_1,b_2, \ldots, b_n+1)}$; we claim that this is the maximum function h in $S_{4^{(n+1)}}$ such that $f > h$. To see this , it suffices to observe that the only way to make h larger is to decrease one or more of the b_i , but this alteration will always make $h > f$.

§16

In this section we consider the transformations in the general case of $S_{4^{(n)}}$. We remarked earlier that there are n functions which are in the equivalence class of a given cardinality of $S_{4^{(n)}}$; unfortunately, it is not always the case that these functions are distinct. For example, if we transform $f_{2,3}$ we get $g_{(3-1),(2+1)} = f_{2,3}$. Hence there is only one function in this equivalence class. What we want is to have a one-to-one correspondence between all n-tuples $(a_1,a_2, \ldots, a_n)$, $a_i \in \mathbb{Z}^+$ and the set of cardinalities of $S_{4^{(n)}}$. The reason why this is problematic is that each function $f \in S_{4^{(n)}}$ can represent more than one n-tuple. For instance, $f_{1,4} = f_{3,2}$ and hence this one function is related to two pairs of numbers. But, on the other hand, we have $f_{4,1} = f_{2,3} \not\approx f_{1,4}$ and thus, although the function $f_{1,4}$ corresponds to two distinct couples of integers, exactly two permutations of $f_{1,4}$ occur on our list, namely $f_{1,4}$ and $f_{2,3}$. Hence, we have a one-to-one correspondence between the distinct permutations of $f_{1,4}$ and the and the distinct transformations of $f_{1,4}$. This property, however is not universal. For example $f_{3,3}$ has only one permutation (itself) but $f_{3,3} = f_{2,4}$. Therefore, we may conclude that there is not, in general, a one-to-one correspondence between the distinct permutations of f and its distinct transformations.

Clearly, there is a one-to-one correspondence between the set of integral n-tuples and the set of cardinalities of $S_{4^{(n)}}$ since both are denumerably infinite, but we leave it as an open question whether or not there is simple correspondence of the sort we were attempting to give, one which will admit extension to $S_{4^{(\infty)}}$ which we briefly consider in the next and concluding section.

§17

The definition of $S_{4^{(\infty)}}$ follows along the lines of our other definitions. If $f \in S_{4^{(\infty)}}$ then f has one of the following two forms:

(1) $f_{a_1(a_2,a_3,\ldots,a_n)}$ for some $n \in \mathbb{Z}^+$;

(2) $f_{a_1(a_2,a_3,\ldots)}$.

Of course, we have not shown that this extension to infinite sequences in the exponent of a function is legitimate. We need some sort of interpretation for what the function $f_{1(1,1,1,\ldots)}$ corresponds to. One possible suggestion would be that it represents $f(x) = [\log(x)]^{(\log(x))^x}$ since the exponent is growing linearly in a sense, i.e. we go from $[\log(x)]^{\log(x)}$ to $[\log(x)]^{(\log(x))^2}$ to $[\log(x)]^{(\log(x))^3}$; there seems to be some relationship between the function $g(x) = x$ and the rate at which the exponent of the exponent is growing. This will perhaps work in this case, but we must also deal with functions such as $g_{1(2,1,2,1,2,\ldots)}$. This should be less than the function f discussed previously since it contains two's where f contains one's. One possible interpretation here would be $g(x) = [\log(x)]^{(\log(x))^{\sqrt{x}} \cdot (\log[\log(x)])^{\sqrt{x}}}$. But of course, this sort of interpretation breaks down in a case such as $h_{1(1,2,1,2,3,1,2,3,4,\ldots)}$ where each natural number occurs infinitely often. Nor will we be able to deal with $p_{1(2,3,4,5,\ldots)}$ where each natural number occurs only once. This is another open problem which we leave for the reader.

We make the observation that *if* (1) there is a one-to-one correspondence between all infinite sequences of natural numbers and *if* (2) we can show that there are functions with cardinalities not occurring in $S_{4^{(\infty)}}$ but 'between' two cardinalities of $S_{4^{(\infty)}}$ then we will perhaps obtain some insight into the structure of the real number system, or more genreally, into what sorts of cardinal structures are possible. As far as (2) is concerned, we make the suggestion that a function like

$f(x) = [\log(x)]^{(\log\log(x))^{\log^{(3)}(x)^{\log^{(4)}(x)^{\cdot^{\cdot^{\cdot}}}}}}$ is probably between two cardinalities of $S_{4^{(\infty)}}$ and perhaps does not correspond to any function in $S_{4^{(\infty)}}$. This is another open problem and is, of course, dependent on an interpretation of an infinite product as an exponent, assuming that such an interpretation is possible.

Two more open problems concern by now familiar topics:

(1) What sorts of transformations leave a function unchanged?

(2) Is F(x) still valid for determining the relative size of two functions?

Again, we see that these questions are both dependent upon an interpretation of the infinite sequence of natural numbers and thus this appears to be the fundamental problem. But we can dismiss (2) without such an interpretation, for when we order the elements of an infinite sequence according to their size, we change the ordinal number of the sequence, and the reason why F(x) is valid in the finite case is that ordering a finite set does ***not change*** its ordinal number. And with this purely negative comment and with these unsolved problems we end this paper with the hopes that the reader will pursue the questions where we leave off.