A Pictorial Exploration of an Exponential First-Order Difference Equation via Complex Iterations

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Abstract

In this paper I report the results of my exploration into the dynamics of a real, first-order difference equation. The exploring was done by looking at the function over the complex numbers and the for diffeerent values of a parameter having a computer plot points in different colors depending on whether they converged to a fixed point, converged to a cycle of some period or diverged. As much as possible, these pictures are verified mathematically and possible results with respect to the appearance of cycles and their change of period are discussed.

INTRODUCTION

The dynamics of real difference equations, both polynomial and exponential, is well documented with respect to stability (1) and also to the appearance of, and change in period of, cycles (2, 3) for specific models as a parameter is varied. What is not known is how these same functions are behaving in the field of complex numbers. Presumably there could be cycles in $\mathbb{C}$ before there are any in $\mathbb{R}$; and perhaps at some point these cycles 'expand' to include intervals of the real line.

This paper relates some computer graphic based research that I did on this possibility as well as the results I came up with. It is organized into three sections. 1: How I generated the pictures. 2: A mathematical verification of the pictures. 3: What do the pictures show?

For the sake of clarity, I've included my definitions of some terms used in the paper. A <u>fixed point</u> of a function is a point $\bar{z}$ such that $\bar{z} = f(\bar{z})$. A <u>cycle of period n</u> is a sequence of n distinct points $(z_1, z_2, \ldots, z_n)$ such that $z_2 = f(z_1)$, $z_3 = f(z_2)$, . . . , $z_1 = f(z_n)$. The <u>iteration</u> of a function means composing the function with itself; i.e. $f(f(z))$ is the second iterate of f. A fixed point is <u>stable</u> (locally stable) if successive iterates of all points "near" the fixed point approach it. It is <u>unstable</u> if the iterates do not approach the fixed point.

How I generated the pictures

I started out with an exponential model suggested by (1); it is $F(x) = x \exp(\, G(x)\,)$ where $G(x) = -1.9(x-1) + \alpha(x-1)^3$. In the paper

this is an example of a function that is locally but not globally stable (globally stable means <u>all</u> points approach or converge to a fixed point). Because there is not global stability, there should be cycles for some values of α. Because of the computer facilities (with color graphics) available and my lack of experience in complex analysis, I attacked this problem experimentally. To do this, I first considered the function as a complex one instead of a real one and then to save execution time I expanded it into real and imaginary parts. In this way I ended up with ($z_t = x_t + iy_t$) $z_{t+1} = f(x_t, y_t) + ig(x_t, y_t)$, where

$$f(x,y) = \exp(\mathcal{B})(x\cos\Gamma - y\sin\Gamma),$$

$$g(x,y) = \exp(\mathcal{B})(x\sin\Gamma + y\cos\Gamma), \text{ and}$$

$$\mathcal{B} = -(x-1)(1.9 + 3\alpha y^2) + \alpha(x-1)^3 \text{ and}$$

$$\Gamma = y(3\alpha(x-1)^2 - 1.9) - \alpha y^3.$$

The program I wrote would get as input a value for the parameter α and a rectangular region and partition size in $\mathbb{C}$. For each of the points in the partition of the region, successive iterations of the function were calculated until either convergence to a fixed point or cycle occurred or the values for x of y got beyond machine capabilities (either too big or too close to zero). The starting x,y position was then plotted in a color corresponding to what happened when the point was iterated.

Rather than mindlessly generate pictures, I made a few restrictions. First I only looked at nonnegative x and y starting positions. I chose nonnegative x because in $\mathbb{R}$, negative x values approach $-\infty$. That is, negative x values don't converge, so naturally there can be no negative reals that converge to a cycle. Only nonnegative y values were used because pictures generated were symmetric when negative y values were used. This can be shown from the function: $\mathcal{B}$ is an even polynomial in y so it's value is

unaffected by a sign change; Γ is an odd polynomial in y so it's sign is just reversed. But in calculating y_{t+1}, cos Γ is even and sin Γ is odd, so y cosΓ and xsin Γ will both have opposite signs, so if $g(x,y) = y_{t+1}$ then $g(x,-y) = -y_{t+1}$. Thus iterating the complex conjugate of any point just yields the sequence of conjugates, i.e. when a point cycles, it's conjugate will also cycle. A natural question then is "what happens to real starting points?" Naturally we expect them to remain in $\mathbb{R}$ because our function is just the complex expansion of a real function, but let's verify this anyway, just to be safe. Setting $y_t = 0$ $\Rightarrow \Gamma = 0$, so $x_{t+1} = x\exp(\mathbb{B})$ and $y_{t+1} = 0\exp(\mathbb{B}) = 0$, so real values do stay real.

Because I was interested in what was happening as real points began to cycle, I focused my attention on these α values. I was able to rule out $\alpha > 0$ (my second restriction) with the use of another program. This program took α as input and then plotted the real function F(x) and it's second iteration for $x \in [0,3]$. When the second iterate first crosses the line y=x at any point other than the fixed point, there will be a cycle of period 2 appearing. I ran the program for many positive α values and found none that would have real cycles; since the function is continuous, I expect there are no positive alpha with real cycles of period 2. To find the $\alpha < 0$ where real cycles appear and change period, I ran the first program on just the real interval [0,3] (imaginary part = 0.0) for α = 0.0, -0.1, -0.2, ... , -1.9 and noted the color changes. As an aside, I also used the second program to verify the α at which a period 2 cycle does appear.

The third restriction I imposed was to limit the region of $\mathbb{C}$ that I iterated the function over. I knew from (1) that a fixed point of

this model is 1 + i0, so I generated the initial pictures to include this point, choosing $0 \le Re(z) \le 2.5, 3.0$ and $0 \le Im(z) \le 2.0$ as the initial region because it fit the graphics screen well and contains the expected "interesting" region of the model. As a point of interest, there are infinitely many other fixed points which I believe are unstable. However, the analysis to show this rigorously has not yet been completed.

Using these restrictions, I set out exploring and found that period 2 cycles appear between $\alpha = -1.14$ and -1.15, and period 4 cycles appear between $\alpha = -1.34$ and -1.35. Naturally then I generated pictures in these regions, as well as others (to get a general idea of the dynamics). Besides these "large scale" pictures, I also generated some "close-ups" of smaller regions whose behavior was unclear at the partition size (0.01) or which looked self-similar. Additionally, I generated a couple pictures for $2.5, 3.0 \le Re(z) \le 5.0$ to see the general form "farther out."

A Mathematical Verification of the Pictures

With all of that as background, let us proceed to the pictures themselves. The majority of those generated during my study have been included and are sorted according to size of α and, when necessary, size of region portrayed. All pictures have been properly labelled. Here is and explanation of the colors seen. The darkest blue/purple color is plotted at all points that converged to the fixed point 1.0 + i0.0 (to within 0.000001). The lighter blue/purple is for points that converged to the fixed point 0.0 + i0.0 (again, within 0.000001). The red, green and pink/purple colors are points that converged to a cycle of period 2, 3 and 4 respectively. White points inside the region are those that "diverged." Here "diverged" means

that the next iteration would cause an overflow error. Experimentally I found that if either $|Re(z)|$ or $|Im(z)| > 5.0$ I could get an error, so I used this as a limit. Yellow points are those that after the indicated number of iterations had not yet converged (again to within 0.000001) or diverged.

At this point let me mention why there are two distinct sets of pictures. Before plotting many of these pictures, I double-checked the algebra involved in forming the equations for $f(x,y)$, $g(x,y)$, $\mathcal{B}$ and Γ and found it to be correct. However, what I failed to verify was that my code correctlly reflected these. There was a small, but significant, sign error in Γ - instead of "$-\alpha y^3$" at the end, the program had "$+\alpha y^3$" - which I didn't notice until later. Fortunately this had no effect on the real line (where $y = 0$), so that while I couldn't trust the previous pictures to be correct, I did know where to look, because cycles in $\mathbb{R}$ were still in the same intervals. It was "only" the behavior in $\mathbb{C}$ that would be affected. These first "incorrect" pictures were included because of an interesting phenomenon that was noticed, which will be discussed later, and because of their general similarity to the "correct" ones. It is worth noting that the $\alpha = 0.00$ picture is the same for both sets (this can be seen by observing that if $\alpha = 0.0$, then $\Gamma = 0.0$ regardless of the sign on the last term), and therefore is included only once.

Having exposed one error, the logical thing to do is to verify mathematically, as much as possible, what is seen in the pictures. The existence of points which converged to zero (other than zero itself) was a surprise. Numerical investigation of many of these points (for example $Re(z) > 4.0$, $\alpha = -1.85$, "incorrect" pictures) showed that what really happens is that a lack of computer precision makes very small numbers <u>look</u> like zero. The nature of the model (1)

indicates that the population dying out won't happen in $\mathbb{R}$, and showing that this can't happen in $\mathbb{C}$ is pretty simple. For a point $x + iy$ (x,y) to converge to 0,0 we need:

$$0 = x_{t+1} = \exp(\mathcal{B})(x\cos\Gamma - y\sin\Gamma) \quad \text{and}$$

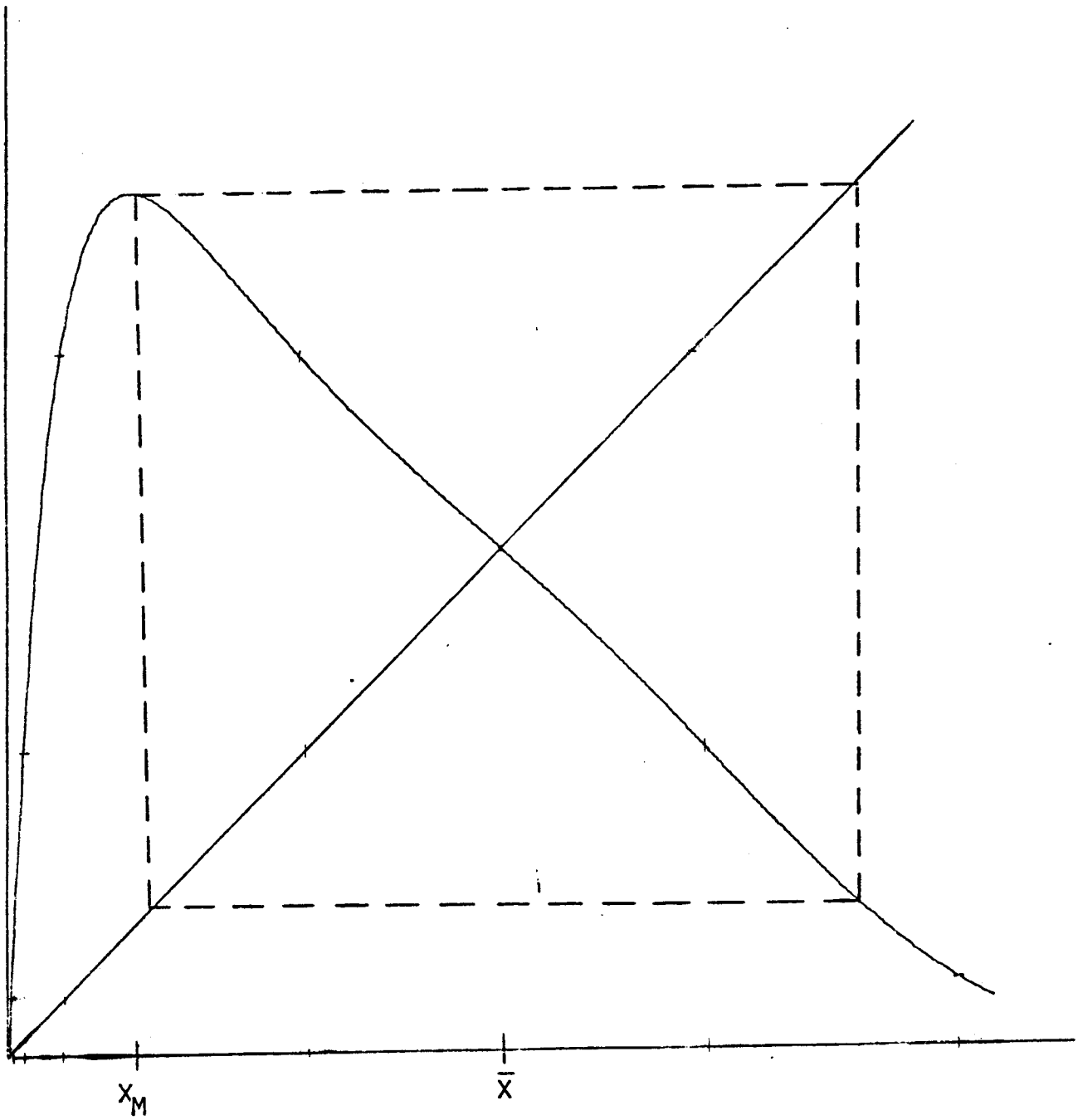
$$0 = y_{t+1} = \exp(\mathcal{B})(x\sin\Gamma + y\cos\Gamma).$$ Because $\exp(\mathcal{B}) \neq 0$, we cancel it. Then consider the following cases. First, if $x,y \neq 0$ and $\Gamma \neq n\pi$ $(n \in \mathbb{N})$ we have $x/y = \sin\Gamma / \cos\Gamma = -y/x$ (one equality from each equation). Let $r = x/y$, then the equalities imply $r = -1/r$ or $r^2 = -1$, which has no real solutions, so no x,y can go to 0,0 under these conditions. Second, if $\cos\Gamma = 0$ we have $y\sin\Gamma = 0 = x\cos\Gamma \Rightarrow y = 0 = x$ since $\sin\Gamma \neq 0$. Similarly if $\sin\Gamma = 0$, $x\cos\Gamma = 0 = y\sin\Gamma \Rightarrow x = 0 = y$. Thirdly, if $x = 0$, $-y\sin\Gamma = 0 = y\cos\Gamma \Rightarrow y = 0$. Similarly if $y = 0$, $x\cos\Gamma = 0 = x\sin\Gamma \Rightarrow x = 0$. So we see that the only point that can converge to 0,0 when iterated is 0,0.

The alternating between regions that converge to the fixed point and to a cycle ($-1.1402 \leq \alpha \leq -1.35$ in the pictures) verifies behavior expected from a population model, at least along the real line (See Figure 1). So the continuation into $\mathbb{C}$ of this pattern is not surprising. Another pattern that is "easy" to account for is best observed for $\alpha = -1.15$ (in the "correct" pictures), although the start of it can be seen in most "full scale" pictures. These alternating 6 white "hearts" (on their sides) and blue/purple and red balls which start about $\mathrm{Re}(z) = 2.0$ and continue for larger $\mathrm{Re}(z)$ are most likely the result of the periodicity of the sine and cosine functions.

As a final point, let's verify divergence for "big enough" x and y, as seen by the white portion in the upper, righthand part of the "full scale" pictures (for both sets). We have that

$$\mathcal{B} = (x-1)[-1.9 - 3\alpha y^2 + \alpha(x-1)^2]$$

Figure 1: The graph of $F(x) = x\exp(G(x))$ and $y=x$ which is commonly used to 'trace' the successive iteration of $F(x)$. The dashed lines represent the cycle of period 2. Blue areas on x are those that converge to the fixed point, $\bar{x}$, and pink areas are those that converge to the cycle; note that the repeating pattern near zero continues (see "incorrect," $\alpha = -1.145$). The boxes help clarify where each region maps to; note that boxes of the same color are at the same height.



$- (x-1)[-1.9 + \alpha(-3y^2 + (x-1)^2)]$. For "big enough" x and y, -1.9 is insignificant, so the use of $\alpha < 0 \Rightarrow \mathcal{B} > 0$ ($\exp(\mathcal{B}) > 1$) **if** $(x-1)^2 < 3y^2$ or $x-1 < \sqrt{3}\,y$. This is the region above a line of positive slope (quite like the one seen in $\alpha = -1.15$). Now let's look at the distance of successive iterates from the origin. Because $\exp(\mathcal{B}) > 1$, we can ignore it, and we have:

$$
\begin{aligned}
x^2_{t+1} + y^2_{t+1} &> (x\cos\Gamma - y\sin\Gamma)^2 + (x\sin\Gamma + y\cos\Gamma)^2 \\
&= x^2(\cos^2\Gamma + \sin^2\Gamma) + y^2(\cos^2\Gamma + \sin^2\Gamma) \\
&= x^2 + y^2.
\end{aligned}
$$

Under these conditions then, we know that successive points are getting farther from the origin (that is, diverging), so we believe the white areas we see. This sounds good, but unfortunately, it fails to show that points which satisfy the first conditions (on α, x and y), still satisfy those conditions after they have been iterated. I was unable to show this, or to show divergence another way. The pictures do show this divergence though, so I haven't given up hope that it can be shown mathematically.

The difficulty in definitely attributing these phenomena to a specific features lies in that Γ is a third-order mixed polynomial in two variables and we need to take (and understand) the sine and cosine of it for general x and y. This as well as the general complexity of the functions f and g are part of the reason the "incorrect" pictures went unchecked for so long. They looked entirely reasonable, having the above mentioned patterns (see $\alpha = -1.85$ and others), but they also had symmetry near the fixed point, which also seemed reasonable. Unfortunately, I wasn't able to show mathematically that it is incorrect, but later I will discuss what seems to be responsible for the difference between the two sets of pictures.

What do the Pictures Show

Now that the mathematical correctness of the pictures has been verified to some extent, what can be discovered from a careful study of them? My main question was "Are there regions in $\mathbb{C}$ that cycle that somehow 'expand' as α changes to include intervals of $\mathbb{R}$ and thereby give real points which cycle?"; and the answer appears to be "No, there are not." Something entirely different is happening as real cycles appear. There seems to be a series of "valleys" of points that diverge, and these valleys get deeper, or closer to the real line (i.e. to each other because the same thing is happening for $\mathrm{Im}(z) > 0$). At some point (between $\alpha = -1.14015$ and -1.1402) these valleys stop growing, and then suddenly, for a slightly different value of α, entire areas between valleys "turn red," that is, cycle instead of converging to a fixed point. This can be seen in the pictures for $\alpha = -0.55$, -1.10 and -1.15 (in this order). Note that the valleys occur everywhere that a red ball will appear and not just along the real line. An interesting observation that I'm not sure what to make of is that the valleys here don't grow straight, but tend to circle in (see especially $\alpha = -1.14015$, a close-up). We aren't done here though; as α continues to get more negative, the once "smooth" (from a full-scale view) balls show valleys too ($\alpha = -1.34$ vs. -1.15). But note this time ($\alpha = -1.348$ close-up) that just after the change to a period 4 cycle appears, the valleys are straight and the entire region that was red turned pink/purple instead of alternating with red. The lack of visible period 2 cycles once period 4 cycles exist is consistent with (2) which states that for "smooth and 'sensible'" real functions, only one cycle is stable (I did not investigate what happens in $\mathbb{C}$ when period 4 cycles double

to period 8, but in $\mathbb{R}$, again all points that converged to a period 4 cycle converge instead to a period 8 cycle).

What I find interesting is that these general patterns also occurred for the "incorrect" pictures. Here the phenomenon is more interesting: as α decreases past -1.10 we see a mushroom type shape appearing and wrapping around inside itself by $\alpha = -1.14$. The close-up for $\alpha = -1.14015$ shows this most clearly and is quite facsinating, if inexplicable. Then just 5/100,000'ths later ($\alpha = -1.1402$) we see in the picture that the entire region of the mushroom has turned red. There are no gradual changes here, as I anticipated finding, rather the real points that cycle appear to "burst" into existence when α gets to a certain value. It is worth observing here that in order to get the fine detail of the pictures, the number of iterations was increased by a factor of ten, so that while this "burst" is sudden, it takes many iterations before it is "certain" (to within 0.000001) where a given point converges to. And notice there are still some yellow points where no convergence is found in 22,000 iterations. Here with the "incorrect" pictures we can also observe the successive valleys "eating away" the circles until cycles of period 4 arise ($\alpha = -1.34$, -1.345 close-up, -1.35). Based on the observation of this pattern in two separate (though admittedly related) functions, I would speculate that these successive valleys are present in all exponential first-order difference equations. Why some of these valleys are straight (period 2 $\rightarrow$ period 4) and some curve in more toward the end(fixed point $\rightarrow$ period 2) is not clear to me.

While the answer to my main question turned out to be "No," in the course of investigating I did find some cycles of period 2 in $\mathbb{C}$ ($\alpha = -0.20$, -0.30). These cycles were a late discovery so they haven't been thoroughly examined; but they seem to appear between

α = -0.18 and -0.19 and then disappear between α = -0.33 and -0.34. The two points being cycled between are not real (notice all real points still converge to the fixed point) but are complex conjugates (this was found by printing out successive iterates of the function and hasn't been verified mathematically). The exact points cycled between seem to depend on α, but not on the starting position. The table contains experimental results (rounded to 4 digits) that were obtained by iterating the point 0.75 + i1.5 until it clearly cycled (to within e-10, about 600 iterations).

alpha	Real part	± imaginary part	
-0.18	1.0000	0.0000	(the fixed point)
-0.19	0.7161	1.8331	
-0.20	0.8026	1.7832	
-0.22	0.9089	1.6975	
-0.24	0.9775	1.6245	
-0.26	1.0000	1.5143	
-0.28	1.0000	1.3766	
-0.30	1.0000	1.2391	
-0.32	1.0000	1.0882	
-0.33	1.0000	0.9438	
-0.34	1.0000	0.0000	(the fixed point)

What happens at the start (between -0.18 and -0.19) is unclear, except that the red regions seem to come out of (or go into) the valleys between humps (α =-0.20). However, it appears that at the end (between α= -0.33 and -0.34) α reaches some value so that the imaginary part gets small enough to be attracted to the fixed point. At this time all the points that cycled instead converge to the fixed point; the regions that were once red appear to have turned blue (compare pictures for α = -0.30 and -0.55).

The other point of interest here is that none of this happens in the "incorrect" model. The complex cycles don't appear

and there isn't the large increase in the number of convergent points (to the fixed point) between $\alpha \approx 0.00$ and ≈ 0.55 that exist in the "correct" pictures. It seems that it is this cycle in the complexes that eventually converges to the fixed point that causes the lack of symmetry around the fixed point in the "correct" pictures. Why this happens in one model and not the other when they seem so similar in other respects is not known to me. Perhaps the answer to this question is part of a fundamental difference between the two functions or between two larger classes of functions that have this general difference.

Conclusion

So, we have seen that for the map studied, cycles of period 2 do not slowly merge into the real line from $\mathbb{C}$; rather when they appear in $\mathbb{R}$, cycles also appear simultaneously in $\mathbb{C}$. Additionally, we found that there are cycles of period 2 which oscillate between complex conjugate pairs for some interval of α values. But what happens at the ends of this interval to "start up" and "kill off" the cycle is not entirely clear pictorially and unexplored mathematically. How, and if, these results generalize to polynomial maps I have not investigated.

References and Notes

1. Cull, P., *Stability of Discrete One-Dimensional Population Models.* Technical Report 86-20-1. Department of Computer Science. Oregon State University.

2. May, R. M., *Nature*, 261, 459-467 (1976).

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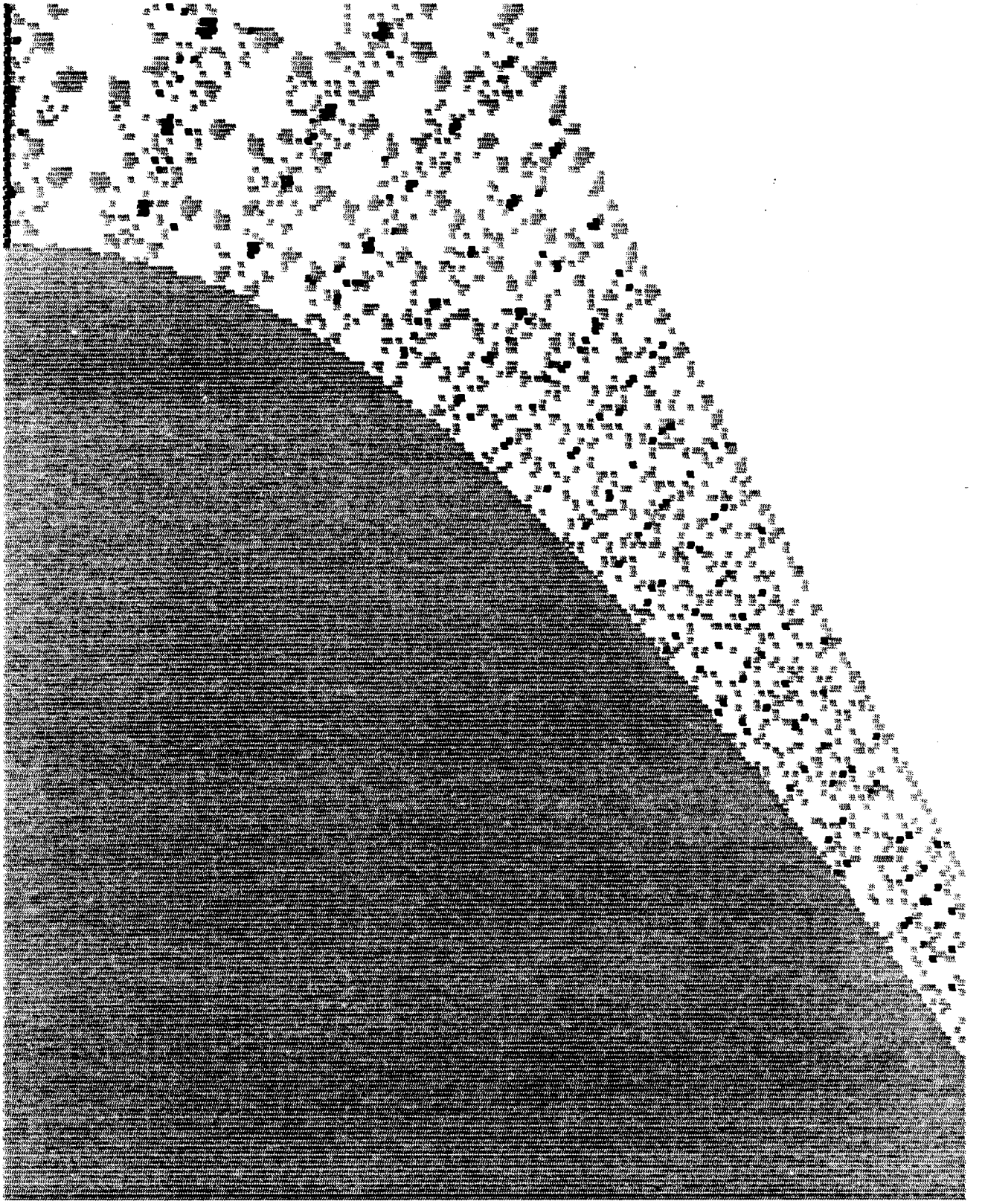
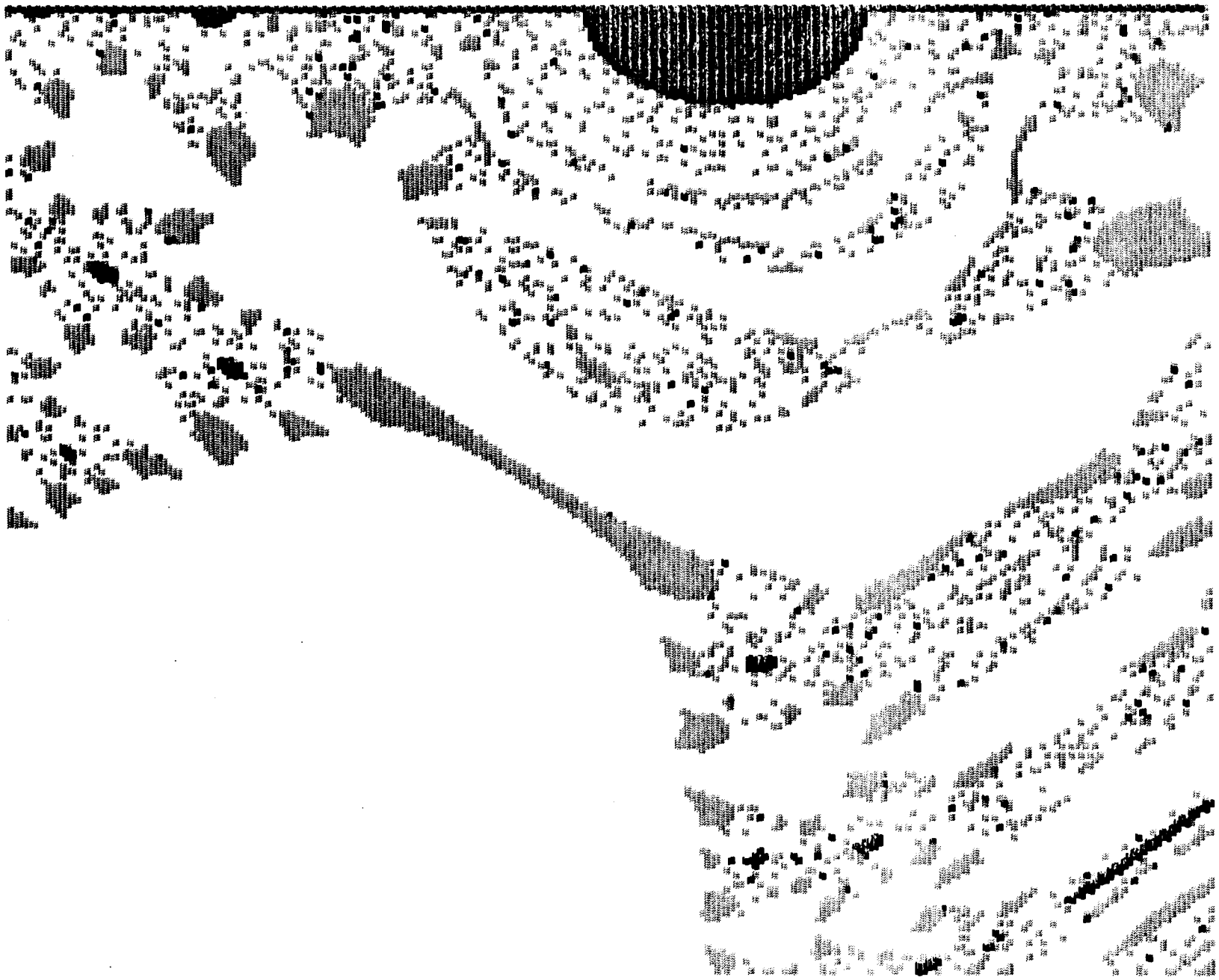
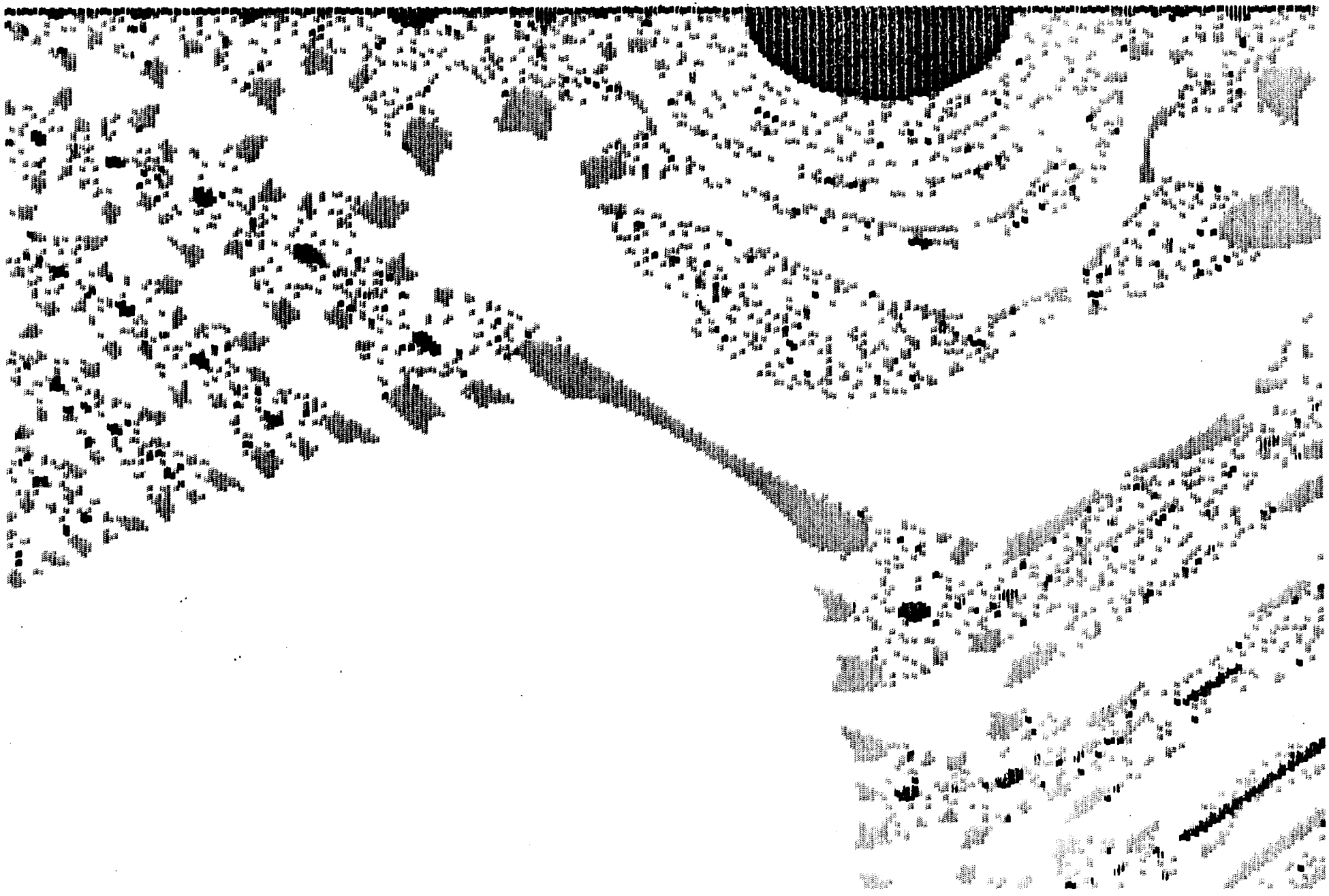
4. I gratefully acknowledge the guidance and assistance of P. Cull in all stages of this project.

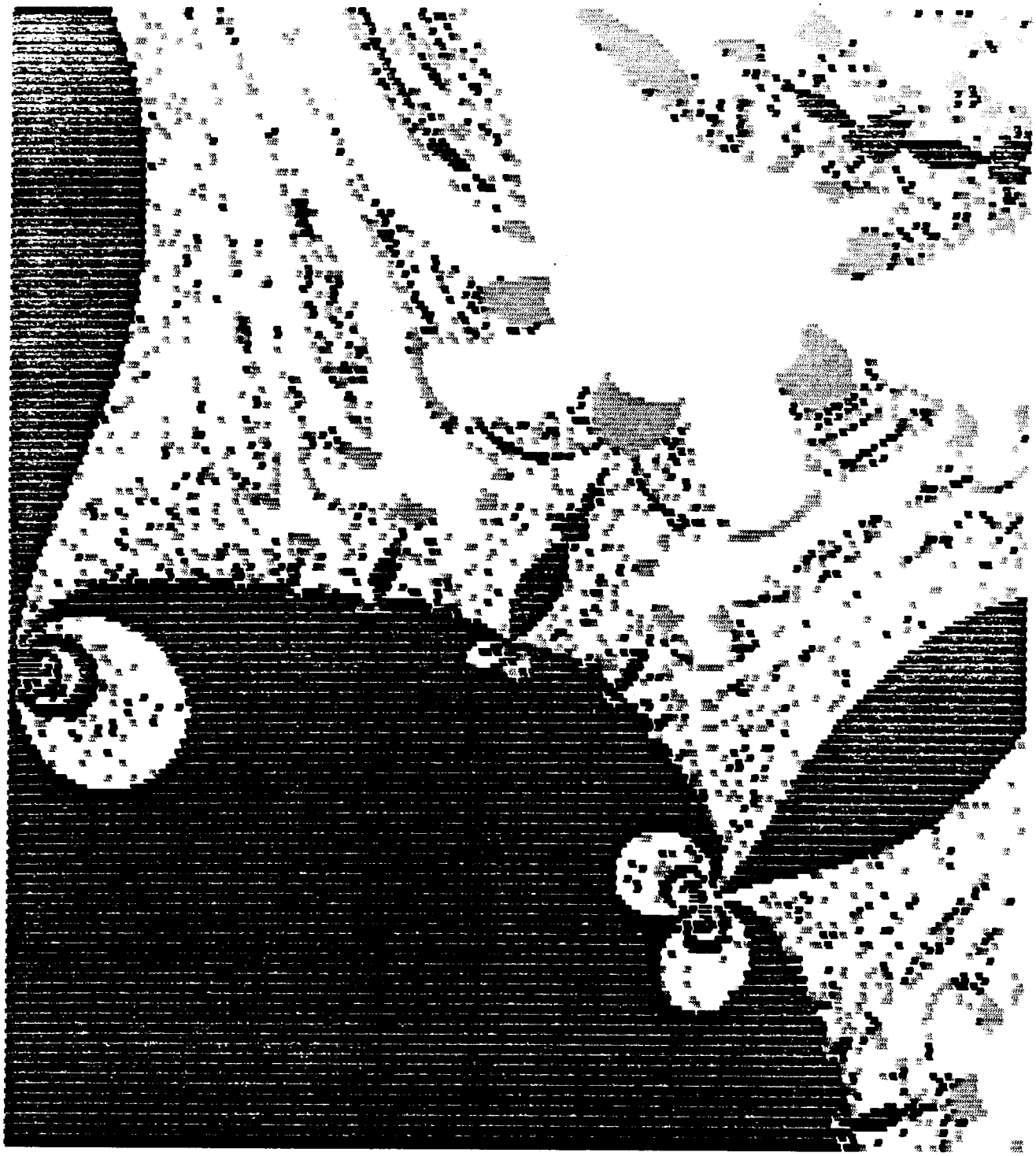






alpha = -1.35

alpha = -1.345



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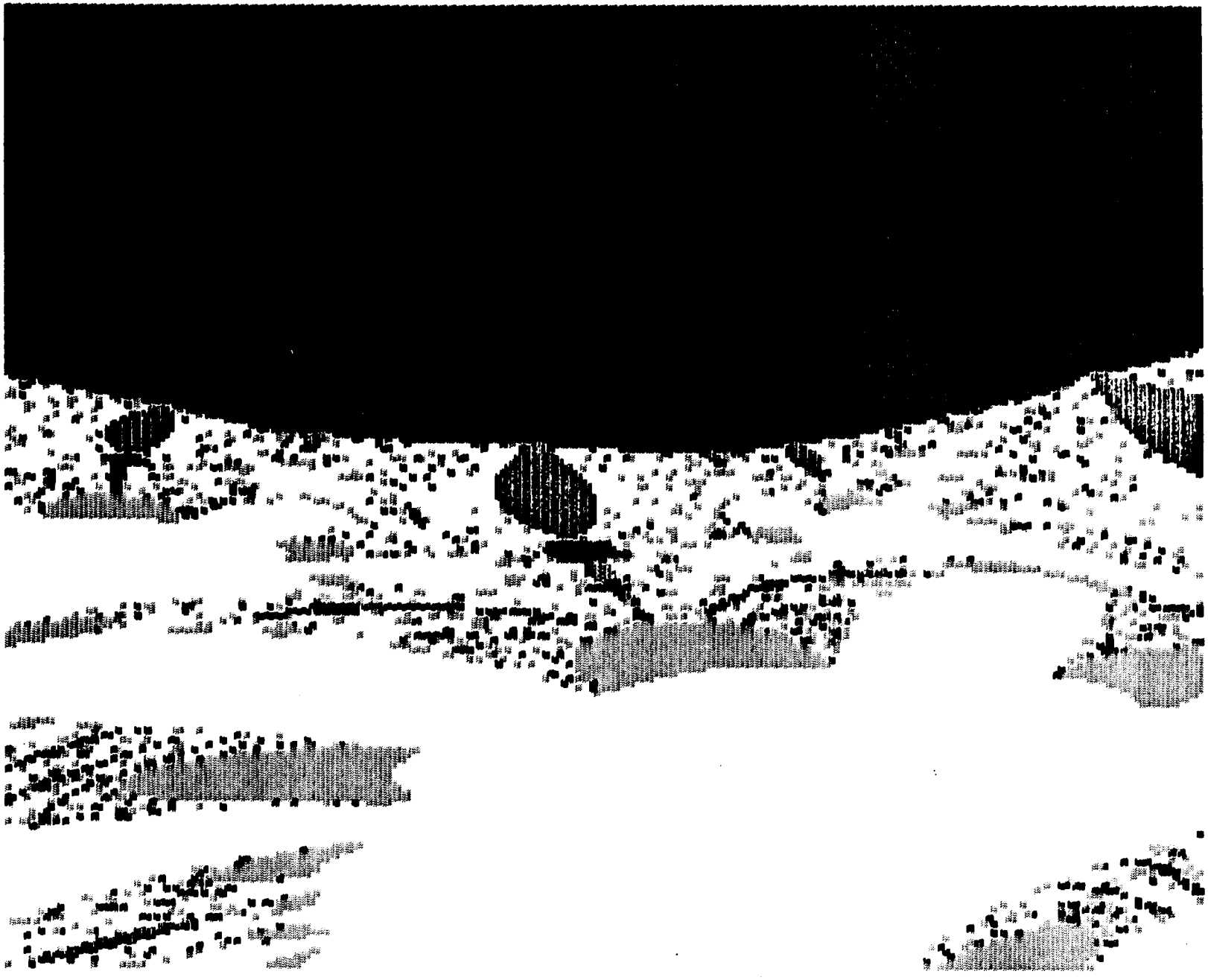

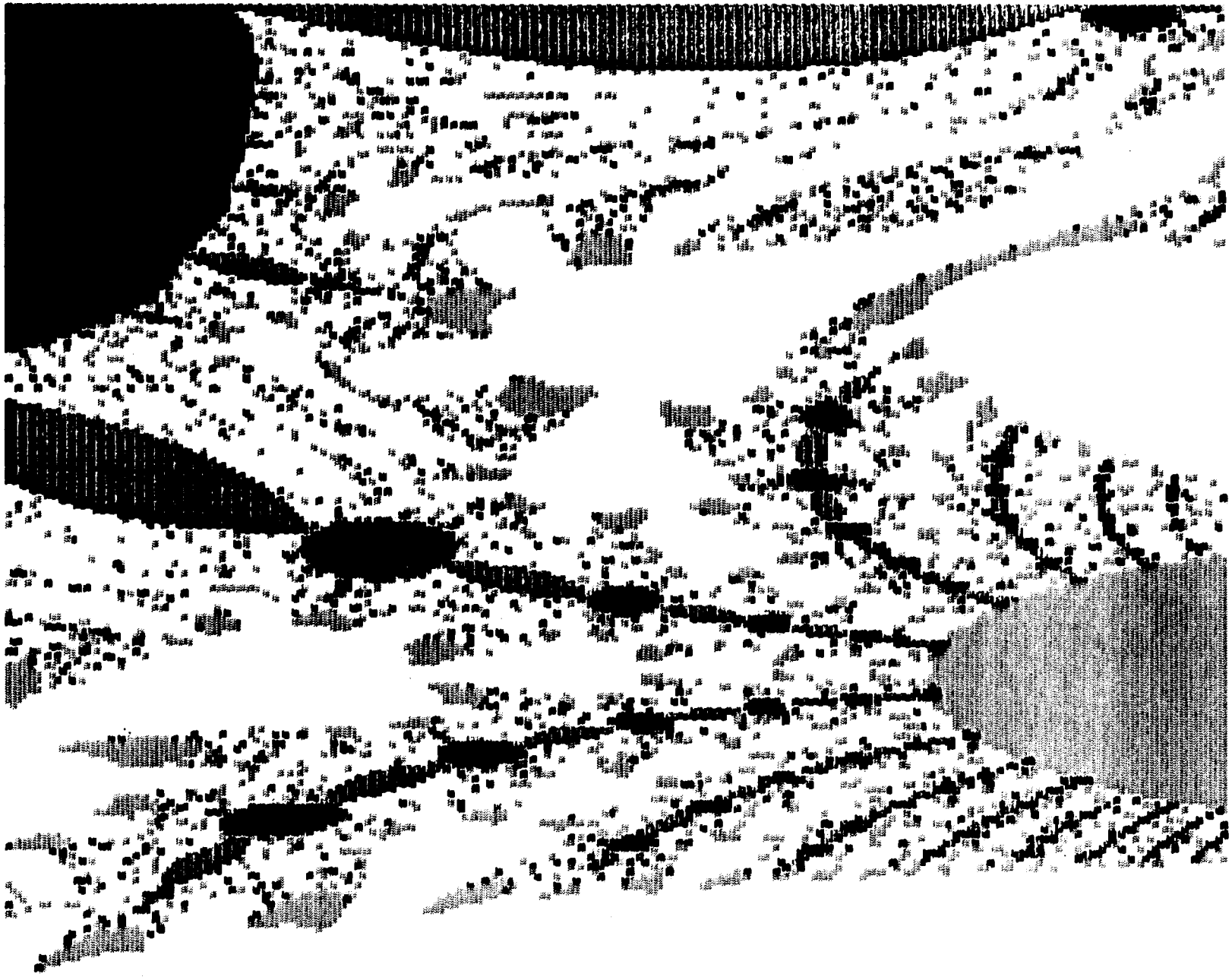

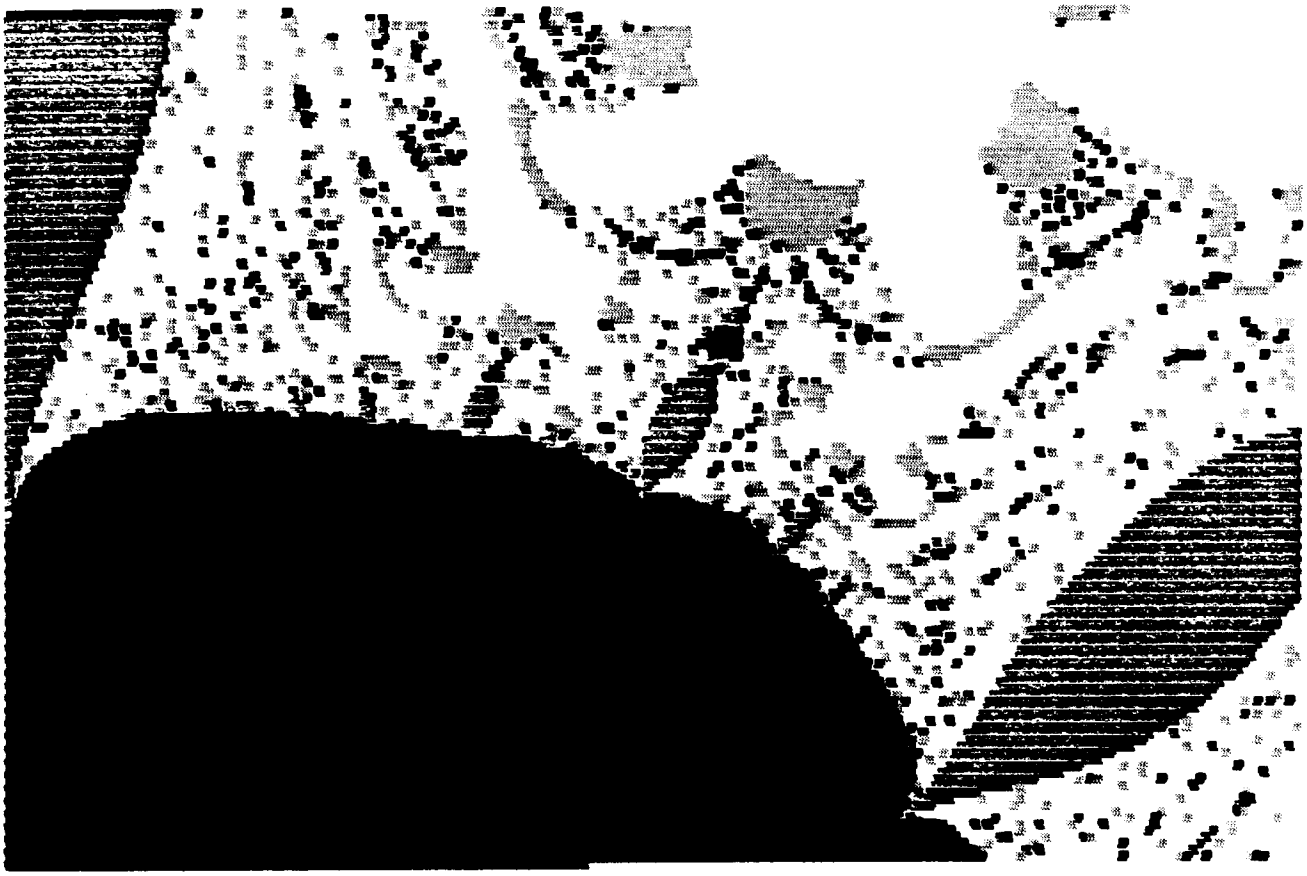

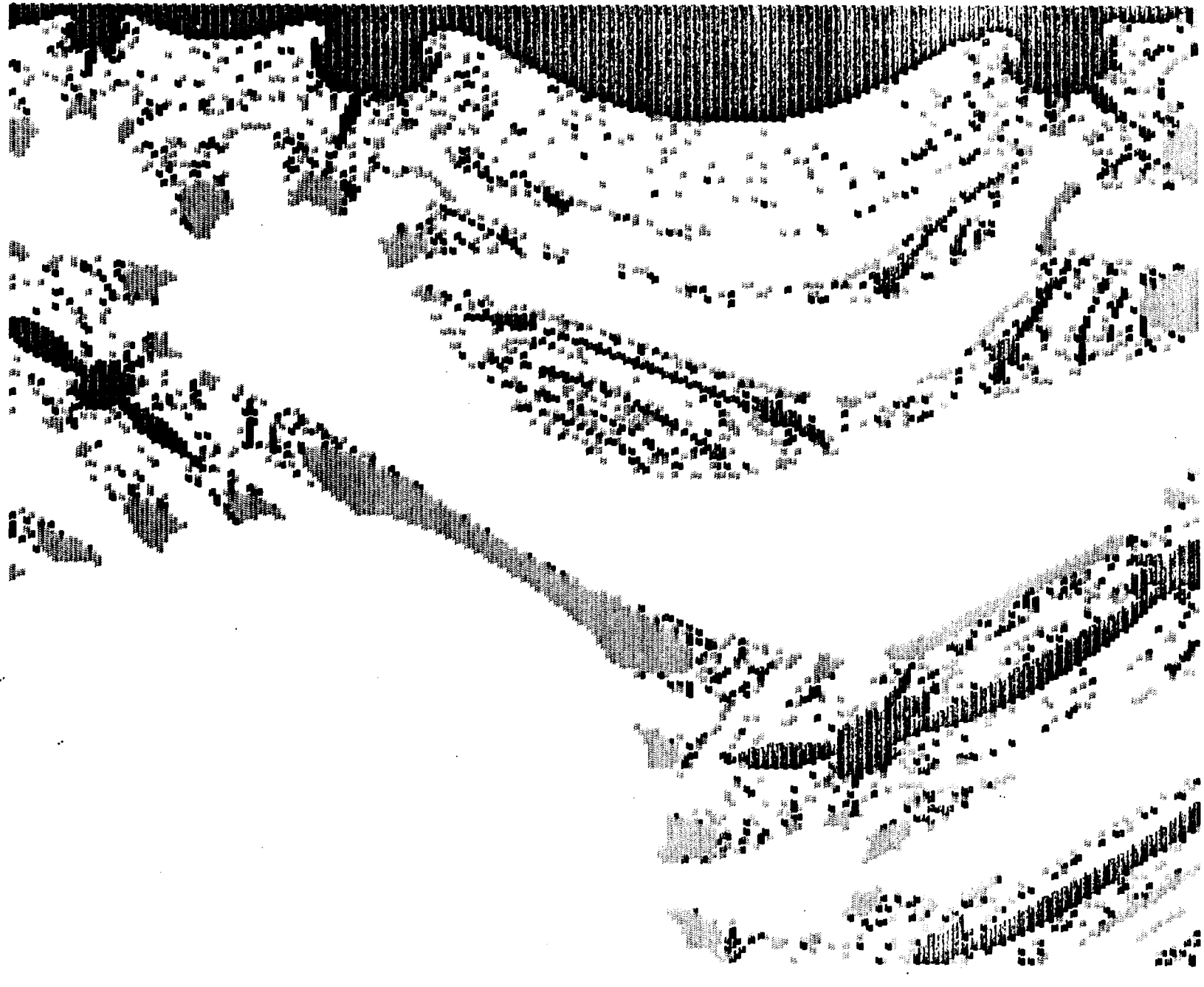

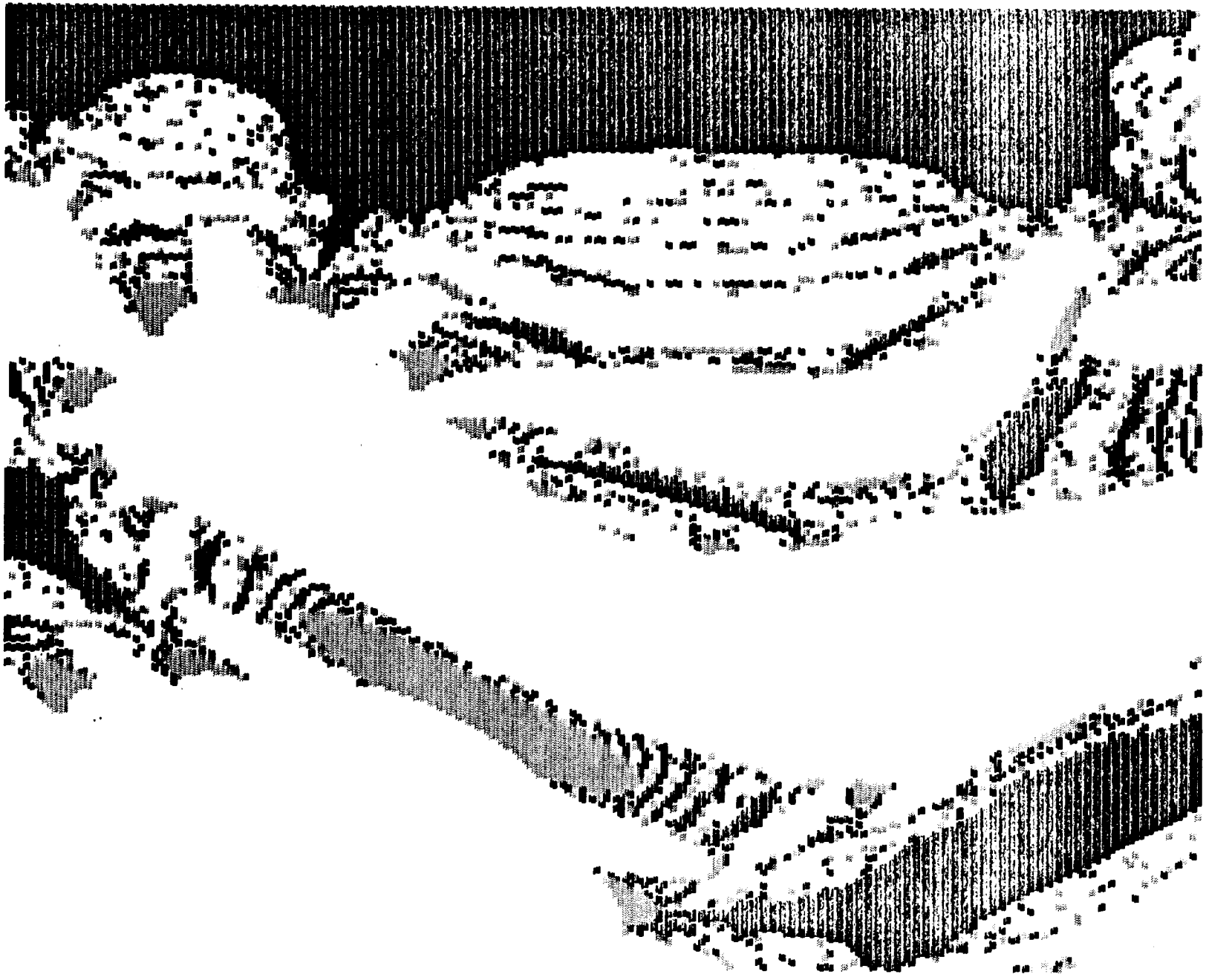

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