Department of Mathematics OSU
Qualifying Examination
Fall 2017

Complex Analysis and Linear Algebra

- Do any two of the three problems in Part CA, *use the corresponding marked blue book* and indicate on the selection sheet with your indentification code those problems which you want to have graded. Similarly, do any two of the three problems in Part LA in the *corresponding marked blue book* and mark those which you want to have graded on the selection sheet.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this exam.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheets back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them and mark whether they are for *complex analysis* or *linear algebra*.

DO NOT WRITE YOUR NAME ANYWHERE – USE ONLY YOUR TEST ID CODE

Part CA: Complex Analysis

1. Let G be the open subset of the complex plane lying inside the circles centered at 0 and 1, each of radius 1. Give a conformal map sending G to the right half plane $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$. (Your answer can be left in the form of a composition of explicitly defined maps.)

2. For $a \in (-1, 1)$, use the calculus of residues to evaluate the integral
$$\int_0^{2\pi} \frac{dt}{1 + a\cos t}.$$

3. Let $f(z)$ be a holomorphic function without zeros inside the unit disc
$$D_1(0) = \{z : \ |z| < 1\}$$
centered at the origin. Prove that there is a function $g(z)$ holomorphic in $D_1(0)$ such that
$$f(z) = e^{g(z)} \qquad \forall z \in D_1(0).$$
(You need to give the construction – not just quote a theorem.)

[Hint: Consider $\int_0^z \frac{f'(\omega)}{f(\omega)}\, d\omega$.]

Exam continues on next page ...

Part LA: Linear Algebra

1. Consider the vector space $\mathcal{Q}$ of all quadratic polynomials with coefficients in $\mathbb{C}$. Consider linear operator $T : \mathcal{Q} \to \mathcal{Q}$ defined by

$$T(f) = f + f' + f'',$$

where f' denotes derivative and f'' denotes second derivative. Find a Jordan basis of $\mathcal{Q}$ for T, that is, a basis such that the matrix of T with respect to this basis is a Jordan canonical form of T.

2. Let T be the complex linear operator from $\mathcal{M}_2(\mathbb{C})$, the space of complex 2-by-2 matrices, to itself given by taking a matrix to its transpose, i.e. $T(B) = B^t$.

   (a) Find the eigenvalues, with multiplicity, of T.

   (b) On $\mathcal{M}_2$ the Frobenius inner product is given by $\langle A, B\rangle = \operatorname{tr} B^*A$, where tr denotes the usual trace function, and B^* denotes the Hermitian transpose (also called the conjugate transpose) of B. Show that T is self-adjoint with respect to the Frobenius inner product, and find an orthormal basis of eigenvectors.

   (c) Find the distance, with respect to the norm coming from the Frobenius inner product, between the matrix $M = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ and the subspace of $\mathcal{M}_2$ consisting of symmetric matrices.

3. Fix an integer $n > 1$ and let $\mathcal{M}_n(\mathbb{C})$ denote the complex vector space formed by the $n \times n$ matrices over the complex numbers. For $M, N \in \mathcal{M}_n(\mathbb{C})$, denote the additive commutator $MN - NM$ by $[M, N]$. Let $\ker M$ denote the kernel (or nullspace) of the linear operator given by left multiplication by $M \in \mathcal{M}_n(\mathbb{C})$.

   Now fix $A, B \in \mathcal{M}_n(\mathbb{C})$ and define

$$\mathcal{N} = \cap_{k=1}^{n} \cap_{\ell=1}^{n} \ker[A^k, B^\ell]\,.$$

   (a) Show that $\mathcal{N}$ is a complex vector subspace of $\mathcal{M}_n(\mathbb{C})$.

   (b) Show that $\mathcal{N}$ is invariant under (left multiplication by) each of A, B.

   (c) Use standard results to conclude that A, B have a common eigenvector if and only if $\mathcal{N}$ contains a non-zero vector.