**Department of Mathematics OSU**
**Qualifying Examination**
**Fall 2008**

## PART II: COMPLEX ANALYSIS and LINEAR ALGEBRA

(1) Do any of the two problems in Part CA and any two problems in Part LA .

(2) Your solutions should contain all mathematical details. Please write them up as clearly as possible.

(3) Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.

(4) You have three hours to complete Part II.

(5) On problems with multiple parts, individual parts may be weighted differently in grading.

**PART : COMPLEX ANALYSIS QUALIFYING EXAM**

1. a) Let $f = f(z)$ be analytic in an open, connected $\Omega \subset \mathbb{C}$. Starting from the Cauchy integral formula show that the modulus $|f(z)|$ cannot attain a maximum in $\Omega$ unless $f$ is a constant function.

   b) Suppose that $f = f(z)$ is a non-constant analytic function defined in an open, connected $\Omega \subset \mathbb{C}$ containing the closure $\overline{\mathcal{D}}_1$ of the unit disk, and that $|f(e^{i\phi})| = 1$ for all $0 \leq \phi \leq 2\pi$. Show that $f$ must have a zero.

   *Hint:* You may want to consider the function $g(z) = 1/f(z)$.

2. a) It is required to expand the function $f(z) = 1/(1+z^2)$ into a series of the form $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-2)^n$. Determine how many different series expansions of this form there are, where each is valid, and find the coefficients $a_n$ for each expansion. You do not need to reduce the numerical expressions for the $a_n$ to simplest form.

   b) Evaluate the integrals

$$I = \int_{|z-2|=2} \frac{dz}{1+z^2}, \qquad J = \int_{|z-2|=4} \frac{dz}{1+z^2},$$

   where the circles are traced in the counterclockwise direction.

3. a) Let $p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_o$ be a polynomial of degree $n \geq 1$. Show that $|p(\mathrm{e}^{i\phi})| \geq 1$ for some $\phi \in \mathbb{R}$.

   b) Let $f(z)$ be entire and suppose that $f(z)$ is real if and only if $z$ is real. Show that $f(z)$ can have at most one zero.

## PART: LINEAR ALGEBRA QUALIFYING EXAM

1. Suppose that $c_0, \ldots, c_n$ are distinct elements of $\mathbb{R}$. For $i \in \{0, \ldots, n\}$, let

$$f_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - c_j}{c_i - c_j} \,.$$

a.) Let $\mathcal{P}_n$ denote the real vector space of polynomials with real coefficients and of degree at most $n$. Show that $\{f_i\}_{0\le i\le n}$ is a basis of $\mathcal{P}_n$ .

b.) Show that

$$\begin{aligned} T : \mathcal{P}_n &\to \mathbb{R}^{n+1} \\ g &\mapsto (\, g(c_0), \ldots, g(c_n)\,) \end{aligned}$$

is an isomorphism of real vector spaces.

c.) Deduce that the following determinant is nonzero:

$$\begin{vmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{vmatrix} \,.$$

d.) Now let $a_1, \ldots, a_{n+1}$ be distinct nonzero elements of $\mathbb{R}$. Deduce that

$$\begin{vmatrix} a_1 & \frac{1}{2}a_1^2 & \cdots & \frac{1}{n+1}a_1^{n+1} \\ a_2 & \frac{1}{2}a_2^2 & \cdots & \frac{1}{n+1}a_2^{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+1} & \frac{1}{2}a_{n+1}^2 & \cdots & \frac{1}{n+1}a_{n+1}^{n+1} \end{vmatrix} \neq 0 \,.$$

**Exam continues on next page ...**

2. Let $\mathcal{M}_n(\mathbb{C})$ denote the vector space of complex $n \times n$ matrices. For $B$ in $\mathcal{M}_n(\mathbb{C})$, denote by $\overline{B^t}$ the conjugate transpose of $B$, and let tr (B) denote the trace. The following defines an inner product:

$$\langle A \,|\, B \rangle = \mathrm{tr}(A\,\overline{B^t}\,)\ .$$

a.) Let $L_D$ denote the linear operator on $\mathcal{M}_n(\mathbb{C})$ given by left multiplication by $D$, thus $L_D(A) = DA$. Now, let $D$ be a diagonal element of $\mathcal{M}_n(\mathbb{C})$. Show that $L_D$ is a normal operator.

b.) For diagonal $D$, express $L_D$ as a linear combination of orthogonal projections.

3. Fix an integer $n \geq 2$ and for $i = 0, \ldots, n-1$, let $c_i = \binom{n}{i}$ denote the standard binomial coefficient. Define the $n \times n$ rational matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & 0 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix}\ .$$

That is, for $1 \leq j \leq n-1$, the $j$th column of $M$ is $e_{j+1}$, the canonical $j+1$st basis vector, and the $n$th column is $(-1)\sum_{j=1}^{n} c_{j-1}\, e_j$ .

a.) Give the Jordan canonical form for $M$.

b.) Give a Jordan basis for $M$.