Department of Mathematics OSU
Qualifying Examination
Fall 2009

PART I : Real Analysis

- Do any four of the six problems in Part I. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete Part I.
- On problems with multiple parts, individual parts may be weighted differently in grading.

1. Use the contraction mapping theorem to prove that, under suitable hypotheses, the equation

$$\phi(x) \;=\; f(x) + \int_a^b K(x,y)\phi(y)\;dy, \qquad a \le x \le b,$$

has a unique solution ϕ. Here, f and K are known functions, and the function ϕ is to be determined. As part of your analysis, develop appropriate hypotheses, including properties of K and f and the specification of the space of functions. Your hypotheses should include a reasonably broad class of functions f and K; for example, do not simply assume $f = 0$ and $K = 0$.

2. Let $\|\cdot\|$ be a norm on $\mathbf{R}^n$. Do not assume any properties of $\|\cdot\|$, other than those that follow from the general definition of norm on a vector space.

   (a) Let $f(x) = \|x\|$ for all $x \in \mathbf{R}^n$. Show that f is continuous on $\mathbf{R}^n$ with respect to the metric ρ defined by $\|\cdot\|$, i.e., $\rho(x,y) = \|x - y\|$ for all x and y in $\mathbf{R}^n$. (Use the triangle inequality.)

   (b) Now define a different norm $\|\cdot\|_1$ by $\|x\|_1 = \sum_{i=1}^n |x_i|$ for all $x \in \mathbf{R}^n$. Prove that the function f defined in part (a) is continuous with respect to the metric defined by $\|\cdot\|_1$. (You do not need to prove that $\|\cdot\|_1$ satisfies all of the axioms of a norm.)

   (c) Show that the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. That is, show that there exist positive constants M_1 and M_2 such that $M_1\|x\|_1 \le \|x\| \le M_2\|x\|_1$ for all $x \in \mathbf{R}^n$. (*Hint:* Consider what happens when f is restricted to the set $S = \{x \in \mathbf{R}^n : \|x\|_1 = 1\}$.)

   (d) Give an example of a linear space (vector space) X and two norms on X that are *not* equivalent, in the sense defined in part (c).

3. Define the convolution of two functions f and g by

$$(f * g)(x) = \int_{\infty}^{\infty} f(x-y)g(y)dy = \int_{\infty}^{\infty} f(y)g(x-y)dy,$$

assuming that the integrals exist. Let ϕ be a continuous function on $\mathbf{R}$ that satisfies $\phi(x) > 0$ for $-1 < x < 1$, $\phi(x) = 0$ otherwise, and $\int_{-\infty}^{\infty} \phi(x)dx = 1$. For each integer $n \ge 1$, let $\phi_n(x) = n\phi(nx)$ for all real x. Then $\int_{-\infty}^{\infty} \phi_n(x)dx = 1$ for all n, ϕ_n is nonzero on the interval $(-1/n, 1/n)$, and as n increases the graph of ϕ_n becomes narrow and tall. The convolution $(f * \phi_n)(x)$ is therefore a weighted average of values of $f(y)$ for y near x.

   Prove that if $f \in L^1(\mathbf{R})$, then $f * \phi_n \to f$ in $L^1(\mathbf{R})$ as $n \to \infty$. (That is, $\|f * \phi_n - f\|_1 \to 0$ as $n \to \infty$.)

   (*Hint:* First consider the case where f is continuous and has compact support, and then extend to $L^1(\mathbf{R})$. You may use the fact, without proving it, that the set of continuous functions with compact support is dense in $L^1(\mathbf{R})$.)

4. For both parts of this problem consider the metric space consisting of the interval $[0, 1]$ equipped with the usual metric $\rho(x, y) = |x - y|$.

   (a) Show that there are no nowhere dense subsets of $[0, 1]$ that have Lebesgue measure equal to 1.

   (b) A set whose complement is a countable union of nowhere dense sets is called a residual set. Show that there exist non-empty residual subsets of $[0, 1]$ with zero Lebesgue measure.

   Hint: You may use without proof that for any $0 \leq \alpha < 1$ there exists a nowhere dense subset E_α of $[0, 1]$ with Lebesgue measure equal to α.

5. Let $1 \leq p < \infty$ and $f_n \in L_p(\mathbf{R})$, $n \in \mathbf{N}$, a sequence of functions that converges pointwise almost everywhere to a function $f : \mathbf{R} \to \mathbf{R}$. Assume that there is a nonnegative function F with $\| F \|_p = \left(\int_{\mathbf{R}} |F(x)|^p \, dx\right)^{1/p} < \infty$ such that $|f_n| \leq F$ for all $n \in \mathbf{N}$.

   (a) Show that $\lim_{n\to\infty} \| f_n - f \|_p = 0$.

   (b) Show by means of a counterexample that the conclusion in part a) need not hold if the hypothesis $|f_n| \leq F$ is omitted.

6. Let f be a nonnegative function defined on a measurable subset E of $\mathbf{R}$. Show that f is measurable if the region $\{(x, y) : x \in E, \ f(x) \geq y\}$ is a measurable subset of $\mathbf{R}^2$.

   Hint: Consider Tonelli's theorem.