Department of Mathematics OSU
Qualifying Examination
Fall 2014

PART I : Real Analysis

- Do any of the four problems in Part I. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete Part I.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book and selection sheet into the unmarked, smaller envelope. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

1. Let H be a separable Hilbert space with orthonormal basis $\{\varphi_k,\ 1 \le k\}$. A continuous function $f : [0,T] \to H$ is represented pointwise in that basis by the Fourier series

$$f(t) = \sum_{k=1}^{\infty} f_k(t)\varphi_k,\ 0 \le t \le T.$$

Show that the series converges *uniformly* on $[0,T]$

2. Let $\{T_n\}$ be a sequence in the space $L(B)$ of continuous linear operators on the Banach space B. Suppose each T_n is compact and that $T_n \to T$ in the norm of $L(H)$. Prove T is compact.

Hint: Let $\{x_n\}$ be a sequence of vectors in the unit ball of H, and show that there is a subsequence $\{x_{n'}\}$ for which $\{T(x_{n'})\}$ is Cauchy.

3. Let $K \in C([0,1]^2)$, and define for all $f \in C([0,1])$:

$$Tf(x) = \int_0^1 K(x,y)f(y)dy, \quad \text{for all } x \in [0,1].$$

Show that $T : C([0,1]) \to C([0,1])$ with respect to the sup-norm, and that T is a compact linear operator.

4. Let A and B be nonempty Lebesgue measurable sets in $\mathbb{R}$ of finite Lebesgue measure, and assume they intersect: $A \cap B \ne \emptyset$. Define

$$A + B = \{a + b\ :\ a \in A,\ b \in B\}.$$

Show that

$$m\big(A + B\big) \ge m\big(A \cup B\big).$$

5. Show that if $\int\limits_0^1 \big|f_n(x)\big|dx\ \le \frac{1}{n^2}$, then $f_n \longrightarrow 0$ a.e. on $[0,1]$.

6. Let $f \in L^p(\mathbb{R})$ for $p \in [1,\infty)$. Set $f_t(x) = f(x+t)$. Show that

$$\|f_t - f\|_p \to 0 \quad \text{as} \quad t \to 0.$$