Department of Mathematics
Qualifying Examination Fall Term 1999

Part I: Real Analysis

Do any four of the problems in Part I. Your solutions should include all essential mathematical details; please write them up as clearly as possible. You have three hours to complete Part I of the exam.

1. Suppose $\{f_n\}$ is a sequence of continuous real-valued functions $[0,1] \longrightarrow \mathbf{R}$ such that $f_n \longrightarrow 0$ almost everywhere on $[0,1]$. Can we conclude that $f_n \longrightarrow 0$ in the L^2-norm on $[0,1]$? If the answer is "yes", give a proof. If the answer is "no", give a counterexample.

2. Let $f : [0,1] \longrightarrow [0,\infty)$ be a Lebesgue measurable function. Prove that
$$\lim_{p\to\infty} ||f||_p = ||f||_\infty \ ,$$
where, as usual, $||f||_p = (\int_0^1 |f|^p)^{1/p}$ and $||f||_\infty = \mathrm{esssup}_{0\le x\le 1} |f(x)|$.

3. Is there an uncountable collection of non-overlapping subsets of the real plane such that each subset has positive Lebesgue measure? Give an example of such a collection or prove that one does not exist.

4. Suppose that f is continuously differentiable on $[0,1]$ such that $\sup_{x\in[0,1]} |f'(x)| = M < \infty$. Prove that
$$| \int_0^1 f(x)dx - \sum_{i=1}^n f(i/n)\cdot 1/n \,| \le M/n \ .$$

5. Let S be the set of equivalence classes of Cauchy sequences in a metric space (X,d) where the equivalence relation is defined by
$$(x_n) \sim (y_n) \text{ if } \lim_{n\to\infty} d(x_n, y_n) = 0.$$
Define a distance function between two equivalence classes of Cauchy sequences $[(x_n)]$ and $[(y_n)]$ by
$$\rho([(x_n)]\,,[(y_n)]) = \lim_{n\to\infty} d(x_n, y_n)\ .$$
Prove that ρ is well defined and is a metric.

6. (a) State the Lebesgue Dominated Convergence Theorem.

   (b) Let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be a Lebesgue integrable function. Prove that $G(x) = \int_0^x g(t)\,dt$ is a continuous function $\mathbf{R} \longrightarrow \mathbf{R}$.

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Part II: Complex Analysis and Linear Algebra

Do any two problems in Part CA and any two problems in Part LA. Your solutions should include all essential mathematical details; please write them up as clearly as possible. You have three hours to complete Part II of the exam.

Part CA

1. Use the residue theorem to evaluate the integral
$$\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\, dx \ .$$

2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in the entire complex plane. Show that for any $r > 0$
$$|a_n| \le \frac{M_r}{r^n}\ ,$$
where M_r is defined as $\max_{|z|=r} |f(z)|$.

3. Let $f:\ \mathbf{C} \longrightarrow \mathbf{C}$ be the the transformation of the complex plane given by $f:\ z \mapsto w = 1/z$. Show that the image of the line $y = x - 1$ is the circle $u^2 + v^2 - u - v = 0$ and the image of the line $y = 0$ is the line $v = 0$. (Here $z = x + yi$ and $w = u + vi$.) Is the mapping f conformal at the point $z = 1$?

Part LA

1. Let H by a 3-dimensional vector subspace of $\mathbf{R^4}$ given by
$$\{(x_1, x_2, x_3, x_4) : x_1 + x_2 - x_3 - x_4 = 0\}$$
and let $T:\ \mathbf{R^4} \longrightarrow \mathbf{R^4}$ be the linear transformation given by reflection across H.

   (a) What is the matrix of T in the standard basis of $\mathbf{R^4}$?

   (b) What are the eigenvalues of T? What are their multiplicities?

2. Let A be a nonzero $n \times n$ nonzero complex matrix and $0_{n\times n}$ be the $n \times n$ zero matrix. Determine whether the following statements are true or false. In each case give a proof or a counterexample.

   (a) If $A^n = 0_{n\times n}$ then $\mathrm{Ker}(A) \cap \mathrm{Range}(A) \ne (0)$.

(b) If $\mathrm{Ker}(A) \cap \mathrm{Range}(A) \neq (0)$ then $A^n = 0_{n \times n}$.

(c) If λ is an eigenvalue of A then λ^2 is an eigenvalue of A^2.

(d) If λ^2 is an eigenvalue of A^2 then λ or $-\lambda$ is an eigenvalue A (possibly both).

3. Let V be a finite-dimensional real vector space and let $(\, , \,)$ be a positive-definite inner product on V. Show that elements $v_1, \ldots, v_m \in V$ are linearly independent if and only if the $m \times m$-matrix $A = (a_{ij})$ given by $a_{ij} = (v_i, v_j)$ is non-singular.