Department of Mathematics
Qualifying Examination
Fall 2002

Part I: Complex Analysis and Linear Algebra

- Do any two problems in Part CA and any two problems in Part LA.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part I of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

Part CA

1. Let C_r be the circle in the complex plane with center at the origin and radius r traversed once in the counterclockwise sense. Let $r > 0$ and $r \neq 2$. Find all possible values of the integral

$$I_r = \int_{C_r} \frac{z^2 + e^z}{z\,(z-2)^2}\, dz$$

2. Let $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$, be a complex function on a open set U in the complex plane.

   (a) If $f(z)$ is analytic in U show that the Cauchy-Riemann equations hold at each point in U.

   (b) **State** but do **not** prove a reasonable converse of part (a).

   (c) Using the Cauchy-Riemann equations prove: If $f(z)$ is analytic and real-valued in a connected, open set U, then f is constant in U.

3. Let $f(z) = \cot z$.

    (a) Determine the region in the complex plane where f is analytic.

    (b) If f has any singularities find them all and state their types (removable, pole, or essential singularity) and in the case of poles find their orders.

    (c) Explain briefly why $f(z)$ has a power series expansion about $1+i$ and find the radius of convergence of the power series.

    (d) f has a Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ in the annulus $\pi < |z| < 2\pi$. Evaluate a_n for $n = -1, -2, -3, \ldots$.

Part LA

1. A matrix B is said to be a square root of a matrix A if $B^2 = A$. A matrix is Hermitian if it equals it transpose conjugate: $\overline{A^T} = A$.

    (a) Give an example of a complex matrix A which does not have a square root. Be sure to show that your example has the desired property.

    (b) Prove that every complex Hermitian matrix has a square root.

2. Determine, up to similarity, all 3×3 complex matrices A such that $A^3 = A^2$.

3. Let V be a finite dimensional complex vector space and let A and B be subspaces of V. You may use the following three standard (and easily proven facts) in what follows: (i) $A+B = \{a+b : a \in A, b \in B\}$ is a subspace of V, (ii) $A \cap B$ is a subspace of V, and (iii) $\dim(A+B) = \dim A + \dim B - \dim(A \cap B)$. Now suppose that A, B, A', B' are subspaces of V such that $\dim A = \dim A'$, $\dim B = \dim B'$, and $\dim(A \cap B) = \dim(A' \cap B')$. Prove that there exists a one-to-one linear operator T on V such that $T(A) = A'$ and $T(B) = B'$.

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Part II: Real Analysis

- Do any four of the problems in Part II.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part II of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

1. Let (X, ρ) be a metric space, D be a dense subset of X, and let $f : D \to \mathbb{R}$ be a uniformly continuous function on D.

   (a) Prove: There exists a uniformly continuous function $g : X \to \mathbb{R}$ such that $g(d) = f(d)$ for all $d \in D$. In other words, f can be extended to a uniformly continuous function on all of X.

   (b) Proof or counterexample: The extension g of f in (a) is unique.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that if A is a Borel set in $\mathbb{R}$, then its inverse image $f^{-1}(A)$ is a Borel set in $\mathbb{R}$.

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function with

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty .$$

   Prove that

$$h(x) = \int_{x}^{1+x} f(t)\, dt$$

   is a continuous function of x.

4. Let X be a metric space.

   (a) Define the term X is **compact**.

   (b) Define two properties of a metric space X that are equivalent to compactness. (Do NOT give any proofs.)

   (c) Let C_n be a sequence of nonempty closed sets in a compact metric space X such that $C_{n+1} \subset C_n$ for $n = 1, 2, 3, \ldots$. Prove or disprove: $\cap_{n=1}^{\infty} C_n$ is nonempty.

   (d) Let $f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k^2}$ for $x \in [0,1]$. Show that the sequence of functions $f_n(x)$ converges pointwise on $[0,1]$ to a limit function $f(x)$.

   (e) Prove or disprove: The limit function $f(x)$ from (d) is continuous.

5. Let ϕ be a positive, smooth (i.e., C^{∞}) function. Suppose ϕ vanishes outside a compact subset of $\{x \in \mathbb{R} : |x| < 2\}$ and satisfies $\phi(x) = 1$ if $|x| < 1$. Let f be a function in $L^2(\mathbb{R})$. Define the convolution operator

$$f * \phi(x) = \int_{\mathbb{R}} f(x-y)\phi(y)dy.$$

   (a) Prove or disprove $f * \phi \in L^1(\mathbb{R})$ for all $f \in L^1(\mathbb{R})$.

   (b) Prove or disprove $f * \phi \in L^2(\mathbb{R})$ for all $f \in L^2(\mathbb{R})$.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bounded Lebesgue measurable function that vanishes (is equal to zero) outside a compact subset of $\mathbb{R}$. Prove that

$$\int_{-\infty}^{\infty} |f(x+t) - f(t)|\, dt \to 0 \text{ as } x \to 0.$$