Department of Mathematics OSU
Qualifying Examination
Fall 2018

Linear Algebra

- Do any four of the six problems. Indicate on the problem selection sheet with your identification number those four which you wish to have graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this exam.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and the selection sheet back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

DO NOT WRITE YOUR NAME ANYWHERE – USE ONLY YOUR TEST ID CODE

1. Suppose that V is a finite dimensional vector space over a field $\mathbb{F}$ and let V^* be the vector space of linear transformations from V to $\mathbb{F}$. Suppose $W \subseteq V$ is a subspace; the annihilator of W, denoted $\mathrm{Ann}(W)$, is the subspace of V^* consisting of the transformations that vanish on W.

   (a) Show that W is the intersection of the nullspaces of the elements of its annihilator.

   (b) Suppose now that $T : V \to V$ is a linear operator. If $V = R_T \oplus W$ (direct sum) and W is T-invariant, show that $W \subseteq N_T$. Here R_T and N_T denote the range and null space (or, kernel) of T, respectively.

   (c) Let $T^t : V^* \to V^*$ be given by $T^t(\ell) = \ell \circ T$. Prove that $Z \subset V$ is a T-invariant subspace if and only if $\mathrm{Ann}(Z)$ is T^t-invariant.

2. Consider the $n \times n$ matrix $F_n = \left(f_{i,j} \right)$ of binomial coefficients $f_{i,j} = \binom{i-1+j-1}{i-1}$. For example,

$$F_1 = [1], \quad F_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}, \quad \ldots .$$

   Prove that $\det(F_n) = 1$ for all $n \in \mathbb{N}$. Hint: Recall $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for $0 < k < n$.

3. Let A and B be $n \times n$ complex matrices. Prove or disprove the following statements:

   (a) If A and B are diagonalizable, so is AB.

   (b) If $A^2 = A$, then A is diagonalizable.

4. Suppose that P is a linear operator on a vector space V such that $P^2 = P$.

   (a) Prove $V = \mathrm{N}_P \oplus \mathrm{R}_P$ (direct sum). Here R_P and N_P denote the range and null space (or, kernel) of P, respectively.

   (b) Further suppose that V is a real inner product space and that P as above has finite dimensional range. Let T be the orthogonal projection from V to the range of P.

   Determine both TP and PT.

   (c) With notation and hypotheses as in (b), find the adjoint operator of T.

Exam continues on next page ...

5. Let V be a 3-dimensional vector space over the field of rational numbers $\mathbb{Q}$. Suppose $T : V \to V$ is a linear transformation and $Tx = y, Ty = z, Tz = x + y$, for certain $x, y, z \in V, x \neq 0$. Prove that $x, y,$ and z are linearly independent.

6.

(a) Suppose that $\lambda_1, \ldots, \lambda_r$ are distinct non-zero complex numbers. Show that the following matrix is invertible.

$$B = \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_r^2 \\ \vdots & \vdots & \ldots & \vdots \\ \lambda_1^r & \lambda_2^r & \ldots & \lambda_r^r \end{bmatrix}.$$

(b) Suppose that A is a complex square matrix such that the trace of A^k is zero for every $k \in \mathbb{N}$. Show that all eigenvalues of A are zero.