

Department of Mathematics OSU
Qualifying Examination
Fall 2017

Real Analysis

- Do any four of the six problems. Indicate on the sheet with your identification number which four you wish to have graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this exam.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheet back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

DO NOT WRITE YOUR NAME ANYWHERE – USE ONLY YOUR TEST ID CODE

Standard Notations and Conventions: \mathbb{R} denotes the reals, \mathbb{C} denotes the complex numbers, m_n is Lebesgue measure on \mathbb{R}^n (the subscript will be omitted when $n = 1$), for any Hausdorff space M , $\mathcal{B}(M)$ denotes the sigma-algebra of Borel sets (the sigma-algebra generated by the open sets), set complementation is denoted with c , i.e. A^c will denote the complement of A with respect to the ambient space, but set difference notation $A \setminus B$ will be used for relative complement. For a measurable set E , $1 \leq p < \infty$, $L^p(E)$ denotes the set of (equivalence classes of) p-th power integrable functions with norm $\|f\|_p = \left(\int_E |f|^p dm\right)^{1/p}$ and for $p = \infty$ denotes the set of (equivalence classes of) essentially bounded measurable functions with essential supremum as norm. Unless otherwise noted, you may assume that functions in L^p are real valued.

1. Assume that $f : \mathbb{R}^k \rightarrow [0, \infty]$ is Lebesgue integrable and $\int_{\mathbb{R}^k} f dm_k = c$ for some $c \in (0, \infty)$. Prove that for all $\alpha \in (0, 1]$ the limit

$$\lim_{n \rightarrow \infty} n \int_{\mathbb{R}^k} \ln(1 + (f(x)/n)^\alpha) dm_k$$

exists (as an extended real number) and compute the limit.

2. Let (X, d) be a metric space with the property

$$f : X \rightarrow \mathbb{R} \text{ is continuous} \quad \Rightarrow \quad f \text{ is bounded.}$$

Prove that X is compact.

3. Let f be a continuous real-valued function on $[0, 1]$. Compute

$$\lim_{n \rightarrow \infty} (n + 1) \int_0^1 x^n f(x) dm.$$

(Of course, you must justify all your claims.)

4. Let $\phi(x) > 0$ be a continuous function on \mathbb{R} which satisfies $\lim_{x \rightarrow \pm\infty} \phi(x) = 0$ and let $\{f_n\}$ be an equicontinuous sequence on \mathbb{R} satisfying

$$|f_n(x)| \leq \phi(x)$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Prove that $\{f_n\}$ has a uniformly convergent subsequence.

The exam continues on the next page

5. Let H be a Hilbert space and let $\{x_j\}$ be a bounded sequence of elements of H which are linearly independent (in the sense of linear algebra). For each $n \in \mathbb{N}$ let $V_n = \text{Span}(x_1, \dots, x_n)$. Assume that for every n , x_n is the nearest element in V_n to x_{n+1} . Prove that the sequence $\{x_n\}$ is convergent.
6. Let $A \subset \mathbb{R}$ be a bounded measurable set of positive (Lebesgue) measure, and let $I = [0, 1]$. Prove that there exists a sequence of real numbers $\{t_j\}$ such that the union of the translates of A by the t_j covers I up to a set of measure zero, that is,

$$m(I \setminus (\cup_j (A + t_j))) = 0.$$