Department of Mathematics OSU
Qualifying Examination
Fall 2018

Real Analysis

- Do any four of the six problems. Indicate on the sheet with your identification number which four you wish to have graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this exam.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheet back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

DO NOT WRITE YOUR NAME ANYWHERE – USE ONLY YOUR TEST ID CODE

Standard Notations and Conventions: $\mathbb{R}$ denotes the reals, $\mathbb{C}$ denotes the complex numbers, set complementation is denoted with ^{c}, i.e. A^c will denote the complement of A with respect to the ambient space, but set difference notation $A \setminus B$ will be used for relative complement. For metric spaces X and Y, $C(X, Y)$ denotes the set of continuous functions from X to Y. If Y is $\mathbb{R}$, we will just write $C(X)$. For $p \geq 1$ denote by ℓ_p the vector space of all sequences $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ such that

$$\|\mathbf{a}\|_p = (\sum_{n=1}^{\infty} |a_n|^p)^{1/p} < \infty$$

and by ℓ_∞ the set of bounded sequences with $\|\mathbf{a}\| = \sup |a_n|$. You may assume the sequences are real-valued unless otherwise stated.

1. For each $n \in \mathbb{N}$, let $f_n : [0, 1] \to [0, \infty)$ be a continuous function, and assume that the sequence $\{f_n\}$ satisfies the property that for all $x \in [0, 1]$ and for all $n \in \mathbb{N}$, $f_n(x) \geq f_{n+1}(x)$.
   (a) Prove that the sequence $\{f_n\}$ is pointwise convergent.
   (b) Set $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in [0, 1]$ and set $M = \sup_{x\in[0,1]} f(x)$. Prove that there exists $t \in [0, 1]$ such that $f(t) = M$.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that
   (i) there exist $x_0, x_1 \in \mathbb{R}^n$ such that $f(x_0) = 0$ and $f(x_1) = 1$,
   (ii) $\liminf_{|x|\to\infty} f(x) \geq 2$.

   Let $K_\alpha = \{x \in \mathbb{R}^n : \ f(x) \leq \alpha\}$.
   (a) Prove that for any $0 < \alpha < \beta < 1$, $d(\partial K_\beta, K_\alpha) = \inf\{|x - y| : \ x \in K_\alpha, \ y \in \partial K_\beta\} > 0$. (Here ∂K_β denotes the boundary of the set K_β.)
   (b) Find an example of $f : \mathbb{R}^n \to \mathbb{R}$, f continuous satisfying (i), such that $d(\partial K_\beta, K_\alpha) = 0$ for some $0 < \alpha < \beta < 1$.

3. Let $\alpha > 0$. A function $f : (-1, 1) \to \mathbb{R}$ is called α-Hölder if
$$C_\alpha(f) = \sup\left\{\frac{|f(x) - f(y)|}{|x - y|^\alpha} : \ x, y \in (-1, 1), \ x \neq y\right\} < \infty.$$
   (a) Find all α-Hölder functions for $\alpha > 1$. Justify your answer!
   (b) Let $\{f_n\}$, $f_n : (-1, 1) \to \mathbb{R}$, be a sequence of α-Hölder functions satisfying $f_n(0) = 0$ and for some $M \in \mathbb{R}$, $C_\alpha(f_n) \leq M$ for all $n \in \mathbb{N}$. Prove that $\{f_n\}$ has a uniformly convergent subsequence.
   (c) Show that the conclusion in (b) may fail to hold if the hypothesis $C_\alpha(f_n) \leq M$ for all n is omitted.

The exam continues on the next page

4. For $x \in \ell_1$, set
$$\|x\|' = 2|\sum_{n=1}^{\infty} x_n| + \sum_{n=2}^{\infty}(1 + \frac{1}{n})|x_n|.$$
   (a) Show that $\| \|'$ is a norm on ℓ_1.
   (b) Show that ℓ_1 is complete with respect to $\| \|'$.
   (c) Are $\| \|_1$ and $\| \|'$ equivalent norms on ℓ_1? (Proof or counterexample needed.)

5. Let (M, d) be a complete metric space, let $T : M \to M$ be a continuous map and let $\varphi : M \to \mathbb{R}$ be a function which is bounded below. Assume that together they satisfy
$$d(x, Tx) \leq \varphi(x) - \varphi(Tx) \quad \text{for all } x \in M.$$
Prove that for every $x \in M$ the sequence $\{T^n x\}$ converges to a fixed point of T. (Here $T^n x$ is the n-th iterate of T applied to x.)

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a C^∞ (infinitely many times differentiable) function such that the sequence of derivatives, $\{f^{(n)}\}$ converges uniformly on any compact set to a function g. Prove that $g(x) = ce^x$ for some constant $c \in \mathbb{R}$.