Department of Mathematics OSU
Qualifying Examination
Spring 2018

Real Analysis

- Do any four of the six problems. Indicate on the sheet with your identification number which four you wish to have graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this exam.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheet back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

DO NOT WRITE YOUR NAME ANYWHERE – USE ONLY YOUR TEST ID CODE

Standard Notations and Conventions: $\mathbb{R}$ denotes the reals, $\mathbb{C}$ denotes the complex numbers, m_n is Lebesgue measure on $\mathbb{R}^n$ (the subscript will be omitted when $n = 1$), for any Hausdorff space M, $\mathcal{B}(M)$ denotes the sigma-algebra of Borel sets (the sigma-algebra generated by the open sets), set complementation is denoted with ^{c}, i.e. A^c will denote the complement of A with respect to the ambient space, but set difference notation $A \setminus B$ will be used for relative complement. For a measurable set E, $1 \le p < \infty$, $L^p(E)$ denotes the set of (equivalence classes of) p-th power integrable functions with norm $\|f\|_p = \left(\int_E |f|^p \, dm\right)^{1/p}$ and for $p = \infty$ denotes the set of (equivalence classes of) essentially bounded measurable functions with essential supremum as norm. Unless otherwise noted, you may assume that functions in L^p are real valued.

1. Construct an open set $U \subset [0,1]$ such that U is dense in $[0,1]$, $m(U) < 1$ and $m(U \cap (a,b)) > 0$ for any non-degenerate interval $(a,b) \subset [0,1]$.

2. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ with $0 < m_n(E) < \infty$ and assume that $f \in L^\infty(E)$ with $\|f\|_\infty > 0$. For each $n \in \mathbb{N}$, set $\alpha_n = \|f\|^n_{L^n(E)}$. Prove the following:

   (a) $\|f\|_{L^n(E)} \to \|f\|_{L^\infty(E)}$ as $n \to \infty$

   (b) $\dfrac{\alpha_{n+1}}{\alpha_n} \to \|f\|_{L^\infty(E)}$ as $n \to \infty$.

3. Let $I = [a,b]$ be a non-degenerate compact interval. Suppose $g : I \to \mathbb{R}$ is a Lebesgue measurable function such that

$$\int_I g(x)e^{\alpha x} dm(x) = 0 \quad \text{for all } \alpha \in \mathbb{R}.$$

   (a) Show that g is Lebesgue integrable on I.

   (b) Prove that $g(x) = 0$ a.e.

4. Let $\mathcal{S} = \{a = (a_j)_{j\in\mathbb{N}} : \ a_j \in \{0,1\}\}$ – the set of all infinite sequences of 0's and 1's. For $a, b \in \mathcal{S}$, $a = (a_j)$, $b = (b_j)$ consider

$$d(a,b) = \sum_{j\in\mathbb{N}} \frac{|a_j - b_j|}{2^j}.$$

   (a) Show that d is well-defined and is a metric on $\mathcal{S}$.

   (b) Prove that $(\mathcal{S}, d)$ is a complete metric space.

   (c) Prove that $(\mathcal{S}, d)$ is a compact metric space.

The exam continues on the next page

5. Let M, d be a compact metric space and let $f : M \to M$ be a contraction, that is $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$. (Do not assume that f is a strict contraction: i.e. for some $c < 1$, $d(f(x), f(y)) \leq cd(x, y)$.) Prove that f has a unique fixed point.

6. Let $f_k : [0, 1] \to \mathbb{R}$, $k \in \mathbb{N}$ be a sequence of Lebesgue measurable functions such that there exists $M > 0$ with

$$\int_{[0,1]} f_k^2(t)\, dm(t) \leq M^2 \quad \text{for all } k \in \mathbb{N}.$$

For each k define

$$F_k(x) = \int_{[0,1]} f_k(t)\, 1_{[a,x]}(t)\, dm(t),$$

where $1_{[0,x]}$ is the indicator function of the interval $[a, b]$.

(a) Show that F_k is a well-defined continuous functions.

(b) Prove that the sequence F_k has a uniformly convergent subsequence.

(c) Let F_{n_k} be the convergent subsequence from (b), and let F be its limit. Prove that for any $\alpha \in (0, 1)$, and any $x \in [0, 1]$,

$$\int_0^1 \frac{1}{|x-y|^\alpha} F_{n_k}(y)\, dy \to \int_0^1 \frac{1}{|x-y|^\alpha} F(y)\, dy.$$