Department of Mathematics OSU
Qualifying Examination
Fall 2020

Real Analysis

- Do any of the four of the six problems. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheet into the envelope in which you received the exam. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

1. Let

$$F(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

for all real $x > 0$, where $a > 0$ is constant. This problem is concerned with sequences $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = F(x_n)$ for $n \geq 0$, with x_0 chosen arbitrarily.

   (a) (3pt) Prove that there exists a closed interval I with positive and finite length such that F maps I into I.

   (b) (7pt) Prove that for every $x_0 \in I$, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a limit that is independent of the choice of x_0. What is this limit?

2. Let $f : [-1, 1] \to \mathbb{R}$ be a Riemann integrable (bounded) function and define

$$F_n(x) = \int_0^x f(\sin(nt))\, dt, \qquad n \in \mathbb{N},\ x \in \mathbb{R}.$$

   (a) (2pt) Prove that F_n are Lipschitz functions for any n. (Recall, g is Lipschitz on I if for an $L > 0$, $|g(x) - g(y)| \leq L|x - y|$ for all $x, y \in I$.)

   (b) (2pt) Prove for any compact set K in $\mathbb{R}$, F_n has a uniformly convergent subsequence on K.

   (c) (6pt) Prove that the entire sequence F_n is uniformly convergent on $\mathbb{R}$ (for a full credit it would be enough to prove uniform convergence on $[0, \infty)$). What is the limit of F_n?

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3. Let (X, d) be a metric metric space.

   (a) (5pt) Suppose $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ are Cauchy sequences in X. Prove that the sequence $\{a_n\}_{n=0}^{\infty}$ defined by $a_n = d(x_n, y_n)$ is convergent in $\mathbb{R}$. (Note: a partial credit will be given if you prove this under the additional assumption that (X, d) is complete.)

   (b) (5pt) Suppose X can be written as a countable union of compact sets: $X = \bigcup_{n \in \mathbb{N}} K_n$ with each K_n – compact in X. Prove that X is separable, i.e. X admits a countable dense set.

4. Assume that a function $\phi : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and satisfies $\phi(x) > 0$ if $|x| < 1$, $\phi(x) = 0$ if $|x| \geq 1$, and $\int_{-1}^{1} \phi(x)\, dx = \int_{-\infty}^{\infty} \phi(x)\, dx = 1$. For each integer $n \geq 1$, let $\phi_n(x) = n\phi(nx)$ for all $x \in \mathbb{R}$.

   (a) (4pt) Assume that a function $f : \mathbb{R} \to \mathbb{R}$ has compact support (i.e., equals zero outside some compact set) and is continuous on $\mathbb{R}$. For each integer $n \geq 1$, define the convolution $f * \phi_n$ by

$$\left(f * \phi_n \right)(x) \; = \int_{-\infty}^{\infty} f(y)\, \phi_n(x - y)\; dy$$

   for all $x \in \mathbb{R}$. Prove that $f * \phi_n \to f$ uniformly on $\mathbb{R}$ as $n \to +\infty$.

   (b) (3pt) Now assume that a function $f : \mathbb{R} \to \mathbb{R}$ has compact support and is continuous except at finitely many points where f has jump discontinuities. Prove that $f * \phi_n$ is uniformly continuous on $\mathbb{R}$, for each $n \geq 1$.

   (c) (3pt) Suppose f satisfies conditions in part (b). Briefly explain why if f has at least one discontinuity, then it is not possible for $f * \phi_n$ to converge *uniformly* to f on $\mathbb{R}$, as $n \to \infty$ (It would be enough to quote a suitable theorem).

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5. Let $p \in [1, \infty]$, and $B = \{\mathbf{f} \in l_p : \|\mathbf{f}\|_p \leq 1\}$ is the "closed unit ball" in l_p.

Recall: l_p is the space of all sequences $\mathbf{a} = \{a_k\}_{k=0}^{\infty}$ such that the p-norm $\|\mathbf{a}\|_p = (\sum_k |a_k|^p)^{1/p} < \infty$ if $1 < p < \infty$ and $\|\mathbf{a}\|_\infty = \sup_k |a_k| < \infty$ if $p = \infty$.

(a) (2pt) Give an explicit proof that B is indeed closed.

(b) (4pt) Prove that every sequence in B has a component-wise convergent subsequence, but B not compact in l_p. (Recall, a sequence of sequences $\mathbf{f}_n = \{f_n(k)\}_{k \in \mathbb{N}}$ converges component-wise if for each $k \in \mathbb{N}$ the sequence $f_n(k)$ converges as $n \to \infty$.)

(c) (4pt) Let $p > q \geq 1$. Prove that for any $\mathbf{a} \in l_p$ $\|\mathbf{a}\|_p \leq \|\mathbf{a}\|_q$, and so $l_q \subseteq l_p$ (a partial credit will be given if instead you prove that a sequence in l_p that converges in l_q-norm is also convergent in l_p). Show that a sequence of sequences in l_q that converges in l_p-norm does not necessarily converge in l_q.

6. Suppose that a function F is defined by a relation of the form $F(t) = \int_a^b f(x,t)\,dx$ for all real t, where a and b are finite, and f and $\partial f / \partial t$ are defined and continuous on the region $D = [a,b] \times \mathbb{R} = \{(x,t) : x \in [a,b] \text{ and } t \in \mathbb{R}\}$. Prove that F is differentiable and

$$F'(t) = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx$$

for all $t \in \mathbb{R}$. That is, in informal terms, show $\frac{d}{dt}\left(\int_a^b f(x,t)\,dx\right) = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx$.

You must give a proof using the theory available *within the syllabus of the qualifying exam*, i.e. use no measure theory or dominated convergence theorems.