

**Department of Mathematics OSU**  
**Qualifying Examination**  
**Fall 2012**

**PART II: COMPLEX ANALYSIS and LINEAR ALGEBRA**

- Do any of the two problems in Part CX, *use the correspondingly marked blue book* and indicate on the selection sheet with your identification number those problems that you wish graded.

Similarly for Part LA.

- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- You have three hours to complete Part II.
- When you are done with the examination, place examination blue books and selection sheets back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

## PART CX : COMPLEX ANALYSIS QUALIFYING EXAM

1. (a) Let  $\Omega \subset \mathbb{C}$  be open and connected. Find all analytic functions  $f(z)$  on  $\Omega$  satisfying  $f(z)^2 = \overline{f(z)}$ .

- (b) Compute the integral

$$\int_{\gamma} \left( \frac{z+1}{z} \right)^2 dz,$$

where  $\gamma$  is the line segment in the complex plane connecting 1 to  $i$ .

2. (a) Let  $f(z)$  be analytic in an open  $\Omega \subset \mathbb{C}$ , where  $0 \in \Omega$ , and suppose that  $f'(0) \neq 0$ . Show that for each integer  $n \geq 2$ , there is an analytic  $g_n(z)$  defined in some open  $\tilde{\Omega} \subset \Omega$  containing the origin so that  $f(z^n) = f(0) + [g_n(z)]^n$ .

- (b) Show that there is no injective analytic function from the punctured unit disk  $D_a^*(0) = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$  onto the annulus  $A(1, 2) = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$ . Are there analytic functions from  $D_a^*(0)$  onto  $A(1, 2)$  (without the injectivity requirement)?

3. (a) Use the calculus of residues to evaluate the integral

$$\int_0^{2\pi} \frac{dt}{1 - 2p \cos t + p^2},$$

where  $p \in \mathbb{R}$ ,  $p \neq \pm 1$ .

- (b) Let  $R(z) = P(z)/Q(z)$ , where  $P(z)$  and  $Q(z)$  are polynomials without common factors, and assume that  $\deg Q(z) \geq \deg P(z) + 2$ . Let  $z_1, z_2, \dots, z_q$  denote the distinct roots of  $Q(z)$ . Show that  $\sum_{i=1}^q \text{Res}_{z_i} R(z) = 0$ .

**Exam continues on next page ...**

## PART LA: LINEAR ALGEBRA QUALIFYING EXAM

1. Let  $T$  be a linear operator on a finite dimensional complex inner product space  $V$ . Prove that  $T$  is an isometry if and only if there is an orthonormal basis of  $V$  consisting of eigenvectors of  $T$  whose corresponding eigenvalues all have absolute value 1.
2. Let  $\phi, \psi$  be linear transformations on a finite dimensional complex vector space  $V$ .
  - (a) Suppose  $\phi, \psi$  commute, and that  $V$  has a basis of eigenvectors of  $\phi$  as well as a basis of eigenvectors of  $\psi$ . Show that  $V$  must have a basis consisting of eigenvectors for both  $\phi$  and  $\psi$ .
  - (b) Show that if  $V$  has a basis of eigenvectors for both  $\phi$  and  $\psi$ , then  $\phi$  and  $\psi$  must commute.
3. Suppose that  $T : V \rightarrow V$  is a linear operator on a complex vector space and that  $\{v_1, v_2, \dots, v_n\}$  is a basis for  $V$  that is a single Jordan chain (in other words, a cycle of generalized eigenvectors) for  $T$ . Determine a Jordan canonical basis for  $T^2$ .