Department of Mathematics OSU
Qualifying Examination
Fall 2011

PART I : Real Analysis

- Do any of the four problems in Part I. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete Part I.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book and selection sheet into the unmarked, smaller envelope. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

1. Let m be Lebesgue measure on $\mathbb{R}$, let $E \subset \mathbb{R}$ be measurable, $f \in L^1(E)$ and suppose that $f > 0$ a.e. on E. Prove that

$$\lim_{k\to\infty} \int_E |f(x)|^{1/k}\, dm = m(E).$$

2. Let f be a non-negative function defined on a Lebesgue measurable subset E of $\mathbb{R}$. Show that f is measurable if the region $\{\,(x,y) \,:\, x \in E,\ f(x) \geq y\}$ is a Lebesgue measurable subset of $\mathbb{R}^2$.

   Hint: Consider Tonelli's Theorem.

3. (a) Prove that if g belongs to $L^1(\mathbb{R})$ then for every $\epsilon > 0$ there exists a $\delta > 0$ such that if A is any measurable set with $m(A) < \delta$ then $\int_A |g|\, dm < \epsilon$.

   (b) Now let $\{f_n\}$ be a sequence in $L^1(\mathbb{R})$ that converges in L^1-norm to f. Prove that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if A is a measurable set with $m(A) < \delta$, then $\int_A |f_n|\, dm < \epsilon$ for all $n \in \mathbb{N}$.

Exam continues on next page ...

4. Let (X, d) be a compact metric space, and let $f : X \to \mathbb{R}$ be continuous. Prove that for every $\epsilon > 0$ there exists a positive real M such that for all $x, y \in X$

$$|f(x) - f(y)| \leq Md(x, y) + \epsilon.$$

5. Let E be a nonempty, closed, convex subset of a Hilbert space. Show that E contains a unique element x of minimal norm.

   Hint. Make use of the parallelogram law.

6. Let $C([0, 1])$ denote the space of continuous functions on $[0, 1]$ equipped with the norm $\| f \| = \sup_{0 \leq x \leq 1} |f(x)|$. Let g be a Lebesgue integrable function on $[0, 1]$ and let $\mathcal{F}$ denote the set of all functions $f \in C([0, 1])$ such that $f(0) = 0$, f is absolutely continuous, and $|f'| \leq |g|$ almost everywhere (with respect to the Lebesgue measure). Show that the closure of $\mathcal{F}$ is a compact subset of $C([0, 1])$.

   Hint. You may use without proof that a function f is absolutely continuous on $[a, b]$ if and only if f is defined on $[a, b]$, f' exists almost everywhere and is Lebesgue integrable on $[a, b]$, and $f(x) = f(a) + \int_a^x f'(t)\, dt$ for all $x \in [a, b]$.