DIGITAL TOPOLOGY: GRAPH-THEORETICAL VS. TOPOLOGICAL

Chris Edwards and Lynelle Lang

INTRODUCTION:

Digital topology has been developed to address problems in image processing, an area of computer science which deals with the analysis and manipulation of pictures by computers. The results from digital topology help provide a sound mathematical basis for image processing operations such as object counting, boundary detection, data compression and thinning. Around the late 1960's Rosenfeld and others began to work on this area from essentially a graph-theoretical point of view. Although this approach did not lead to a consistent topology due to contradictions [7], working around the problems still brought many useful results [5, see bibliography]. In the last couple of years, a consistent topology, which was published by E. Khalimsky in Russia in 1970, is beginning to be used and some have approached the same image problems topologically [3, 4, 6]. This paper gives a brief sketch of each approach and the advantages and disadvantages we discovered in working from these concepts.

GRAPH-THEORETICAL DIGITAL TOPOLOGY:

The graph-theoretical approach begins by defining connectedness out of adjacency ideas. If x = (a,b) is a point in $\mathbf{Z} \times \mathbf{Z}$ then the points (a ± 1, b) and (a, b ± 1) are **4-adjacent** to x . These points and (a ± 1, b ± 1) are **8-adjacent** to x . The following holds for k = 4 or 8. A **k-path** from point P to Q is a sequence of points $P = P_0, P_1, \ldots, P_n = Q$ where P_{i-1} is k-adjacent to P_i. Points P and Q are **k-connected** in a set **S** if there exists a k-path from P to Q consisting entirely of points in **S**. **Components** are maximal connected sets. The one infinite component of $\bar{S}$ (complement of **S**) is the **background**. **S** is a **k-curve** iff **S** is k-connected and every point of **S** is k-adjacent to exactly two points in **S**.

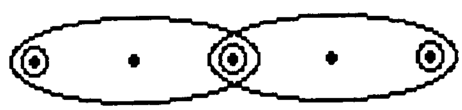
To prove a digital Jordon curve theorem in this context requires the set, **S**, and its complement, $\bar{S}$, to have different ideas of adjacency. Usually **S** is defined as 4-connected, forcing $\bar{S}$ to be 8-connected. With these definitions the following can be proved [10, Sect. 2.4]:

Digital Jordon Curve Theorem I: The complement of a curve γ has exactly two components, namely the inside and the outside (the one that meets the background) of γ. Moreover, every point of γ is 4-adjacent to both of these components.

A **border point** in **S** is any point adjacent to a point in S.

KHALIMSKY TOPOLOGY:

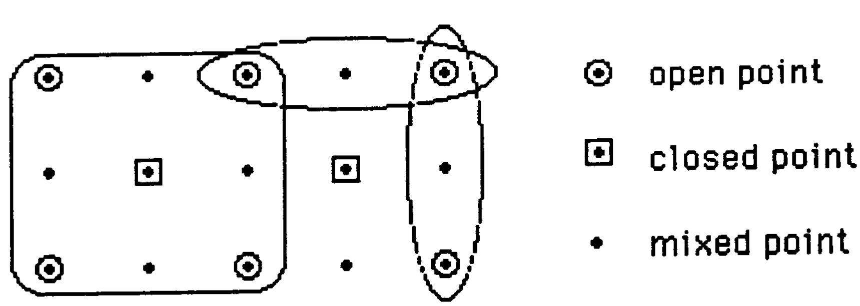
To define a topology on the digital plane, we first define a topology on the integers. This topology can be defined in terms of the minimal neighborhood **N(x)** of each point x. (A set is **open** iff it contains N(x) for every point x in the set). Define a topology on the integers by letting N(k) = {k}, for k even and {k-1, k, k+1} for k odd (See below).



It is valid to regard this space as a digital line because it satisfies the condition of being a connected ordered topological space (COTS).

Definition 1: A **connected ordered topological space** is a connected topological space X, with this property: if Y is a three point subset, there is a y ε Y such that Y meets two connected components of X-{y}.

For a topology on the digital plane we use the product topology. The topology on the integers and the associated product topologies are called the Khalimsky topologies. The Khalimsky topology in the plane consists of three kinds of points: **open points** (open x open), **closed points** (closed x closed) and **mixed points** (open x closed or closed x open). The minimal open set for each point is the product of the minimal open sets on the integers.



Definition 2: A **digital path** (**digital arc**, respectively) in a topological space is the range of a continuous function (homeomorphism) from a finite COTS, i.e., from a finite interval in **Z**.

Definition 3: A **digital Jordon curve** is a finite connected set J with $|J| \geq 4$ such that J - {j} is a digital arc for each j ε J.

Using these definitions the following can be proven:

Digital Jordon Curve Theorem II: If J is a digital Jordon curve in a digital plane X × Y then X × Y - J has exactly two components. The infinite component is called the outside, the other is called the inside.

The relation $x \leq y$ iff y ε N(x) defines a partial order on this space called the **specialization order**. In a topological space we call x a **boundary point** of a set **S** iff each neighborhood of x intersects both S and $\mathbb{S}$. For the Khalimsky topology this means that x is a **boundary point** of **S** iff N(x) intersects both **S** and $\mathbb{S}$. Note that an open point cannot be in the boundary.

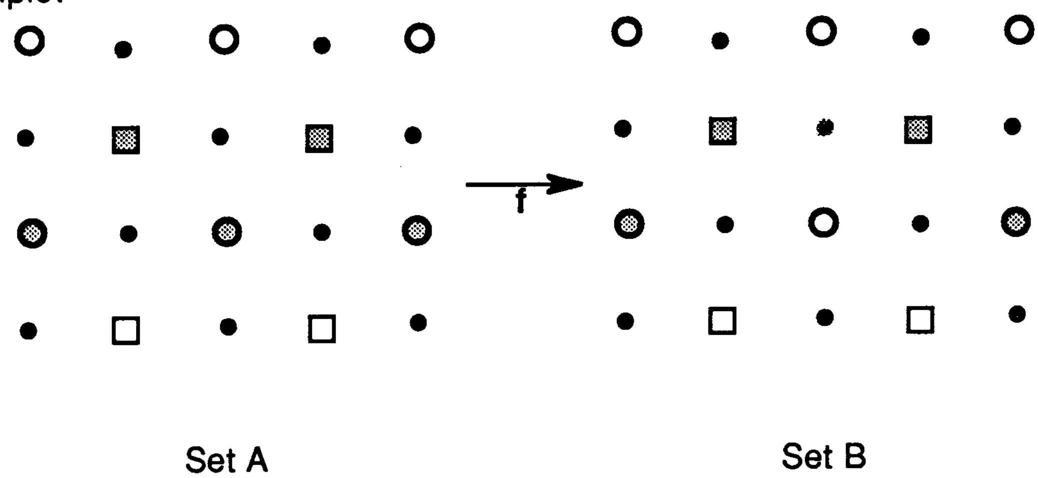
For applications with computers, the **open screen**, where the pixels correspond to the open points in the topology, is usually used. The mixed and closed points do not show up on the screen. The advantages of the open screen over the **pure screen** (both open and closed points correspond to pixels) are mentioned later.

COMPARISON:

Working in the Khalimsky topology provides a topological proof of the Jordon Curve Theorem and other properties on the digital plane, such as a characterization of which curves separate the digital plane into connected components. A topological approach leads naturally to constructions which may be possible in Rosenfeld's theory but have not been considered. Since the topology is a product topology, generalization into n-dimensional digital spaces is possible. Although the graph-theoretical system is called digital topology, essential notions such as open subset, open neighborhood, and boundary of a set are difficult, if not impossible to transfer.

However, there does appear to be some advantages to working in Rosenfeld's theory. For example, he uses a class of metrics to define continuous functions which preserve the connectedness of areas [9]. In Khalimsky's topology continuous functions preserve connectedness in the topology but since only open points serve as pixels, "pixel"-connectedness is not guaranteed to be preserved.

Example:



If **f** is the function that maps A to B then **f** is continuous, but when examining the image on the screen the three open points in Set A are "pixel"-connected while the open points in Set B are not.

This problem can be avoided by requiring open points to be taken to open points but this only guarantees 8-connectedness in the image of either a 4 or 8-connected set.

Definition 4: A function **f** is **specially continuous** in the Khalimsky topology on the digital plane if **f** is continuous and open points are not mapped to mixed points.

Definition 5: If X is a set in Khalimsky topology then $\mathbf{X_p}$ is the open points or pixels in X considered in the Rosenfeld topology.

Definition 6: A set X is **specially connected** if $\forall x,y \in X_p \; \exists z \in X$ such that $z \leq x$ and $z \leq y$. (This would be true for any connected set when using the common global face memberships discussed later).

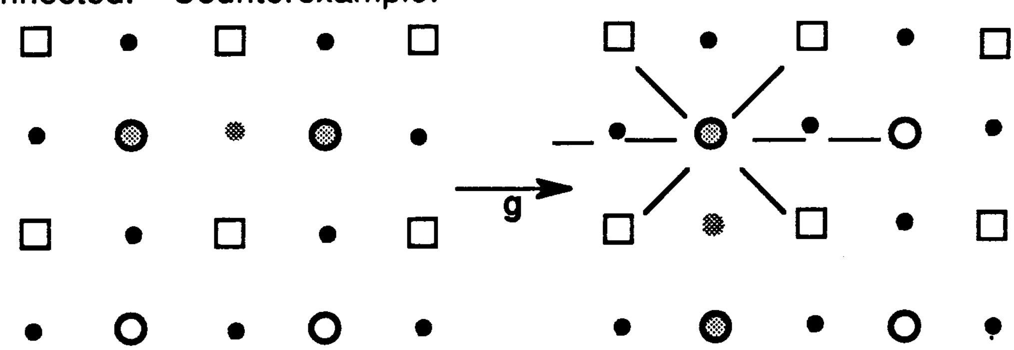
Theorem 1: Let X and Y be digital planes in the Khalimsky topology with X specially connected. Let $\mathbf{f} : X \rightarrow Y$ be a specially continuous function. If X_p is (4 or 8)-connected then Y_p is 8-connected.

Proof: Let x,y ε X_p be (4 or 8)-adjacent. Then since X is specially connected ∃ z ε X such that z ≤ x and z ≤ y in the specialization order, where z is a nonopen point. Then f(z) ≤ f(x) and f(z) ≤ f(y) [Prop. 2.1, 6], and f(z) is a nonopen point so f(x) and f(y) are 8-adjacent. Therefore Y_p is 8-connected.

Theorem 2: Let X and Y be digital planes in the Khalimsky topology. Let **f** : X→Y be a homeomorphism and X a specially connected set in the Khalimsky topology. If X_p is (4-)8-connected then Y_p is (4-)8-connected.

Proof: Let x,y ε X_p be 4-(8-)adjacent. Then since X is specially connected ∃ z ε X such that z ≤ x and z ≤ y in the specialization order, where z is a mixed (closed or mixed) point. Then f(z) ≤ f(x) and f(z) ≤ f(y) [Prop. 2.1, KKM], and f(z) is a mixed (closed or mixed) point so f(x) and f(y) are 4-(8-)adjacent. Therefore Y_p is 4-(8-)connected.

If one tries to use a pure screen with a 6-adjacency relation, then even a homeomorphism does not guarantee that images of connected sets will be connected. Counterexample:



Set C   Set D

The two open points in set C are 6-adjacent while the open points in set D are not, even though g is a homeomorphism.

An important concept in digital topology is that of data compression. One way to compress data is to designate a shape by storing a code for only its outline rather than information for each pixel. In Rosenfeld's topology there are difficulties with transferring the idea of a boundary or border. From the definition the borders of a set **S** and **S** are different and using 4 or 8 adjacency also gives different results. Moreover, with any one of these borders, the border is a set of pixels which , counter-intuitively, has a nonzero area. In Khalimsky's topology the boundary is unique and does not show on the screen when considering the open points as pixels.

Here is an example of a boundary detection algorithm in the Khalimsky topology:

> Given a point $x_0 \varepsilon$ **S**, such that A(x) meets S. Beginning with x_0, check all non open points of A(x) to see wether they are boundary points or not. Let the first new boundary point be x_1, and store any other boundary points for future use. Check all points in $A(x_1)$ for boundary points, find an x_2 and store any others. Continue this process. If at any step no new boundary points are determined by studying $A(x_i)$, then denote the most recently stored unused boundary point as x_{i+1}, and continue.

The boundary can be stored in the same way for either Khalimsky or Rosenfeld. Given a starting point in the boundary, we can define a sequence of numbers such that each number tells the reader the position of the following point. One numbering method starts at the given point and numbers the eight surrounding points of that point from 0 to 7 beginning with the right horizontal neighbor as 0, and proceeding clockwise.

In the Khalimsky topology we must keep the information about not only the pixels (open points) but also the closed and mixed points. For an m x n rectangle, this means approximately 3 to 4 times more set points in Khalimsky's topology as compared to Rosenfeld's. For Khalimsky's the total number of stored points is (2n-1)(2m-1) as compared to m•n in Rosenfeld's. Storing the boundary points explicitly requires about twice as many in Khalimsky's as Rosenfeld's.

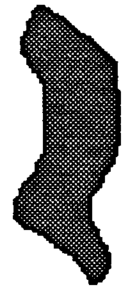
<u>m x n rectangle compression ratio: boundary/total</u>

Khalimsky's	4(m+n) / (2n-1)(2m-1)
Rosenfeld's	2(m+n-2) / mn

An example of the compression of an arbitrary image is given here:

<u>compression ratios</u>

Khalimsky's 589/1005 ≈ .059
Rosenfeld's 238/2631 ≈ .090



A way to make storing boundary points practical in Khalimsky's is to define a **global face membership**. This is a rule which assigns a set membership to the nonpixel (closed and mixed) points according to the set membership of an adjacent pixel (open point). With a global face membership rule one stores only the pixel information explicitly, while the values of the other points are stored implicitly, requiring no memory space. The problems with this method is that with low resolution pictures not every membership rule will correctly represent the connectivity information. Some knowledge of the picture beforehand may indicate what membership rule will work but in some cases each point's membership must be encoded explicitly. But in some of these cases it is possible to store more information than in Rosenfeld's.

In situations where the right information is known about the pictures beforehand, using Khalimsky's topology could be useful. However, when working with relatively arbitrary images Rosenfeld's border may be the most efficient.

SUMMARY:

The topological viewpoint of Khalimsky and others is appealing because it is consistent with the ideas of general topology. This consistency not only allows the use of standard topological ideas and definitions, but also gives this approach an easily defined n-dimensional generalization. Unfortunately, this viewpoint appears to have some disadvantages in certain applications. In data compression there is a trade-off between the quality of the image resolution and the amount of information that must be stored. Unlike Rosenfeld's approach which deals only with the actual pixels, Khalimsky's requires information about other points as well. Also continuous functions on Khalimsky's topology do not preserve the properties we would like in the image. These are two of the current areas in digital topology research. One advantage in Rosenfeld's approach is the amount of past research which has already been done. Algorithms for data compression have been analyzed and continuous functions have been defined which preserve connectedness. Unfortunately, this approach cannot be defined consistently in terms of classical topological concepts. This means that many ideas must be redefined in terms of the graph-theoretical basis. Further research in digital topology from a topological point of view may overcome the current disadvantages.

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