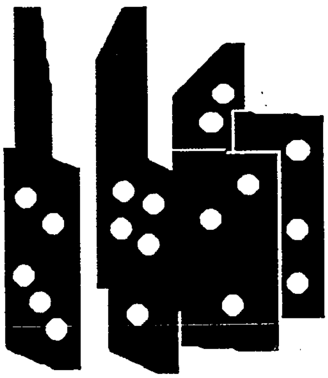
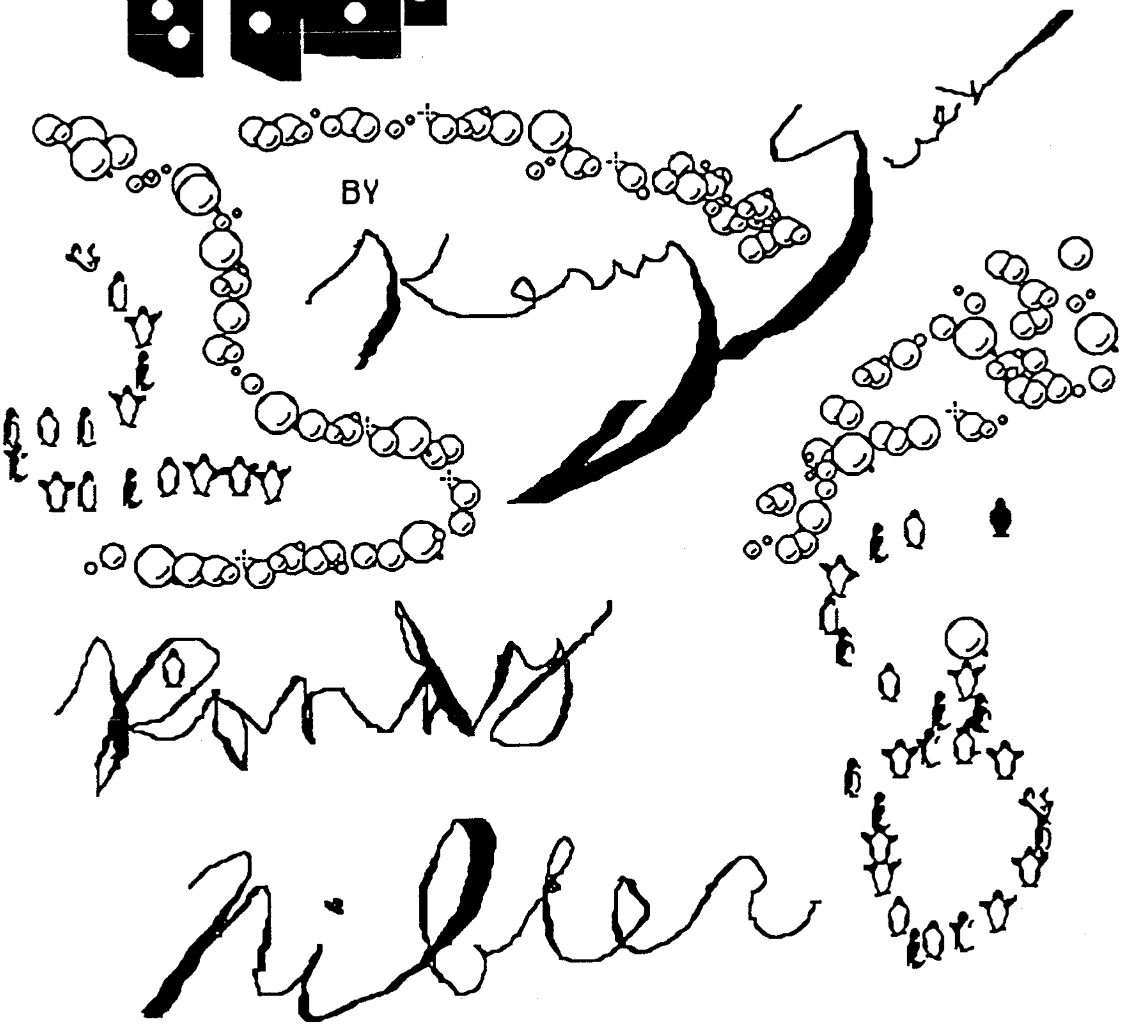


DOMINO TILINGS

BY



Project for NSF REU Program OSU, Summer, 1990

Domino Tilings

We attempted to delve into the answers to questions that arose while dealing with the tilings of 2n x 2n checkerboards. One such question was why the number of distinct tilings was always a square (if n was even), or twice a square (if n was odd). This question, however, became insurmountable. However, we thought that other interesting properties of checkerboards with other dimensions would offer insight into the peculiar characteristic of the 2n x 2n checkerboards.

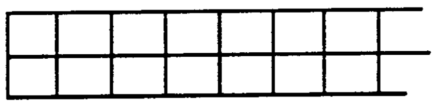
After many conjectures (most of which became dead ends or it was discovered that they were obvious or had already been proven and were well documented), we stumbled upon the idea of coming up with pseudo-recursion relations for 1,2,3,4,5,6 x n (2n for the odds) checkerboards. This proved to be encouraging and fruitful.

Obviously, a 1 x 2n board can only be tiled one way. We, also assumed that the number of tilings for a non-dimensional board was one. For a 2 x n board, something interesting jumped out. We found that the number of tilings, T_n, for a 2 x n board are the Fibonacci numbers.

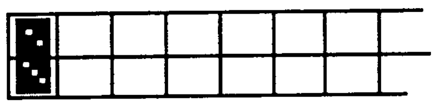
$2^{T_0} = Fib_0 = 1$
$2^{T_1} = Fib_1 = 1$
$2^{T_2} = Fib_2 = 2$
$2^{T_3} = Fib_3 = 3$
$2^{T_4} = Fib_4 = 5$

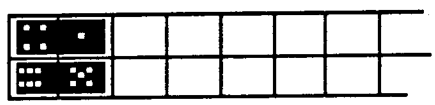
..... and so on.

The relationship can be easily illustrated.



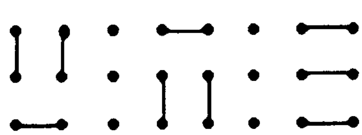
We will call this our $_2T_n$ tiling.



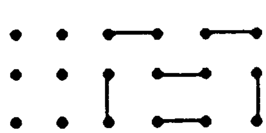


There are only two ways to "end" a 2 x n board, which are illustrated above. Therefore, $_2T_{n+1}$ must be equal to $_2T_n + {_2T_{n-1}}$. This is, of course the exact relationship between the Fibonacci numbers.

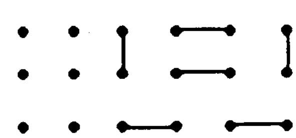
For the 3 x 2n board we found the recursive relationship $_3T_{2n} = 4\ ({_3T_{2n-2}}) - ({_3T_{2n-4}})$. For this recursion, $_3T_0=1$, and $_3T_2=3$. These numbers were easily observed.



The three ways to tile a 3 x 2 section.



And it's reflection:



are the only ways to tile a 3 x 4 tiling so that you don't have a 3 x 2 tiling also, so that it won't be already included.

Following this pattern for each ${}_3T_{2n-2m}$, we arrive at a series:

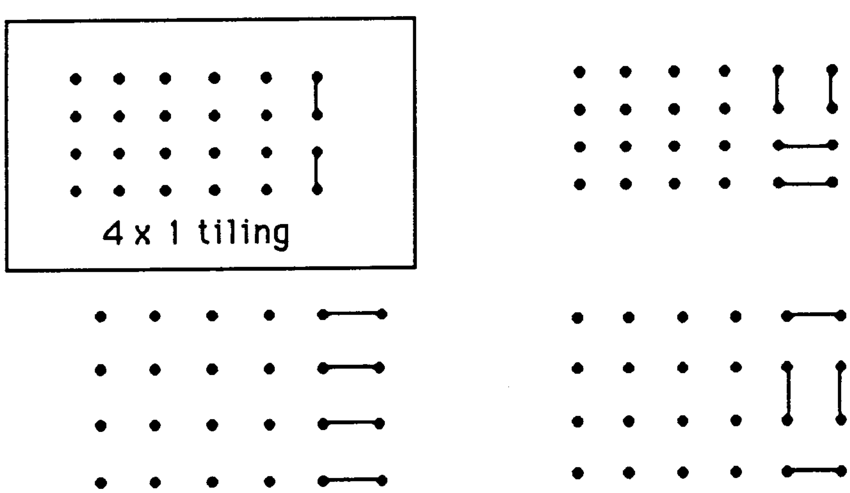
$${}_3T_{2n} = 3({}_3T_{2n-2}) + 2({}_3T_{2n-4}) + 2({}_3T_{2n-6}) + 2({}_3T_{2n-8}) + \ldots\ldots\ldots\ldots$$

Now, we merely add a ${}_3T_{2n-2}$ and then subtract the series for ${}_3T_{2n-2}$, and we arrive at the concise recursion relation:

$${}_3T_{2n} = 4({}_3T_{2n-2}) - ({}_3T_{2n-4}).$$

The recursion relation for 4 x n boards gets a little less concise, but was not too much more difficult to discover.
The relation had more terms and was found to be :

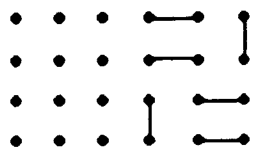
$${}_4T_n = ({}_4T_{n-1}) + 5({}_4T_{n-2}) + ({}_4T_{n-3}) - ({}_4T_{n-4}).$$



4 x 1 tiling

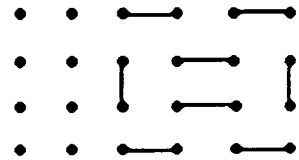
And it's reflection.

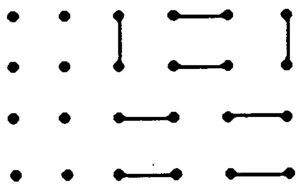
4 x 2 tilings.



And it's reflection.

4 x 3 tiling





And it's reflection.

4 x 4 tilings.

One can see that this leads to the following series:

$$4T_n = (4T_{n-1}) + 4(4T_{n-2}) + 2(4T_{n-3}) + 3(4T_{n-4}) + 2(4T_{n-5}) + 3(4T_{n-6}) + \ldots\ldots$$

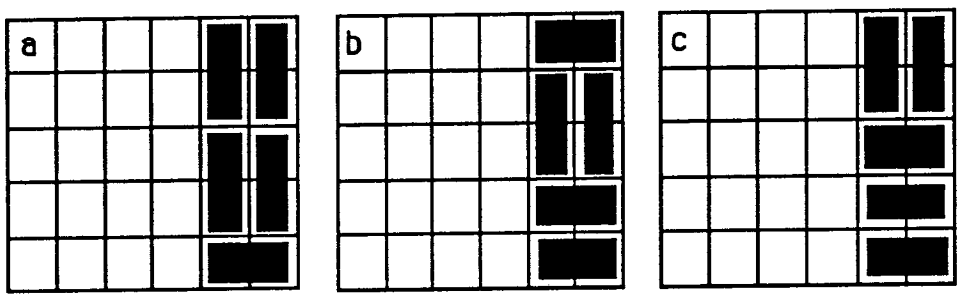
The 4 x 1 tiling contributes the first term. The four 4 x 2 tilings contribute the next term. After this, the tilings with reflections give rise to two extra tilings with each increase in dimension (n). The symmetric tilings lead to an extra tiling with every other increase in dimension.

By adding a $4T_{n-2}$ and then subtracting the series for it, we arrive at the condensed recursion formula:

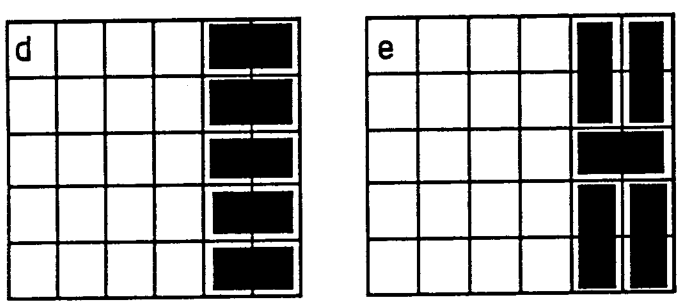
$$4T_n = (4T_{n-1}) + 5(4T_{n-2}) + (4T_{n-3}) - (4T_{n-4}).$$

When we stepped up to 5 x n tilings, the coefficients grew rapidly and we were unable to find a recursion relationship, but we came up with an algorithm which generated the coefficients of a series solution. As before, we started by counting the number of different tilings and the tilings which followed from them.

For the 5 x 2 case, there were:



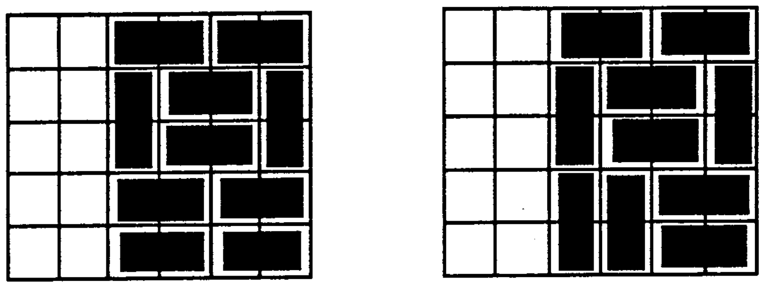
And their inverses.



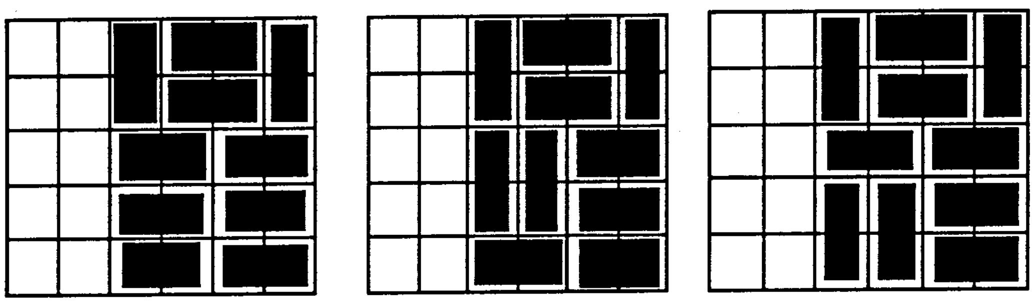
And these, the symmetric ones.

By only turning vertical dominoes, these tilings generated the following 5 x 4 tilings:

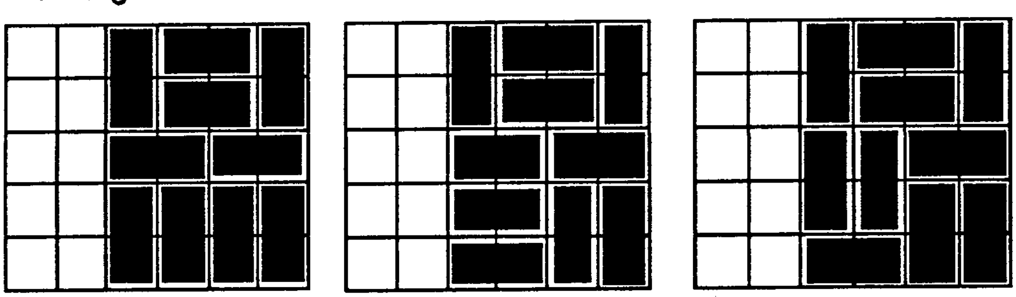
(b) generated:



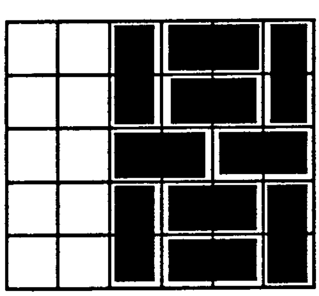
(c) generated:



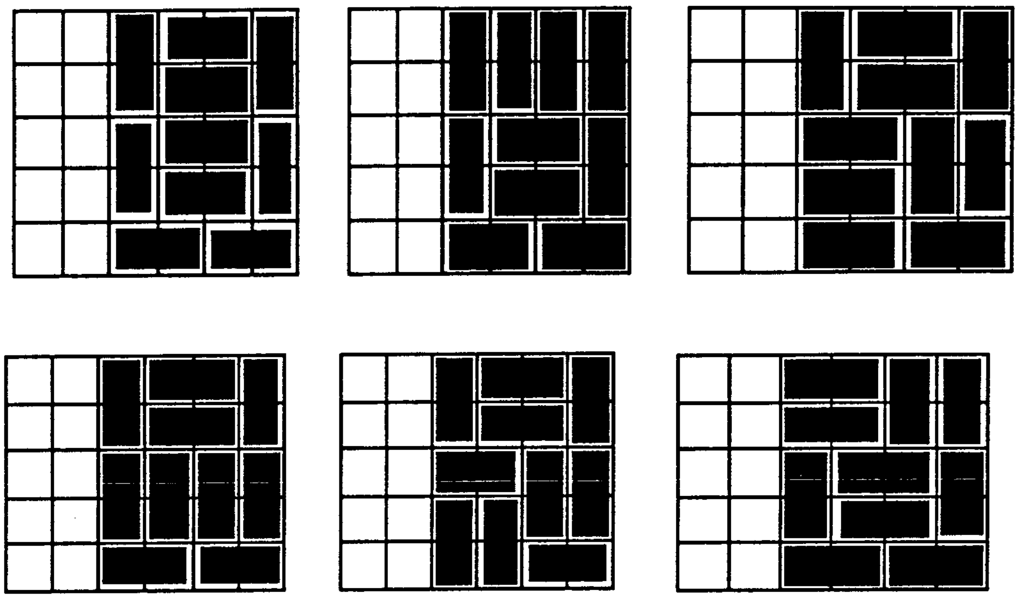
(e) generates :



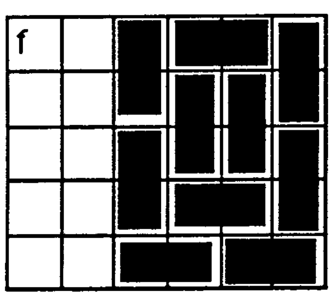
And their inverses, and a symmetric tiling:



(a) generates the following:



And the special tiling:



In order to generate additional tilings, we noted that there was a correspondence between the 5 x 4 tilings that we just listed and the 5 x 2 tilings, from which they were generated, and the subsequent tilings that are generated.

The (b) tiling generates two patterns of tilings, one of which will be similar to (b) for creating further tilings, and one of which will be like (a) for creating further tilings.

The (c) tiling generates three patterns of tilings: one like (a), one like (c), and one like (e) for further tilings.

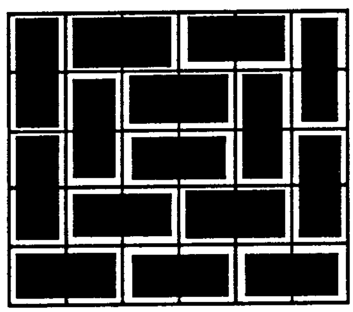
The (d) tiling generates no tilings.

The (e) tiling generates seven patterns: two like (a), two like (c), and three like (e).

The (a) tiling generates three which look like (a), one like (b), one like (c), one like (e), and the special tiling, (f).

The (f) tiling will generate everything that the (a) tiling will generate, plus a "special" tiling, which is also of (f) type.

Only tiling (f) generates the following 5 x 6 tiling:



After examining the generations of tilings, we came up with equations which governed the creation of the next set of tilings.

In short,

(a) => 3 (a)' and (b)' and (c)' and (e)' and (f)'
(b) => (a)' and (b)'
(c) => (a)' and (c)' and (e)'
(e) => 2 (a)' and 2 (c)' and 3 (e)'
(f) => 3 (a)' and (b)' and (c)' and (e)' and 2 (f)'

where the prime denotes the next generation from that set.

So, we can derive the relationships:

$(a)' = 3(a) + (b) + (c) + 2(e) + 3(f)$
$(b)' = (a) + (b) + (f)$
$(c)' = (a) + (c) + (f) + 2(e)$
$(e)' = (a) + (c) + (f) + 3(e)$
$(f)' = (a) + 2(f)$

Thus, we can generate a double prime set from the prime set using these same formulae, and so on, and so on,..............

The sum of the elements of this set will be a coefficient in the series for ${}_5T_{2n}$.

We define $S^m = (a)^{(m)} + (b)^{(m)} + (c)^{(m)} + (e)^{(m)} + (f)^{(m)}$

Then,

${}_5T_{2n} = 8({}_5T_{2n-2}) + S'({}_5T_{2n-4}) + S''({}_5T_{2n-6}) + ...+ S^{(m)}({}_5T_{2n-2m-2}) +....$

The initial values are:

$(a) = 2$
$(b) = 2$
$(c) = 2$
$(e) = 1$
$(f) = 0$

One can rewrite all the ${}_5T_{2n}$ terms in terms of $S^{(m)}$ and can derive an equation of the form:

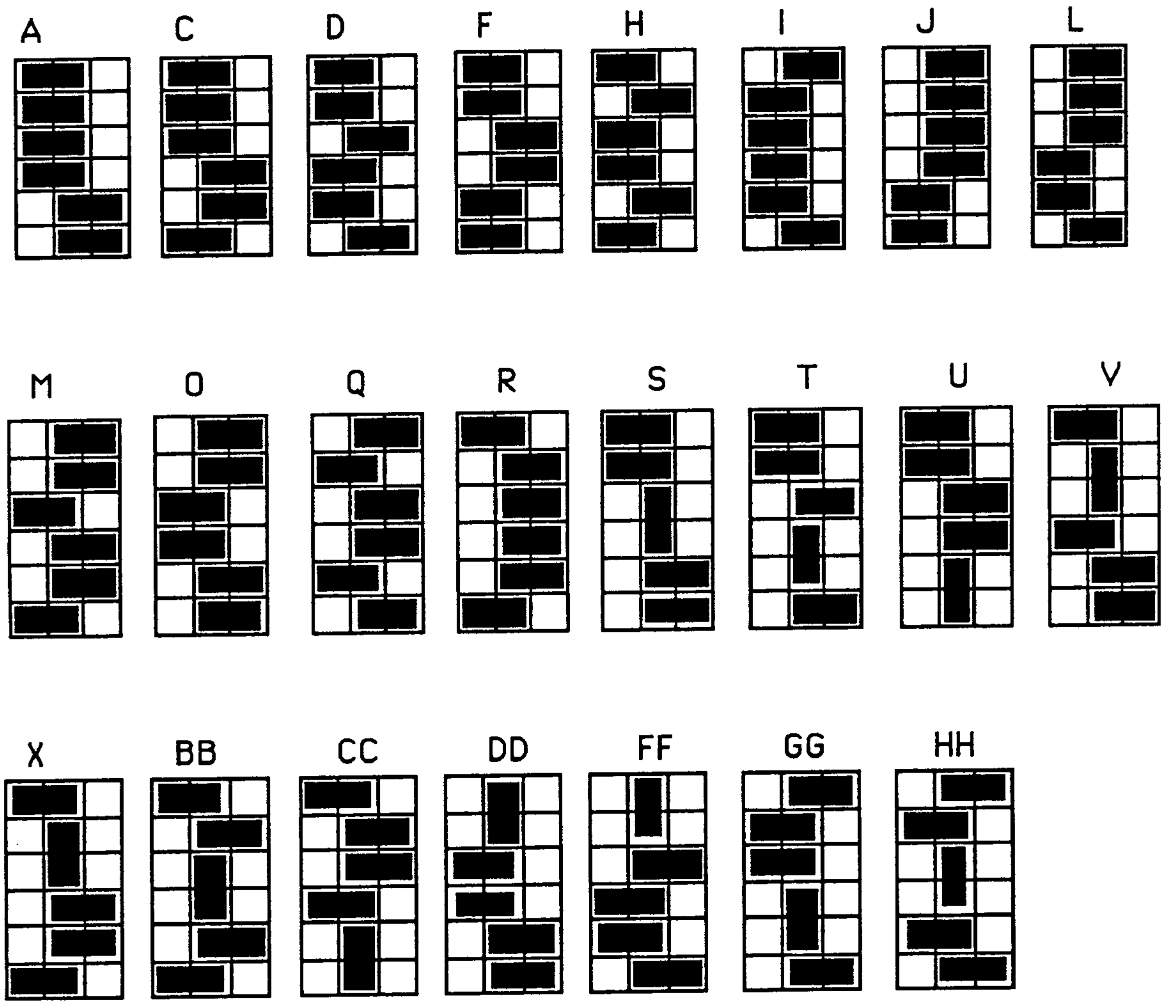
$$ {}_5T_{2n} = \sum_{j=0}^{n-1} S^{(j)} a_j $$

$$ \text{for which,} \quad a_m = \sum_{k=m+1}^{n-1} S^{(k-m-1)} a_k $$

$$ a_{n-1} = 1 $$

With 6 x n tilings, we pursued the same sort of idea about counting the tilings recursively using the different cases of dominoes crossing one column of the board.

We found several orientations that worked, and some that didn't.



These tilings all have two or four dominoes going to both the left and the right. All the letters that one would expect to find in our alphabet and did not appear above, corresponded to a tiling that disallowed the complete tiling of the rest of the checkerboard. These tilings left a different number of black and white squares on a checkerboard pattern to the left and the right of the column tiling. This fact meant that the rest of the board could not be tiled, so the orientation had to be disallowed, hence the absence of some of the lettered tilings.

II will denote the column tilings with two or four bonds to the right but none to the left.

As before, we will show how the tilings transform to higher generations.

A => J' and II'
C => L' and II'
D => M'
F => O' and II'
H => Q'
I => R'
J => A' and S' and V' and DD' and II'
L => C' and X' and CC'
M => D' and T' and FF' and II'
O => F' and 2U' and II'
Q => H' and BB'
R => I' and 2GG' and HH' and II'
S => A' and S' and V' and DD' and II'
T => A' and S' and V' and DD' and II'
U => A' and S' and V' and DD' and II'
V => D' and T' and FF' and II'
X => I' and 2GG' and HH' and II'
BB => I' and 2GG' and HH' and II'
CC => D' and T' and FF' and II'
DD => F' and 2U' and II'
FF => C' and X' and CC'
GG => C' and X' and CC'
HH => H' and BB'

It is obvious that the following equalities hold since they have dominoes going to the right in an identical way.

A' = S' = V' = DD'
GG'= 2 HH' = 2 I'
H' = BB'
U' = 2 F'
C' = X' = CC'
D' = T' = FF'

In order to eliminate some of the variables, we combine each set above into one variable and achieve the following equations:

A => A' and D' and J' and F' and 4 II'
C => L' and I' and D' and 3 II'
D => M' and A' and C' and II'
F => 2 A' and O' and 3 II'
H => Q' and I' and II'
I => R' and 2 C' and H'
J => A' and II'
L => C'
M => D' and II'
O => F' and II'
Q=> H'
R=> I' and II'

The primed terms are then:

II' = 4 A + 3 C + D + 3 F + H + J + M + O + R
A' = A + D + 2 F + J
C' = D + 2 I + L
D' = A + C + M
F' = A + O
H' = I + Q
I' = C + H + R
J' = A
L' = C
M' = D
O' = F
Q' = H
R' = I

The initial values are:

A= 2, C= 0, D= 2, F= 1, H= 0, I= 1, J= 2, L= 2, M= 0, O= 1, Q= 0, R= 0.

The series is found to be:

$$6T_n = 6T_{n-1} + 12(6T_{n-2}) + II'(6T_{n-3}) + II''(6T_{n-4}) + \ldots + II^{(n-3)}(6T_1) + II^{(n-2)}$$

In general, we can find ${}_{m}T_n$ by replacing 6 by m, and just finding the II factors for that m.

$$ {}_{6}T_n = \sum_{k=-1}^{n-2} II^{(k)} a_k $$

for which, $$ a_m = \sum_{j=m+1}^{n-2} II^{(j-m-2)} a_j $$

and $a_{n-2} = 1$, $II^{(-1)} = 1$, $II^{(0)} = (Fib_6) - 1 = 12$

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