# Research Experience for Undergraduates

## A Combinatoric Approach to Hamiltonian Uniform Subset Graphs

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Daniel L. Viar
University of Arkansas

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## A Note to the Reader

During the summer of 1990, I had the opportunity to work in graph theory at Oregon State University trying to solve a conjecture concerning Hamiltonian uniform subset graphs. This paper is the culmination of that work. I would like to take this opportunity to express my sincere thanks to anyone and everyone who either made such a summer program possible or helped me to become part of such a program. I would also like to give a special thanks to Jim Simpson for telling me about this problem, being at what seemed like my beck and call, and showing patience above and beyond the call of duty.

Thanks,

Daniel L. Viar
undergraduate

## Introduction

Chen and Lih first developed the concept of a uniform subset graph in 1987 [1]. A uniform subset graph G(n,k,t) is defined to have all k-subsets of an n-set as vertices and edges joining k-subsets intersecting at t elements. They showed that G(n,k,t) had already appeared in the literature with special values of n, k, and t. (see for instance, [2], [3], [4], or [5] ) They also conjectured that except for the Peterson graph, (G(5,2,0) or G(5,3,1)) G(n,k,t) is Hamiltonian. In this paper we give a summary of their results and discuss a way of approaching the conjecture from a purely combinatorial point of view.

## Defining G(n,k,t)

To define the uniform subset graph G(n,k,t) we consider the n-set $V(n)=\{1,2,...,n\}$ and define

$$G(n,k,t) = ( V(G), E(G) ) \text{ where}$$
$$V(G) = V(n,k) = \{x \mid x \subseteq V(n) \text{ and } |x| = k \} \text{ and}$$
$$E(G) = \{ (x,y) \mid x,y \in V(n,k) \text{ and } | x \cap y | = t \}.$$

Clearly, V(n,k) is just all the subsets of V(n) with k elements and we see that there are $\binom{n}{k}$ of these. The way E(G) is defined, it simply says take two elements of V(n,k) (They are k-sets) and if they have t elements in common then there is an edge connecting them. So far we have not specified a way to choose appropriate n, k, and t. Define the triple (n,k,t) to be admissible for G(n,k,t) if

$$n > k > t \quad \text{and} \quad n > 2k \quad \text{if} \quad t = 0$$
$$n \geq 2k - t \quad \text{if} \quad t > 0$$

This guarantees that G(n,k,t) always has cycles and is always connected.

## Examples

Perhaps the most trivial example is G(3,2,1). Here V(3) = { 1,2,3 } and V(3,2) = { 12, 13, 23 }. Clearly this makes sense since $\binom{3}{2} = 3$. Note that E(G) = { (12,13),(12,23),(13,23) }. Hence the graph makes a triangle. (fig. 1) Another example is G(5,2,0) or G(5,3,1) which gives the Peterson graph. (fig. 2) Note: One reason that these two graphs can be the same is that they have the same number of vertices or $\binom{n}{k} = \binom{n}{n-k}$

## What is Known

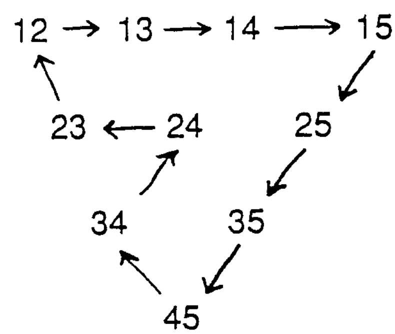
As we stated earlier, the restriction to admissible triples not only makes sense for | V(n,k) | = $\binom{n}{k}$ but it also guarantees that G will be connected. Chen and Lih were able to show that G is regular of degree $\binom{k}{t}\binom{n-k}{k-t}$ which is also its connectivity. G(n,k,t) is both vertex and edge transitive. In dealing with hamiltonian cycles Chen and Lih became interested in finding out information about the length of the longest cycle of G(n,k,t). Let c(n,k,t) denote the length of the longest cycle in G(n,k,t). This is also called the circumference of G(n,k,t). Table 1 contains a summary of known facts about c(n,k,t). Chen and Lih were able to find two functions e(k) and f(k) such that G(n,k,0) and G(n,k,1) are Hamiltonian if $n \geq e(k)$ and $n \geq f(k)$ respectively. Table 2 list some facts about e(k) and f(k). Using the properties of e(k), f(k), and c(n,k,t) Chen and Lih were able to prove the facts presented in Table 3. Notice that from (10) we have that G(n,k,k-1) is Hamiltonian and in particular G(n,2,1) is Hamiltonian when k = 2. In the following discussion of how the problem can be formulated in terms of combinatorics we give a proof of G(n,2,1) and then G(n,k,k-1), which differs from the way Chin and Lih prove it.

## Uniform Subset Graphs and Combinatorics

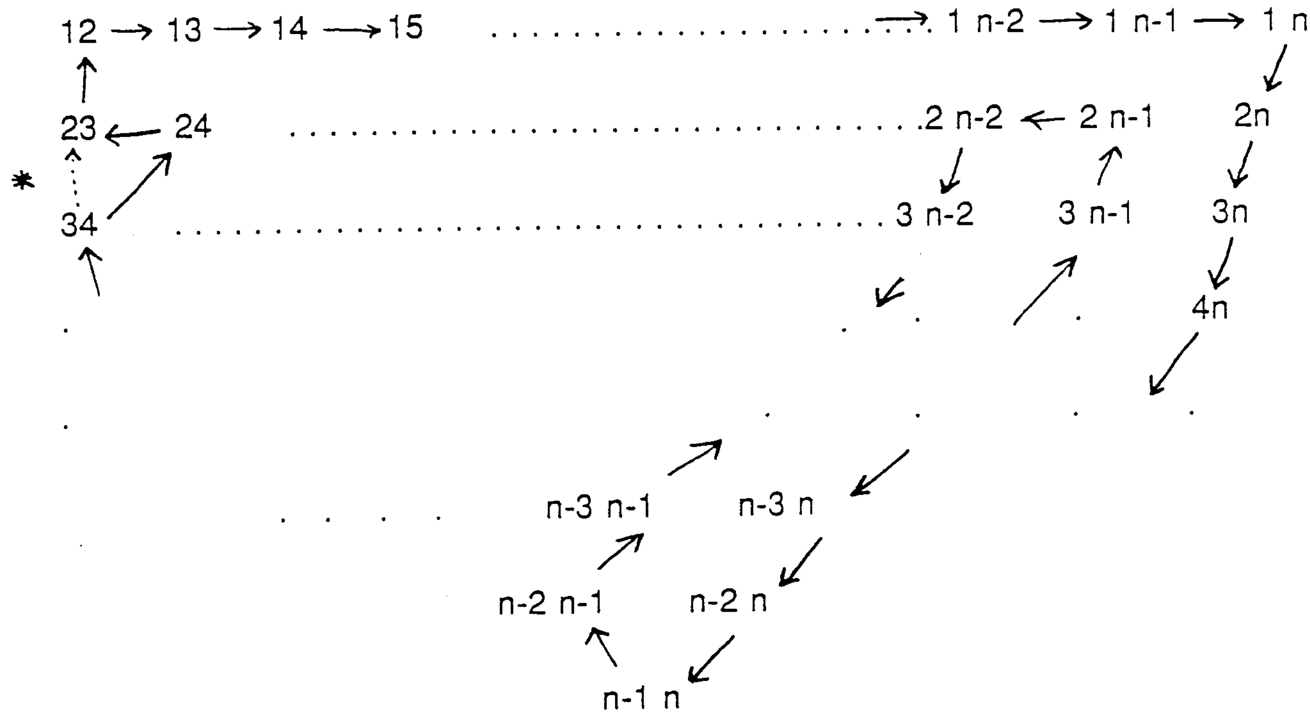
The problem: "Prove that except for the Peterson Graph G(n,k,t) is always Hamiltonian" can be reformulated in the following way. Define a triple (n,k,t) to be admissible in exactly the same way as it was for G(n,k,t). The problem then becomes: "Find an algorithm or way of constructing all the members of $\binom{n}{k}$ so that the $i^{\underline{th}}$ member of $\binom{n}{k}$ has t elements in common with the (i + 1)th of $\binom{n}{k}$ and do it in such a way that the last member has t elements in common with the first." For example, the members of $\binom{3}{2}$ in order are 12, 13, and 23.

**PROPOSITION** The graphs G(n,2,1) and G(n,k,k-1) are Hamiltonian.

Proof: Take for example G(5,2,1). Its vertex set is V(5,2) = { 12,13,14, 15, 23, 24, 25, 34, 35, 45 }. We can construct a Hamiltonian circuit by writing down the V(5,2) in the following way:

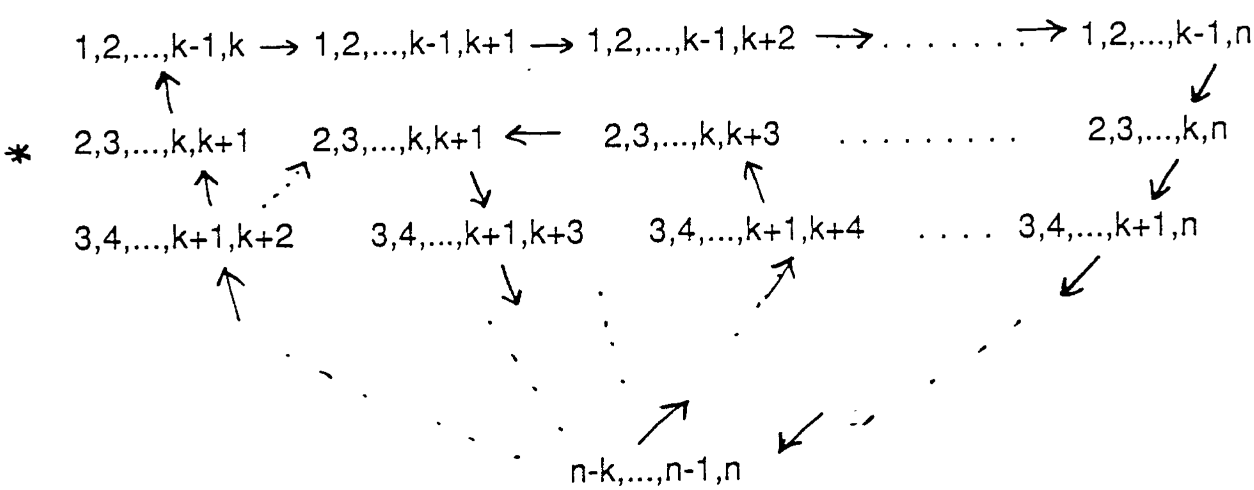



First we connect every element of V(5,2) that has a 1 in common and then every element that has a 5 in common and then a 4 and then a 3 and finally a 2 ending with 23 which connects to 12. To do the proof of G(n,2,1) we do exactly the same thing. First write down all the elements of $\binom{n}{2}$ in the form above. Connect everything that has a 1 in common starting with 12. The last of these is 1n. Next connect all of the n's. The last of these is n-1 n . Next connect all of the n-1's and continue to work backwards. Eventually one reaches 23 which connects to 12. The construction looks like the following:*




*There are actually two different cases, one when n is odd and one when n is even. However, the only real difference in the diagram is at the end before 23 connects to 12. If n is odd then 24 connects to 23 if n is even then 34 connects to 23.

We can now do a similar construction for G(n,k,k-1). First we write down the elements of as shown below.



Connect the (1,2,...,k-1)'s together until (1,2,...,k-1,n) is reached. Then connect the n's together and since $n - 1 = k + x$ for some x, connect the n-1's and so on. The construction is essentially the same as before. QED

## Summary and What's Next

We have given an overview of uniform subset graphs and shown how the problem of solving Chen and Lih's conjecture can be formulated into finding special ways of writing down the elements of $\binom{n}{k}$ . Using this formulation we gave an alternate proof of G(n,k,k-1). Unfortunately, at present the combinatorial formulation has not given any new results, Its only originality is that it gives us a new way of looking at G(n,k,t).

A Bipartite graph H(n,k,t) can be constructed from G(n,k,t). [6] Whether or not our approach can be applied to learn something about H(n,k,t) remains to be seen. It is the author's opinion that no one has found the correct approach to solve the conjecture because of one unanswered question. "Why is it possible that the Peterson Graph could be the only G(n,k,t) which is not Hamiltonian?" Answering this question may be equivalent to solving Chen and Lih's conjecture.

## References

[1] B-L. Chen and K-W. Lih, Hamiltonian uniform subset graphs, *J. Combin. Theory Ser.B* **42** (1987), 257-263

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[3] M. Mather, The rugby footballers of Croam, *J. Combin. Theory Ser.B* **20** (1976), 62-63

[4] A. Moon, The graphs G(n,k) of the Johnson schemes are unique for $n \geq 20$, *J. Combin. Theory Ser.B* **37** (1984), 173-188

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[6] J. E. Simpson, Hamiltonian bipartite graphs, *preprint*

## Table 1: Facts about c(n,k,t)

$$c(n,k,t) \leq c(n+1,k,t)$$
$$c(n,k,t) \leq c(n+1,k+1,t+1)$$
$$c(n,k,t) = c(n,n-k,n-2k+t)$$
$$c(n,k,t) + c(n+1,k,t) \leq c(n+1,k+1,t+1)$$

## Table 2: Facts about e(k) and f(k)

$$a(n,k) = \binom{n-1}{k-1} \Big/ \binom{n-k}{k} \leq 1$$

$$e(k) = \min \{ n \mid n > 2k \text{ and } a(n,k) \leq 1 \}$$

$$f(k) = \min \{ n \mid n \geq 2k-1 \text{ and } \binom{n}{k} < 3k \binom{n-k}{k-1} \}$$

for k = 2,3, . . . ,16
f(k) = 3,6,10,15,22,29,39,49,61,74,88,104,121,139,159

for $k \geq 16$ $a(n(k+1),k+1) \leq a(n(k),k)$

## Table 3: When G(n,k,t) is Hamiltonian (Ham.)

1. G(2k-1,k-1,0) Ham k = 2,4,5,6,7,8
2. G(n,k,0) Ham if $n \geq k + \left( \frac{k 2^{1/k}}{2^{1/k} - 1} \right)$
3. G(n,k,0) Ham if k=1 $n \geq 3$
   k=2 $n \geq 6$
   k=3 $n \geq 7$
4. Induction Theorem

$G(n,k,t)$ and $G(n,k+1,t+1)$ Ham imply $G(n+1,k+1,t+1)$ Ham

5. $n \geq e(k)$ imply $G(n,k,0)$ Ham

6. $k \geq 16$ and $n \geq \frac{k(k+1)}{2}$ imply $G(n,k,0)$ Ham

7. $n \geq f(k)$ imply $G(n,k,1)$ Ham

8. $n \geq k^2-k$ imply $G(n,k,1)$ Ham

9. let $n_0$ and $k$ be such that $G(n,k,0)$ is Ham for all $n > n_0$ and $G(n_0,k+r,r)$ is Ham for $r=0,1,\ldots,n_0-2k$. Then $G(n,k+r,r)$ is Ham for all $n>n_0$ and $r=0,1,\ldots,n-2k$.

10. $G(n,k,k-1)$ Ham

11. $G(n,k,k-2)$ Ham

12. $G(n,k,k-3)$ Ham

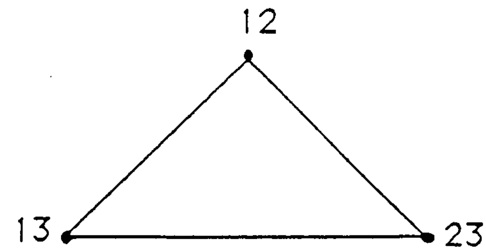


Figure 1: G(3,2,1)

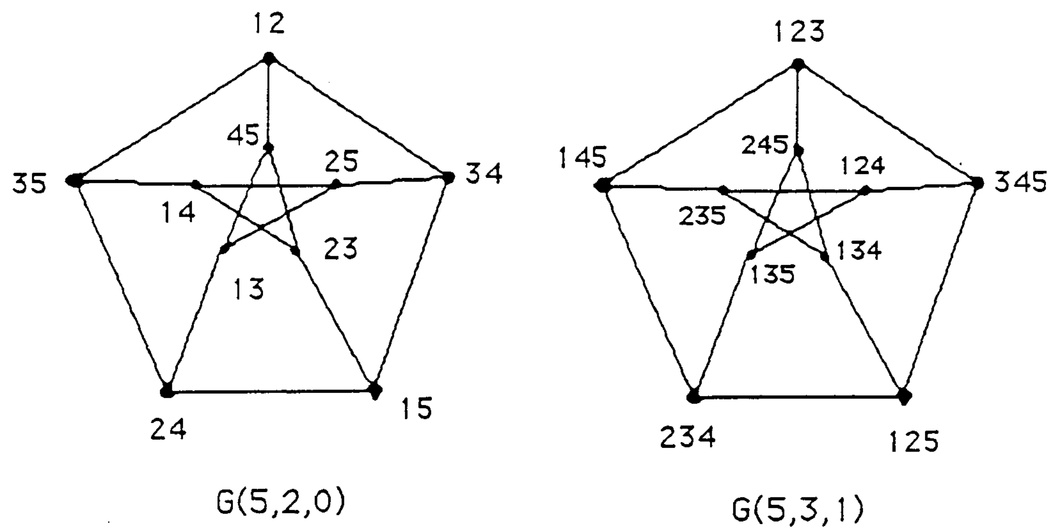


Figure 2: The Peterson Graph