The Existence of Dense Functions and Dense Partitions

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A. INTRODUCTION. Some functions are 'nice' and some are not. For example, suppose we have a real function on the real line satisfying $f(x+y) = f(x) + f(y)$. It is known that if such a function is continuous at zero, then it is uniformly continuous everywhere, and in fact it is linear – which is very 'nice.' But if the function is not continuous at zero, then it is not 'nice' at all: it is continuous nowhere, and in fact its graph is scattered throughout the entire plane. To be more precise, its graph intersects every disk (arbitrarily small) in $\Re^2$. We say that such a function is **dense** (this matches the ordinary definition of a dense set in a topological space). More generally,

1. Definition. A function (or partial function) from a topological space D to a topological space R is **dense** if the function (considered as a set of ordered pairs) is dense in the product topology $D \times R$.

Theorem 2 gives some equivalent formulations, which follow directly from the definition.

2. Theorem. A function $f{:}D \to R$ is dense $\Leftrightarrow$ for each nonempty open set A in D, and each nonempty open set B in R, there exists $x \in A$ and $y \in B$ such that $y = f(x)$ $\Leftrightarrow$ for each nonempty open set A in D, $f(A)$ is dense in R. Also, this statement holds if we replace the words "nonempty open set" with "nonempty basic set."

B. EXISTENSE OF DENSE FUNCTIONS. A natural question to ask is whether there exists a dense function between two given topological spaces, and if so, how to construct one. We begin by giving a necessary condition for the existence of a dense function.

3. Theorem. Let D and R be topological spaces. If there exists a dense function from D to R, then the size of the smallest nonempty open set in D is at least the size of the smallest dense set in R. (By "smallest nonempty open set in D" we mean any nonempty open set in D whose cardinality is a lower bound for cardinalities of nonempty open sets in D.)

Proof: Let A be the smallest nonempty open set in D. Then $f(A)$ is dense in R and clearly $|A| \geq |f(A)|$.

We find that this condition is also sufficient in certain cases; Theorems 4, 5, 7, and 9 are partial converses to Theorem 3.

4. Theorem. Let D and R be topological spaces, and suppose D has a finite basis. Then, if the size of the smallest nonempty open set in D is at least the size of the smallest dense set in R, there exists a dense function from D to R.

Proof: Let S be the smallest dense set in R, and let $\{A_1, A_2, A_3, \ldots, A_n\}$ be the finite basis. Without loss of generality we may assume that the list $\{A_i\}$ contains no repetitions and is ordered in such a way that if either $A_i \subsetneq A_j$ or $|A_i|<|A_j|$ then $i<j$. By theorem 2 it suffices to find a map $f{:}D \to R$ such that $S \subseteq f(A_i)$ for all $i=1\ldots n$.

First we assign function values to the points in A_1. Since $|A_1| \geq |S|$ by hypothesis, it is easy to assign these function values in such a manner that $S \subseteq f(A_1)$. Now we proceed by recursion. Assume function values have been assigned to all the points in $A_1 \cup A_2 \cup \cdots \cup A_k$, and that $S \subseteq f(A_i)$ for all $i=1\ldots k$. We will now extend the domain of the function to include A_{k+1}. Suppose, for the first case, that $A_{k+1} \cap A_i \neq \varnothing$ for some $i \in \{1,\ldots,k\}$. Then $A_{k+1} \cap A_i$ is a nonempty open proper subset of A_{k+1}, so there must exist some j such that $A_{k+1} \cap A_i = A_j$. By the ordering we have chosen for the list $\{A_n\}$, we know that $j < k$. Therefore, we have already assigned function values to A_j in such a way that $S \subseteq f(A_j)$. So, to each point of A_{k+1} to which we have not already assigned a function value, we may assign an arbitrary point of S. Then $S \subseteq f(A_j) \subseteq f(A_{k+1})$.

Now we turn to the second case, in which $A_{k+1} \cap A_i = \varnothing$ for all $i=1\ldots k$. Since we have assigned function values only to those points in $A_1 \cup A_2 \cup \cdots \cup A_k$, none of the points in A_{k+1} have been assigned function values. Thus we are free to assign function values to the points of A_{k+1} in any way we choose. Since $|A_{k+1}| \geq |S|$ by hypothesis, we know that we can map A_{k+1} onto S, just as we did with A_1, so that $S \subseteq f(A_{k+1})$.

In this manner we assign function values to all the points in $A_1, A_2, \ldots, A_n$.

5. Theorem. Let D and R be topological spaces, and suppose the size of the smallest nonempty open set in D is at least the size of the smallest basis for D and at least the size of the smallest dense set in R. Then there exists a dense function from D to R.

Proof: Let S be the smallest dense set in R, and let $\mathcal{U}$ be the smallest basis for D. Our assumption is that each open set in D has cardinality greater than or equal to both $|\mathcal{U}|$ and $|S|$. Now, if $\mathcal{U}$ is finite, then we are done, by theorem 4. Assume then that $\mathcal{U}$ is infinite. In that case, $|\mathcal{U} \times S| = \max\{|\mathcal{U}|, |S|\}$, so our assumption is that each open set in D has cardinality at least $\kappa \equiv |\mathcal{U} \times S|$.

Now, the cardinal number κ is also an ordinal number, and we know that there is a well-ordering of $\mathcal{U} \times S$ that has ordinal number κ. Thus $\mathcal{U} \times S$ can be written as a κ-sequence of ordered pairs $\{ (A_\alpha, p_\alpha) \}_{\alpha \in \kappa}$.

Now we define a sequence of points $\{ x_\alpha \}_{\alpha \in \kappa}$ using transfinite recursion, as follows. Let $B_\alpha = \{ x_\xi : \xi < \alpha \}$. Clearly $|B_\alpha| \leq |\alpha|$. Since $\alpha < \kappa$ and κ is a cardinal number, $|\alpha| < |\kappa|$. Since A_α is an open set in D, we have by assumption that $|\kappa| \leq |A_\alpha|$. Therefore $|B_\alpha| < |A_\alpha|$. Thus there is a point in A_α which is not in B_α. Let x_α be that point.

Now, each x_α is distinct, by our construction. Thus $f \equiv \{ (x_\alpha, p_\alpha) : \alpha \in \kappa \}$ is a partial function on D. To verify that f is dense, we need only show that $S \subseteq f(A)$ for all $A \in \mathcal{U}$. But this is clear, since for every $p \in S$ we have $(A, p) = (A_\alpha, p_\alpha)$ for some α, and $x_\alpha \in A_\alpha = A$ with $f(x_\alpha) = p_\alpha = p$. Obviously, any dense partial function can be extended to a dense function by arbitrary assignment of function values, so we are done.

6. Definition. A topological space X is **relatively countable** if it has a basis with cardinality no larger than that of X itself.

7. Theorem. Let D and R be topological spaces, and suppose every subspace of D is relatively countable. Then, if the size of the smallest nonempty open set in D is at least the size of the smallest dense set in R, there exists a dense function from D to R.

Proof: Let $\{A_\alpha\}_{\alpha \in \gamma}$ be a well-ordered sequence of all the open sets in D, ordered in increasing cardinalities. We shall define using transfinite recursion a sequence of functions $\{f_\alpha\}_{\alpha \in \gamma}$ each of which satisfy one of the conditions: (1) $f_\alpha \neq \varnothing$ and f_α is a dense function from A_α to R, or (2) $f_\alpha = \varnothing$ and A_α intersects some A_ξ, with $\xi < \alpha$ and $f_\xi \neq \varnothing$.

Assume f_ξ has been thus defined for all $\xi<\alpha$, and consider A_α. For case (I), suppose that A_α intersects some A_ξ, with $\xi<\alpha$ and $f_\xi \neq \varnothing$. Then we simply define $f_\alpha = \varnothing$, thus satisfying condition (2). For case (II), assume that A_α is disjoint from every A_ξ with $\xi<\alpha$ and $f_\xi \neq \varnothing$. We claim that in this case, A_α has no open subset of smaller cardinality than A_α itself.

To see this, suppose there were an open subset B of A_α with smaller cardinality. Since B is an open set, we must have $B=A_\beta$ for some $\beta<\alpha$. If $f_\beta \neq \varnothing$, we contradict our assumption for case (II). If $f_\beta = \varnothing$ then condition (2) must hold for f_β, which means that A_β intersects some A_ξ, with $\xi<\beta$ and $f_\xi \neq \varnothing$. But since $A_\beta = B \subseteq A_\alpha$, this implies that A_α also intersects A_ξ (with $\xi<\beta<\alpha$ and $f_\xi \neq \varnothing$). Again we have a contradiction to our assumption for case (II).

Thus we have established our claim that every open subset of A_α has the same (for it cannot be greater) cardinality as A_α. Now, A_α is relatively countable, as a subspace of D. Therefore if B is any open subset of A_α, $|B| = |A_\alpha| \geq |\text{smallest basis for } A_\alpha|$. Then by Theorem 5, there exists a dense function from A_α to R. We let f_α be that function. Thus condition (1) is satisfied.

Our recursive definition is now complete. Notice that it is only in case (II) that we define f_α to be nonempty. But this is precisely the case in which A_α is disjoint from all the ranges of the functions f_β for $\beta<\alpha$. Therefore the ranges of the functions are all disjoint, so $f \equiv \bigcup_{\alpha\in\gamma} f_\alpha$ is a partial function.

To verify that f is a dense partial function from D to R, we need only to verify that $f(A)$ is dense for every open set A in D. We know $A=A_\alpha$ for some $\alpha \in \gamma$. If $f_\alpha \neq \varnothing$ (condition 1), then $f(A)$ contains $f_\alpha(A)$, which is dense in R because f_α is a dense function from A to R. If $f_\alpha = \varnothing$ (condition 2), then A intersects some A_ξ, with $\xi<\alpha$ and $f_\xi \neq \varnothing$. Then $f(A) \supseteq f(A \cap A_\xi) \supseteq f_\xi(A \cap A_\xi)$, which is dense in R because f_ξ is a dense function from A_ξ to R. Thus f is a dense partial function from D to R. Of course, any dense partial function can be extended to a dense function by arbitrary assignment of function values, so we are done. ∎

8. Lemma. Every metrizable space is relatively countable.

Proof: If M is finite, then every metric yields the discrete topology, so the set of all singleton points forms an basis. If M is infinite, then (for any arbitrarily chosen metric) the set of all balls of rational radius forms an basis, with cardinality $|M \times \mathbb{Q}| = |M|$. ∎

9. Theorem. Let D and R be topological spaces, and suppose D is metrizable. Then, if the size of the smallest nonempty open set in D is at least the size of the smallest dense set in R, there exists a dense function from D to R.

Proof: Theorem 7, Lemma 8, and the fact that metrizability is hereditary. ∎

At this point, one might wonder if every topological space is relatively countable. If so, then we could drop the condition of metrizability in Theorem 9, and between Theorems 3 and 9 we would have a necessary and sufficient condition for the existence of a dense function. Unfortunately, we have been unable to prove or disprove the statement that every topological space is relatively countable. We speculate that the statement is related to (perhaps equivalent to) the Generalized Continuum Hypothesis.

C. DENSE PARTITIONS.

10. Definition. A **dense partition** of a topological space X is a family of pairwise-disjoint subsets of X, each of which is dense, such that the union of all the subsets is equal to X. The **size** of a partition is simply its cardinality (the number of subsets).

For example, the family of subsets of the real line (with the usual topology) consisting of the set of irrationals and the set of rationals is a dense partition with size 2. We consider the question: given a topological space, does there exist a dense partition of a given size? For example, does the real line have an uncountable dense partition? We shall see that this is related to the question of the existence of dense functions.

11. Theorem. Let D be a topological space every subspace of which is relatively countable. The following are equivalent:

i. Each open set in D has cardinality at least κ.

ii. D has a dense partition of size κ.

iii. There exists a dense function from D to R, where R is any topological space such that the smallest dense set in R has cardinality κ.

Proof: (i $\rightarrow$ ii) Let κ be given the discrete topology. Then the smallest dense set in κ is κ itself, so the conditions of Theorem 7 are met. Therefore there exists a dense function f from D to κ. Define $A_\alpha = f^{-1}(\{\alpha\})$ for all $\alpha \in \kappa$. Each A_α is dense in D because it is the inverse image of an open set in κ, therefore $\{A_\alpha: \alpha \in \kappa\}$ is a dense partition of size κ.

(ii $\rightarrow$ iii) Let $\{A_\alpha: \alpha \in \kappa\}$ be a dense partition of size κ, and let $\{p_\alpha: \alpha \in \kappa\}$ be a dense set in R of cardinality κ. Define $f(x) = p_\alpha$, for all x in A_α. Then f is a dense function from D to R, since given any nonempty open set P in D and any nonempty open set Q in R, there is some p_α in Q (since $\{p_\alpha: \alpha \in \kappa\}$ is dense), and A_α must intersect P, (since $\{A_\alpha: \alpha \in \kappa\}$ is a dense partition).

(iii $\rightarrow$ i) Theorem 3.