The Game of Zigzag
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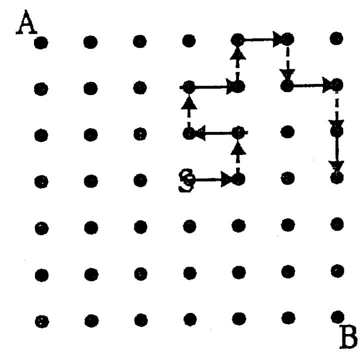
1992 REU Project

I Introduction

We discovered the game of Zigzag in The Amazing Mathematical Amusement Arcade by Brian Bolt. He gives the following description:

> *"Use a 7x7 or 9x9 square array of dots. Start at the centre dot, S. The first player draws an arrow either across or up and down to the nearest dot. The second player follows with an arrow to form a continuous path. The players move alternately. The object is to form a path from S to the home base (A for the first player, B for the second player) without visiting any point more than once. The first player to home base wins."* [1]



This game seems restricted by the size of the board and it is unclear whether A wins if B moves into A's goal (and visa versa). Therefore, a more generalized, complete definition of the game is needed in order for a worthwhile analysis to be performed.

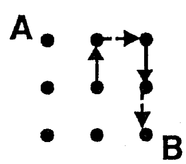
Definition 1. *The game of **Zigzag** is played on a mxn grid of dots where m and n are both odd. The goals, labeled A and B, are in diagonally opposite corners. Play begins at the center dot, S. The two players alternate making a horizontal or vertical move from the dot they're positioned on to an adjacent dot, forming a continuous path from S. Each dot can be visited, at most, one time. The player whose goal is reached wins, no matter which player makes the final move.*

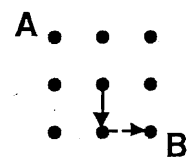
Later we will relax some of the restrictions we have placed on the game in order to expand our analysis.

II Analysis

A. Preliminaries We emphasize that Zigzag is a finite, two person, zero-sum game of perfect information. The game is finite because the board is finite. It is zero-sum because either one player wins and the other loses, or else a tie occurs. It contains perfect information because all moves are visible to both players. Therefore, a pure strategy solution to the game exists. This means that if both players are using their best stategy, then only one outcome of the game is possible. [2]

To begin our analysis, we reduce the size of the game to a 3x3 grid, and find that the second player has a winning strategy. This can be shown in two cases. (Note: From now on A refers to the first player, B refers to the second player, S refers to the starting square, A's goal will always be in the upper left-hand corner, and mxn will refer to m rows and n columns.)





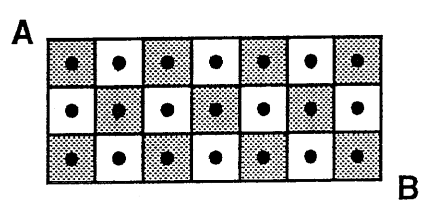
Since the grid is symmetric, we need only consider two of A's four possible first moves. B's strategy when A moves up is equivalent to that when A moves left. Likewise, B's strategies when A moves either down or to the right are also equivalent. When A moves up from S, B moves right to avoid A's goal. A is forced to move down. B moves into B's goal and wins the game. When A moves down from S, B moves right to win the game. Therefore, no matter what strategy A chooses, B can win the game.

Using similar arguments, it can be shown that A has a winning strategy on a 3x5 board while B has a winning strategy on a 3x7 board. This alternating pattern for the value (outcome) of the game leads us to the following lemma:

Lemma 1. *Given a 3xn board where n>0, A's and B's goals are diagonally opposite, and S is the center square, if n=1(mod 4) then A has a winning strategy. If n=3(mod 4) then B has a winning strategy.*

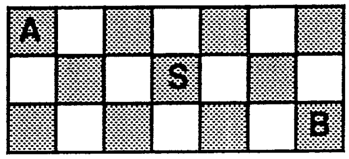
This lemma will be proven in conjunction with the proof of Lemma 4.

When analyzing the game, it is helpful to arrange the grid on a checkered board so that each square represents a dot.



Notice that a player can only move to squares of the same color throughout the play. For instance, if play begins on a black square then A can only move to white squares. We say that A 'controls' the white squares. Likewise, B controls the black squares. From now on, we will represent the game of Zigzag by a checkered board rather than a grid of dots.

An advantage exists for the player who controls a corner. Consider the following board:

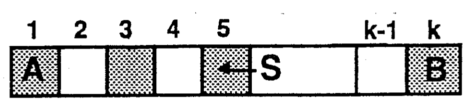


In this board, B controls the corners. The player who is able to move into a corner (other than a goal) is in control of three consecutive moves. For instance, if B moves into the top right corner from the right, then A has no choice but to move down. Now B can make another move.

Another possible advantage exists by controlling the goals. In the above board, B controls both goals. Therefore, the only way that A can win is if B moves into A's goal. This will only happen if B's alternative moves are blocked so that B has no choice but to move into A's goal. This is called a forced win and it is much more difficult to acheive than a win by simply moving into one's own goal. This suggests that the player who controls the goals may have an advantage. In all 3xn boards where n=3(mod 4), B controls both corners and goals and B has a winning strategy. Similarly, in all 3xn boards where n=1(mod 4), A controls both corners and goals and A has a winning strategy.

B. Variations We will now relax some of the restrictions in our original definition of Zigzag by changing the position of the starting square. Given Lemma 1 regarding 3xn boards, we find that A will continue to have a winning strategy when S is any square of the same color as the center square on a 3xn board where n=1(mod 4) Similarly, on a 3xn board where n=3(mod 4), B continues to have a winning strategy when S is any square of the same color as the center square. We would like to relax our restrictions even further by allowing n to be either odd or even. However, before continuing with the 3xn board cases, let's first consider the 1xn and 2xn board cases.

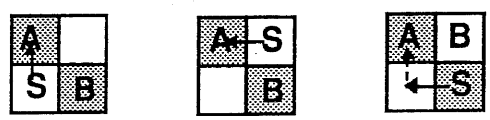
Lemma 2. *On a **1xn board** where n is any natural number, A's and B's goals are at opposite ends of the board, and S is any square other than a goal, A has a winning strategy.*



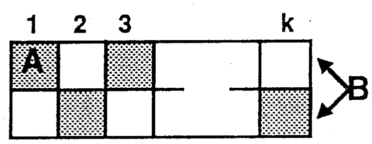
Proof: If A moves in the direction of A's goal from S, then B is forced to continue in this direction. Play is forced to A's goal.❑

Lemma 3. *On a **2xn board** where n is any natural number, S is any square other than a goal, and the goals are either diagonally opposite or in opposite corners of the same row, A has a winning strategy and play never moves to the right of S.*

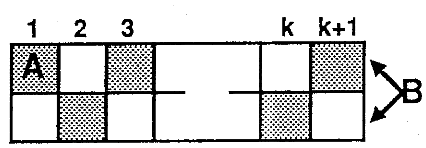
Proof: (by Induction) This statement is obviously true in the base case when n=2.



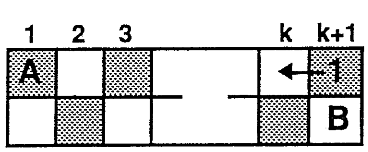
Assume the statement holds when n=k.

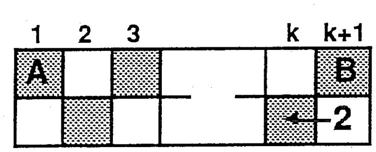


(Note: A can win from either square in column k, given that the other square in the column is B's goal.) Now, create a 2x(k+1) board by adding one column to the 2xk board on the side that has B's goal.



A continues to have a winning strategy from any square on the 2xk board because play never moves to the right of S so the additional column does not affect play. Therefore, we need only to consider squares 1 and 2 in the additional column.





If S is square 1, then A moves left onto the 2xk board. B's only options are squares on the 2xk board, and from any of these squares, A has a winning strategy according to our assumption. The same argument holds when S is square 2. Therefore, the statement holds.❑

C. 3xn Board We now return to the 3xn board in which the outcome of the game depends on the number of columns (whether odd or even) and on the color of the starting square. (Note: From now on we will refer to the color of A's goal as color(A), the color of B's goal as color(B) and the color of the starting square as color(S).)

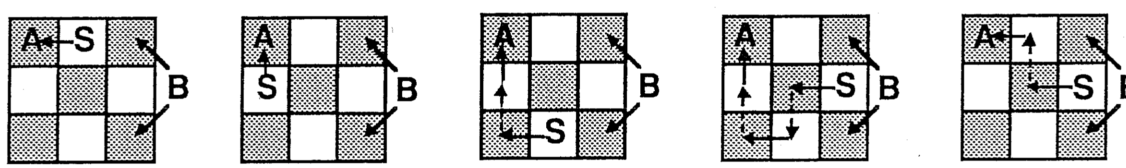
Lemma 4. *On a 3xn board, the following cases result:*

1) If n is any natural number, the goals are diagonally opposite or in opposite corners of the same row, and color(A)≠color(S), then A has a winning strategy.

2) If n is odd, the goals are diagonally opposite, and color(A)=color(S), then B has a winning strategy.

3) If n is even, the goals are diagonally opposite, and color(A)=color(S), then A has a winning strategy.

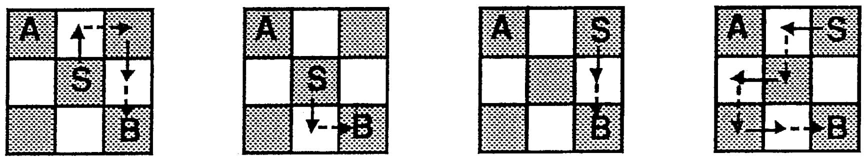
Case 1 Proof: (by Induction) We begin with the 3x3 board base case. When play starts on any white square, A moves left until it can move into its goal. Play never moves to the right of S. Also, notice that it doesn't matter whether B's goal is in the top or bottom corner of the last column.



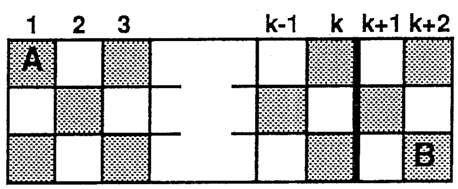
Assume the statement is true when n=k, also noting that play never moves to the right of S. To create a 3x(k+1) board, add an additional column to the 3xk board on the end that contains B's goal.

A will continue to have a winning strategy from the starting squares on the 3xk board because play never moves to the right of S so the added column doesn't affect A's strategy. From the starting square on the k+1 column, A moves right, onto the 3xk board. B's only options are squares on the 3xk board, and from any of these squares A has a winning strategy according to our assumption. Therefore the statement holds.

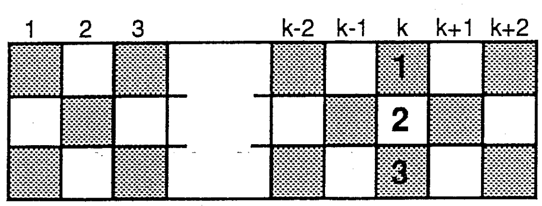
Case 2 Proof: (by Induction) The statement is true for the base case, a 3x3 board.



If S is the center square, then by symmetry B's strategies when A moves either up or left are equivalent. B's strategies when A moves down or right are also equivalent. When S is the right corner square A can move down, towards B's goal, in which case B moves into its goal and wins. Otherwise A can move left from start. B moves down into the center square causing A to move left, away from B's goal. B then moves down into the corner, forcing A to move right, out of the corner, and allowing B to move into its goal. Now assume the statement is true for n=k, where k is odd. Create a 3x(k+2) board by adding two columns to the 3xk board on the end that contains B's goal.



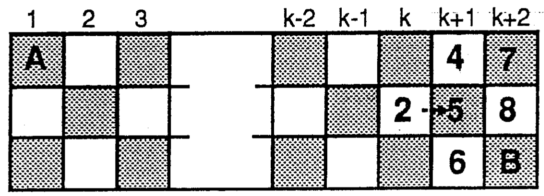
First consider the case when S is on one of the first k columns. By the assumption we know that B can force a play from S to B's goal on a 3xk board. (Note: B's goal on the 3xk board is square 3 on the following diagram.)



However, on a 3x(k+2) board, A can force play off the 3xk board by moving right at square 1 or 3, and B can move off the board at square 2. Therefore, when playing on a

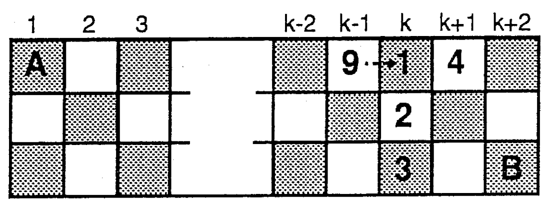
3x(k+2) board where S is on one of the first k columns, B can force a play to either square 1 or 2 or 3.

If B is on square 2, then its strategy is to move right to square 5. From square 5, A has a choice of moves to squares 4, 6 and 8.



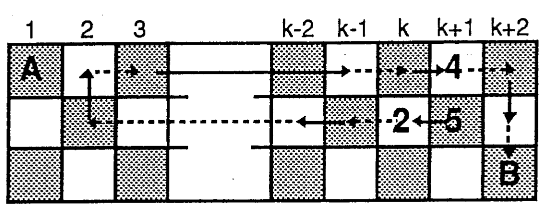
If A moves to either square 6 or 8 then B can move into its goal. If A moves to square 4, then B can move to square 7, forcing A to square 8 from which B can win. Therefore, from square 2, B has a winning strategy.

If A is on square 1, then A has a choice of moves to square 2 or 4 (but not square 9 since B must have moved off of square 9 onto 1. B would not have moved off of square 2 onto 1 since its strategy from square 2 is to move right. B could not have moved off of square 4 since play, up until this point, has remained on the 3xk board).

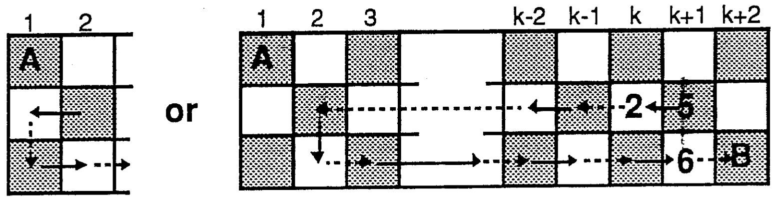


B has a winning strategy from both squares 2 and 4. The same strategy holds when A is on square 3. Therefore, from any S on the first k columns, B has a winning strategy.

To complete the argument, we must show that B has a winning strategy when S is either square 5 or 7. If S is square 5, then A has four possible moves, squares 2, 4, 6, and 8. From squares 4, 6, and 8, B has a winning strategy. If A moves left to square 2, then B's strategy is to continue moving left as long as A does so.

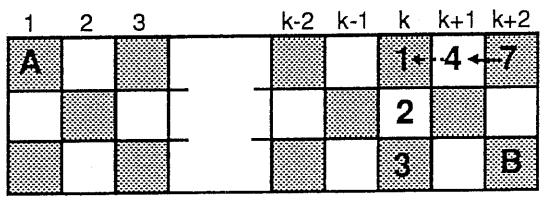


If A moves up before reaching the left edge of the board (as in the above diagram), then B should move right, forcing play toward B's goal. A is ultimately forced to square 4 from which B has a winning strategy.



If A does not move up, but rather hits the left edge of the board, B should move down, forcing A right. However, if A moves down, then B's strategy is to go right, forcing play toward B's goal. A is ultimately forced to square 6 from which B has a winning strategy.

When S is square 7, if A moves down, then B can move into its goal and win. If A moves left to square 4, then B can move left to square 1.

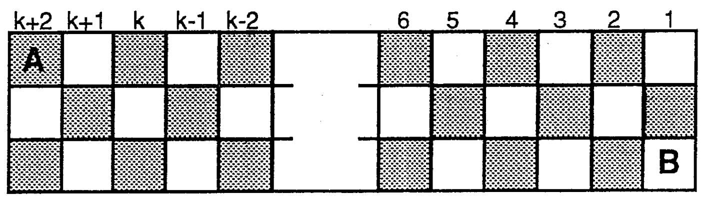


This can be considered a starting square on the 3xk board. From square 1, B can force play to square 3 (B's goal on a 3xk board) or square 2, due to our assumption. As demonstrated before, B has a winning strategy from both squares 2 and 3. Therefore B has a winning strategy from any S on the 3x(k+2) board.

Case 3 Proof: (by Induction) In this case, A will use the following strategies: (1) If S is in the same row as A's goal (row 1 in this case), then A's first move is right. (2) If S is in row 2, then A's first move is up. (3) If S is in the same row as B's goal (row 3 in this case), then A's first move is up. (4) During play, A never moves left while in row 2. This is a stronger hypothesis than is stated in the lemma.

The statement is true for a 3x2 board, our base case.

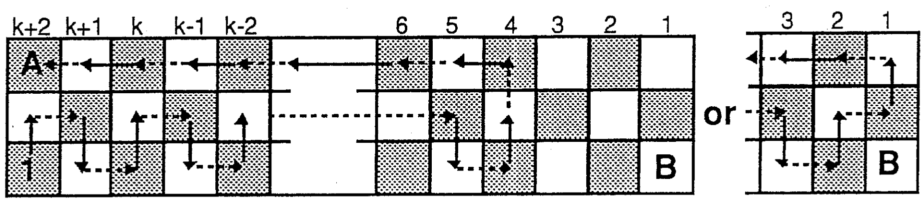
Assume the statement is true for n = k, where k is even. To create a 3x(k+2) board, add two columns to a 3xk board on the end with A's goal.



We now break the proof into two parts:

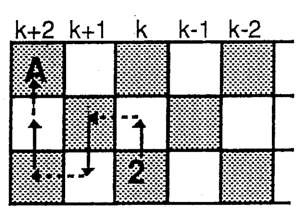
Part 1: S is in one of the last three columns (k, k+1, or k+2).

If S is square 1, then A begins by moving up (strategy 3), forcing B to move right to avoid A's goal. A continues to zigzag up and down between row 2 and row 3 towards B's goal until either (1) B moves up, or (2) B hits the right edge.

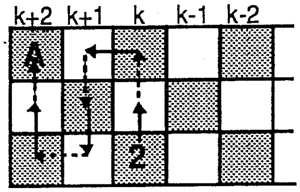


If B moves up, then A moves left and play is force to A's goal. If B hits the right edge, then A moves up, forcing B to move left and play is forced to A's goal. Therefore, A has a winning strategy from square 1.

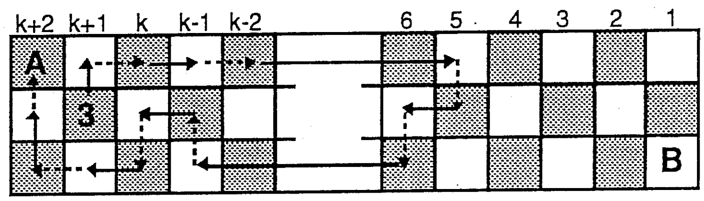
If S is square 2, then A's first move is up (strategy 3). If B moves right, then play proceeds with the same strategy as when S is square 1. If B moves left, then A moves down, forcing B left and A up. B is forced into A's goal.



If B moves up, then A moves left onto a 3x2 board in which A has a winning strategy.

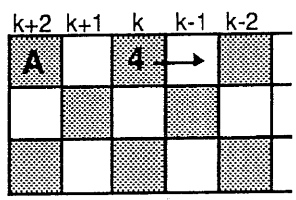


If S is square 3, then A moves up (strategy 2). B moves right to avoid A's goal. A continues right until B moves down (which it may choose to do or will be forced when it hits the right edge). A continues to move left, no matter whether B stays in row 2 or row 3.



Eventually row 2 will be blocked by S so B is forced to row 3 if it is not already there. Now A continues left and play is forced to A's goal.

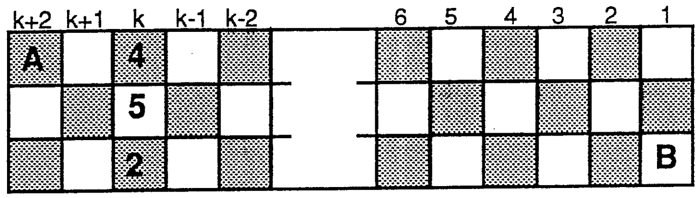
If S is square 4, then A moves right (strategy 1).



A follows the same strategy as when S is square 3. Eventually play is forced to A's base. Therefore, A has a winning strategy from any S in columns k, k+1, and k+2.

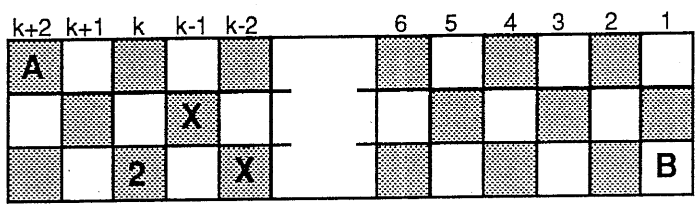
Part 2: S is in one of the first k-1 columns.

On a 3xk board, A can force play to square 4, its home base, by our assumption.. However, on a 3x(k+2) board, A's strategies for the 3xk board will force play to column k but once in that column, play can be forced off the 3xk board. Begin by following A's 3xk board strategies from our assumption until play moves to column k. We must consider the following cases: (1) The first move to column k is to square 2, and (2) The first move to column k is to square 4.

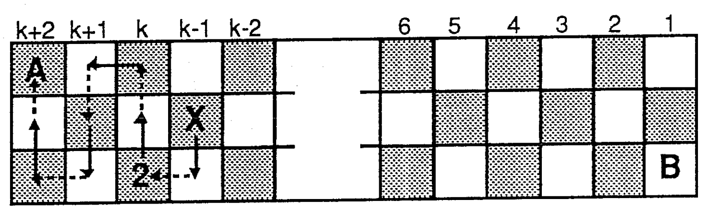


Notice that the first move to column k will never be to square 5 since A would have to move there and A never moves left in row 2 (strategy 4).

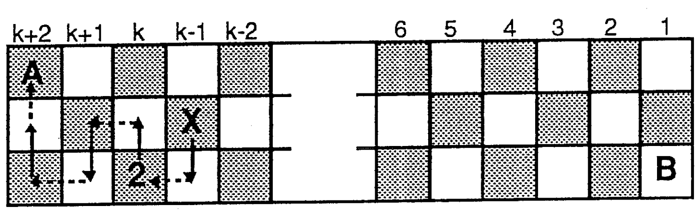
Consider the first case, when B moves left in row 3 onto square 2. In this case one of the squares marked 'X' must be taken since play starts on a black square.



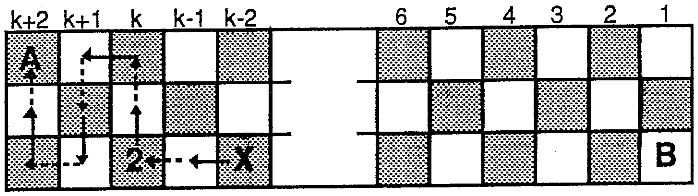
First consider the case when the square in row 2 is taken. A moves up, from square 2 to square 5. Now B has two choices, up or left. If B moves up then A moves left onto a 3x2 board in which A has a winning strategy.



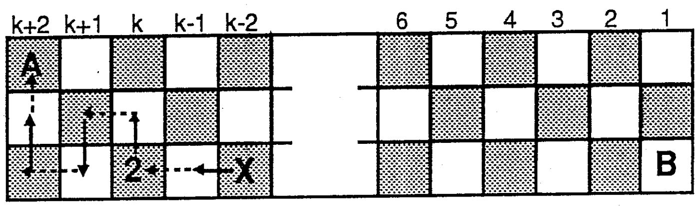
If B moves left then A moves down forcing B left. Play is forced up into A's goal.



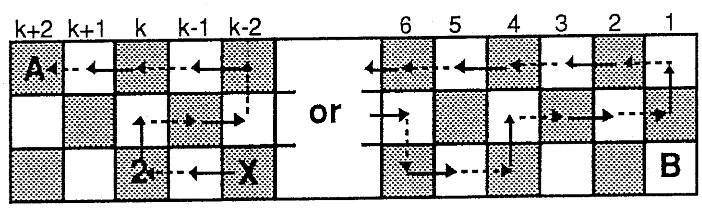
Now go back to the original situation where B moves to square 2 and consider the case where the square in row 3 is taken. A moves up so B has the choice of up, left, or right. If B moves up then A moves left onto a 3x2 board in which A has a winning strategy.



If B moves left then A moves down forcing B left, around the corner, and into A's goal.

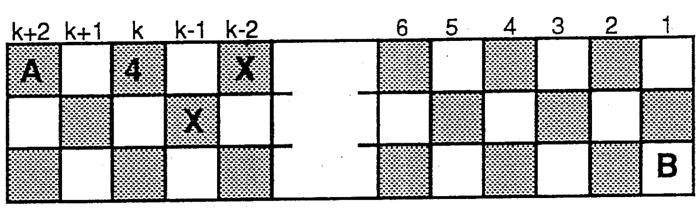


If B moves right then A continues right until either (1) B moves up into row 1, or (2) the black square directly above and right of A is taken (or doesn't exist if play is on the right edge). If (1) occurs, then A moves left and play is forced to A's goal. If (2) occurs, then A moves up and B is forced left, forcing play to A's goal.

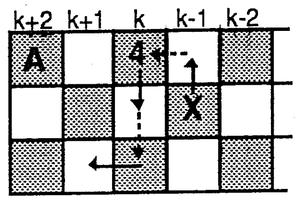


These are the only two possibilities that can occur because A never moves left on row 2 (strategy4). For this reason, either B is blocked in row two and forced up or A is blocked in row 2 and moves up but then B is blocked in row 1 and forced left. In any case, A has a winning strategy.

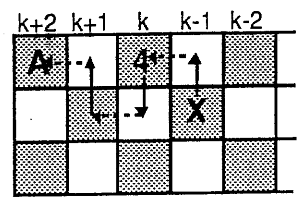
Now, consider the second case when the first move to column k is to square 4. If this occurs, then one of the two squares marked 'X' must be taken.



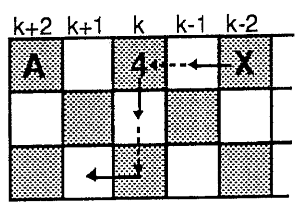
Consider the case when the square in row 2 is taken. A moves down from square 4 to square 5 allowing B to move either left or down. If B moves down, then A moves left onto a 3x2 board in which A has a winning strategy.



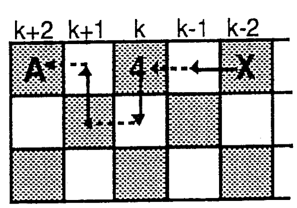
If B moves left then A moves up forcing B into A's goal.



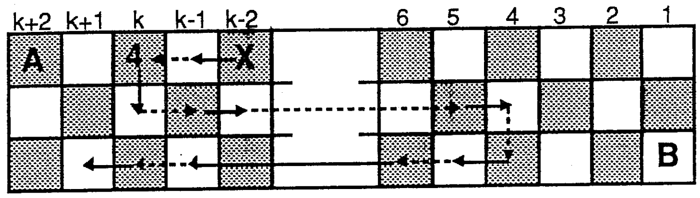
Now go back to the original situation where B is on square 4 and the square in row 1 is taken. A moves down so B has the choice of moving left, right, or down. If B moves down, then A moves right onto a 3x2 board where it has a winning strategy.



If B moves left, then A moves up and B is forced into A's goal.



If B moves right, then A continues right until B moves down or a block is hit.



Row 2 is always blocked for the following reason: Either play started in row 2 or 3 and is eventually forced to row 1 (since the square marked 'X' is taken), or else play started in row 1 but A's strategy in this case is to move right from start. So play must have eventually moved out of row 1 and wrapped back around to eventually hit the square marked 'X'. If B hits a block and moves down (or chooses to move down) then A moves left and forces play back to a 3x2 board where A has a winning strategy. The only time A will hit a block inrow 2 is if the entire column is blocked (since A never moves left in row 2 by strategy 4). Therefore, A is forced down and B is forced left. Play is forced to a 3x2 board where A has a winning strategy. Therefore, we can conclude that in every case, A has a winning strategy.❑

III Strategies

A strategy for the winning player is implied in the proofs of the 1xn, 2xn, and 3xn boards. This strategy is composed of all the correct moves the player should make, given any situation in the game. If the player uses this strategy, then he/she will win regardless of the opponent's strategy. However, it is much too long and tedious a task to determine the correct move in *every* situation of the game. Therefore, it is more helpful to have a simple algorithm which describes the player's strategy.

Conjecture 1. *In a game of Zigzag, if the goals are in diagonally opposite corners then the algorithm for the winning player's strategy is:*

1) *On a* ***1xn board*** *where n is any natural number and S is any square other than a goal, A moves left, towards A's goal.*
2) *On a* ***2xn board*** *where n is any natural number and S is any square other than a goal, A moves left, towards A's goal. If it becomes possible, A moves into A's goal.*
3) *On a* ***3xn board*** *where n is any natural number and color(S)≠color(A), A moves left. When it becomes possible A moves into A's goal.*
4) *On a* ***3xn board*** *where n is odd and color(S)=color(A), B avoids moving into A's goal. B avoids moving onto an edge but stays on the edge once there (except to avoid A's goal). B always moves right or down (unless this contradicts one of the previous heuristics).*

At this point, we have only a lengthy algorithm for a 3xn board where n is even and color(S)=color(A). In outline, this algorithm is: (1) If S is in the same row as A's goal, then A's first move is right. (2) If S is in row 2, then A's first move is up (to the row that contains A's goal). (3) If S is in the same row as B's goal, then A's first move is up to row 2. (4) A never moves left (towards A's goal) in row 2.

IV Further Research

The 4xn boards, and all boards larger than these, have yet to be solved. We have determined that A has a winning strategy on the 4x4 board, which brings up an interesting possibility. A has a winning strategy on the 2x2 and 4x4 boards. It is possible that this pattern continues on all square boards of even length sides or on square boards of side length equal to a power of two.

We can also show that on a 5x5 board, where S is the center square, B has a winning strategy. Recall that this is also true on the 3x3 board. In fact, B has a winning strategy for all 3xn boards where n=3(mod 4) while A has a winning strategy for all 3xn boards where n=1(mod 4) (Lemma 1). This implies the following conjecture.

Conjecture 2: *On a mxn board where m and n are odd and S is the center square:*
If (m+n-2)/2 is odd, then A has a winning strategy.
If (m+n-2)/2 is even, then B has a winning strategy.

Notice that (m+n-2)/2 is the minimun number of moves needed to reach a goal. We have shown that this conjecture holds for the 3xn and 5x5 cases. Conjecture 2 can be expanded to include 3-dimensional boards as well. The minimum number of moves on a mxnxs board is (m+n+s-3)/2. If (m+n+s-3)/2 is odd, then A may have a winning strategy. If (m+n+s-3)/2 is even, then B may have a winning strategy. If we can prove this expanded conjecture for the 3-D case, we will be able to do so for any higher dimension.

There are also some other questions about the analysis of Zigzag which we would like to answer. They include:

* *What is the strategy for each size board? Is there more than one strategy for some of the boards? What are.they? Is there a general algorithm?*
* *If the player with the winning strategy makes a mistake, will it necessarily cost him the game? Are all mistakes equal?*
* *What results when we change the position of the goals?*
* *What if there are more than 2 players and each player has a goal?*
* *What happens when the grid is not rectangular?. What if each point not on an edge has degree d=3 or d≥4? Is the degree of the point related to the results? Can graph theory help solve our problems?*

* *On (even x even) or (odd x even) boards in the 2-D case, does the winning player control his own goal or his opponents, or is such a generalization impossible?*

We plan on doing further research and answering some of these questions.

V Conclusion

We have analyzed the game of Zigzag and proven which player has a winning strategy in the following cases: 1xn, 2xn where the goals are diagonally opposite or in opposite corners of the same row, 3xn where the goals are diagonally opposite or in opposite corners of the same row and color(A)≠color(S), 3xn where the goals are diagonally opposite and color(A)=color(S). We have made progress in analyzing the game of Zigzag in the past few weeks, but much more remains to be done. We plan to continue working on the project and will hopefully discover more about the game. One of the first things we'll do is expand cases 2 and 3 of lemma 4 to account for the goals in opposite corners of the same row. Our goal is to consider as many possibilities as we can and find a simple algorithm, if one exists, for the game.

We would like to thank Gus Waters and Greg Perry for writing computer programs for us. Special thanks to Paul Cull for all of his ideas, advice, help, and encouragement.

References

[1] Bolt, Brian. The Amazing Mathematical Amusement Arcade. New York: Cambridge University Press, 1984, p. 24.

{2] Davis, Morton D. GAME THEORY: A Nontechnical Introduction. New York: Basic Books, Inc., Publishers, 1970.