Colorings of the Plane

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In this paper we discuss two open problems in Euclidean Ramsey Theory:

The first concerns the existence of monochromatic triangles in a two-colored plane. We require only that the colors of the vertices of the triangle be the same. It is known that the plane can be two-colored so that there are no monochromatic equilateral triangles of a particular size. However, it is conjectured in [Erdös et al. 1973 A], the seminal paper on Euclidean Ramsey Theory, that all other kinds of triangles always exist monochromatically under any two-coloring.

The second problem asks for the minimum number of colors, r, such that an r-coloring of the plane exists that contains no two points a unit distance apart of the same color. This value is known to be greater than three, and at most seven.

While we were not able to solve these problems we were able to show:

1. For every two-coloring there exist an uncountable number of sizes of monochromatic equilateral triangles.

2. For every four-coloring there exist an uncountable number of d's such that there exist two point distance d apart of the same color.

3. If there exists a four-coloring that does not contain two points a unit distance apart of the same color then each of the four largest monochromatic sets is totally disconnected.

4. If for a fixed five-coloring, S is the set of distances d such that there exist no two points distance d apart of the same color, then S is totally disconnected.

Additionally, an alternate proof was found to an existing theorem of Woodall that states if the plane is "tiled" by five sets, then one of the sets contains two points a unit distance apart.

We will discuss these problems in order, noting some interesting connections between them.

Section 1

In this section we are concerned with questions of the following type: Is it true that for every two-coloring of the Euclidean Plane, $\mathbb{E}^2$, there exists a monochromatic copy of a given triangle T? If this property holds, then T is said to be Ramsey in $\mathbb{E}^2$ with two colors. The following conjecture was made in 1973.

Conjecture 1: [Erdös et al., 1973 B] *All non-equilateral triangles are Ramsey in* $\mathbb{E}^2$ *with two colors* [The Triangle Problem].

There are several useful results known regarding this problem. The following are the three that we found to be most useful.

Theorem 1: [Erdös et al., 1973 B] *Let K be a triangle with sides a, b, and c. Then for any two-coloring, K exists monochromatically if and only if at least one of the equilateral triangles of side a, b, or c exists monochromatically.*

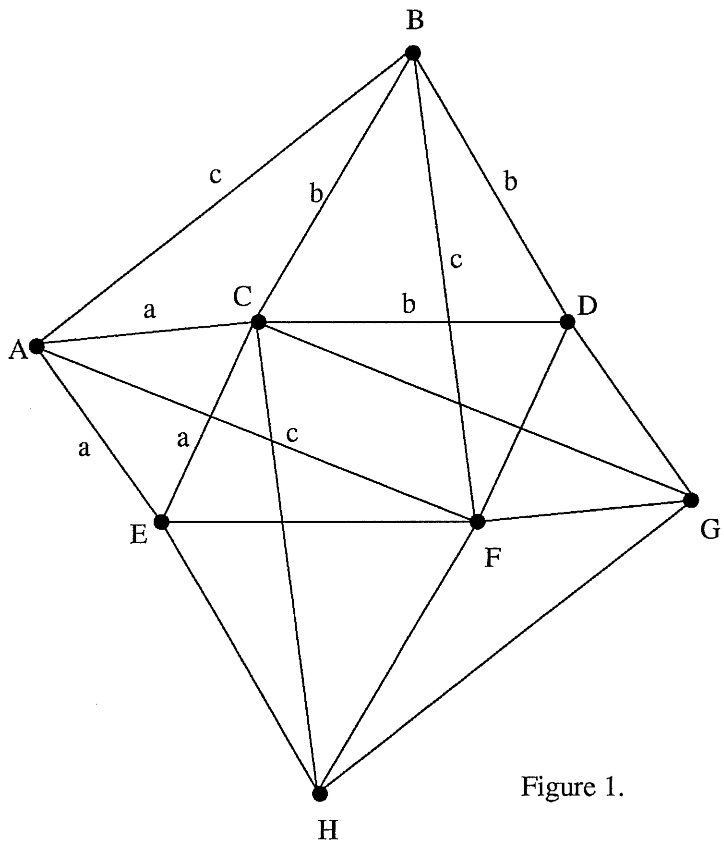


Figure 1.

Proof: Consider Figure 1. Triangles *ABC, AEF, BDF, FHG, HEC,* and *GDC* are congruent copies of each other, containing sides of length a, b, and c. Also, triangles *AEC, GFD, BCD, FHE, ABF, HGC* are all equilateral triangles of side a, a, b, b, c, c respectively. It is easy to see that if one of the triangles of sides a, b, and c is monochromatic, then one of the equilateral triangles must also exist monochromatically. The converse follows from a similar argument, thus proving the assertion.□

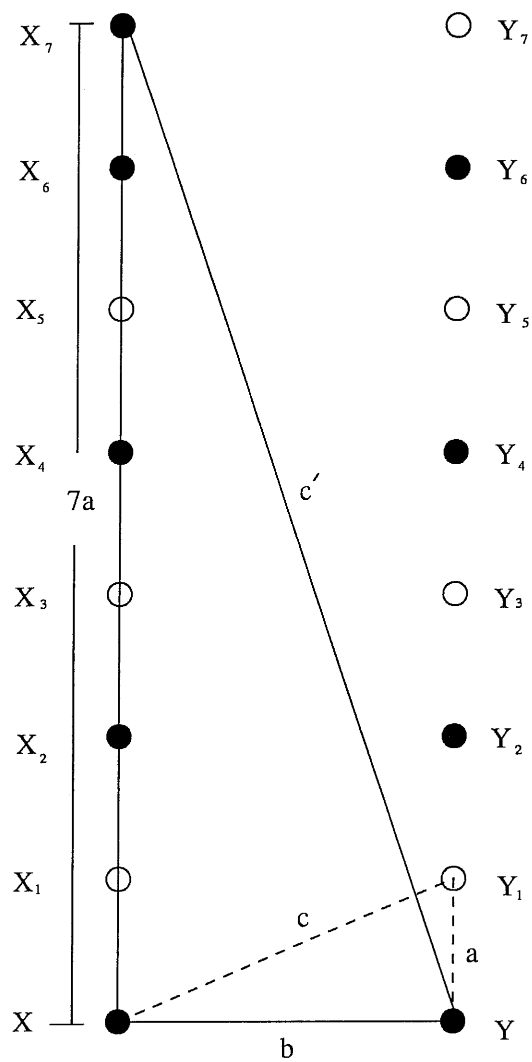
The strongest statement to date was shown in 1976 by Shader. This result relies on the following Lemma, the proof of which can be found in [Shader, 1976].

Lemma 1: For every $a \in \mathbb{R}$ and for any two-coloring of the plane, there exists a monochromatic equilateral triangle of side $a, 3a, 5a$, or $7a$.

Theorem 2: *All right triangles are Ramsey in* $\mathbb{E}^2$ *with two colors.*

Proof: Consider any triangle with sides a, b, and c such that $a^2 + b^2 = c^2$. By Lemma 1, there exists a monochromatic equilateral triangle of side $a, 3a, 5a$ or $7a$. If a monochromatic equilateral triangle of side a exists, then we are done since Theorem 1 would guarantee the monochromatic existence of any triangle containing a side of length a. Therefore, three more cases need to be accounted for.

Case 1: Suppose that an equilateral triangle of side $7a$ exists monochromatically. Therefore, by Theorem 1, there exists a monochromatic right triangle with sides $7a, b, c'$ where $(7a)^2 + b^2 = (c')^2$.



The above figure represents this monochromatic right triangle where the points X, Y, and X_7 are all the same color, say blue. Now, in order to avoid a monochromatic right triangle of sides a, b, c the points X_1 and Y_1 must both be colored red. Similarly, the points X_2 and Y_2 must both be blue, etc. However, this will force the points X_6, Y_6 and X_7 to be blue thus creating a monochromatic right triangle of sides a, b, and c. The other two cases are similar, hence proving the assertion.□

If one follows an analogous argument and considers the skew lattice determined by a parallelogram P, the following result is immediate.

Theorem 3:[Shader, 1976] *For every two-coloring of $\mathbb{E}^2$ and every parallelogram P, there exists a congruent copy of P, say P', such that P' has three vertices of the same color.*

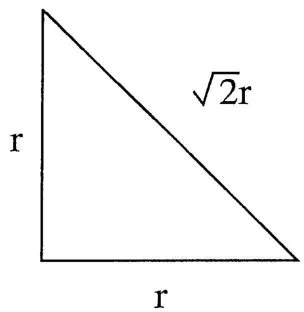
Given these three results, there are several alternate problems that immediately imply the Triangle Problem. Consider another conjecture proposed in 1973.

Conjecture 2:[Erdös et al., 1973 B] *If $\mathbb{E}^2$ is two-colored so that no monochromatic equilateral triangles of unit length exist, then all other sizes of equilateral triangles exist monochromatically under that coloring.*

The proof that Conjecture 2 implies the Triangle Problem is immediate in light of Theorem 1. The following theorem is as strong a statement as we have been able to show in support of Conjecture 2.

Theorem 4:[Brown] *For every two coloring of* $\mathbb{E}^2$*, there exist an uncountable number of values of* r*, where* $r \in \mathbb{R}^+$*, such that an equilateral triangle of side* r *exists monochromatically.*

Proof: Consider the following right triangle.



We know that this triangle is Ramsey in $\mathbb{E}^2$ for every value of $r \in \mathbb{R}^+$. Now consider the multiplicative subgroup of $\mathbb{R}^+$, call it M, that is generated by $\sqrt{2}$. Thus, $M = \{2^{I/2} | I \in \mathbb{Z}\}$. Therefore the cosets of this subgroup will partition $\mathbb{R}^+$ into equivalence classes under the following equivalence relation.

$$X \text{ is related to } Y \Longleftrightarrow \frac{X}{Y} \in M$$

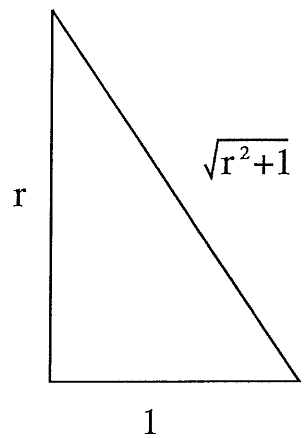
Thus, the equivalence class of 1 is $\{...1/2, \sqrt{2}/2, 1, \sqrt{2}, 2...\}$. Now in light of the previous right triangle and Theorem 1, it is easy to see that at least every other element of each equivalence class must have a monochromatic equilateral triangle associated with it. Therefore, at least "half" (using half very loosely) of each equivalence class has an associated monochromatic equilateral triangle. Furthermore, $\mathbb{R}^+$ is equal to the uncountable union of these countable sets. Therefore, at least "half" of the values in $\mathbb{R}^+$ must also have associated monochromatic equilateral triangles. Although this is far from a rigorous definition, the fact that an uncountable number of values in $\mathbb{R}^+$ describe monochromatic equilateral triangles is obvious.□

The following statements, if shown, would immediately imply Conj. 2. However, they remain elusive and as yet unanswered.

S1: For any right triangle with sides a, b, c, two of the sides have have associated monochromatic equilateral triangles. By Theorems 1 and 2, it is known that a monochromatic equilateral triangle of either side a, b, or c exists.

S2: All isosceles (non-equilateral) triangles with at least one side equal to a unit are Ramsey in $\mathbb{E}^2$.

Proof that S1 $\Rightarrow$ Conj. 2: Consider the following right triangle.



Since all right triangles are Ramsey, **S1** would force monochromatic equilateral triangles of two of the sides 1, r, or $\sqrt{r^2+1}$. Therefore, if no unit equilateral triangles exist monochromatically then Conj. 2 directly follows from Theorem 1.□

The proof that **S2** $\Rightarrow$ **Conj. 2** is even more immediate. Again, if we assume that no unit monochromatic equilateral triangles exist, then Conj. 2 must immediately follow. It is also worth noting that since the Triangle Problem implies S2, it is actually equivalent to S2 and Conj. 2 as well.

There is one other Conjecture made in [Erdös et al. 1973 B] that also implies Conjecture 2 and is as follows.

Conjecture 3: [Erdös et al., 1973 B] The only way to two-color $\mathbb{E}^2$ such that no unit equilateral triangles exist monochromatically is with strips of alternating color having width $\sqrt{3}/2$, allowing some freedom in coloring the edges.

It is trivial to show that this coloring only prohibits monochromatic equilateral triangles of unit length thus implying Conj. 2. We were unsuccessful in making any progress on this problem and mention it only because it would solve not only the Triangle Problem but could also be used to show the infamous Four-color problem, the proof of which we reserve until the next section. Of course, it would be very nice to solve both of these problems with a single result. However, this final conjecture seems to be the most difficult to prove. Never the less, due to it's application, it is worthy of consideration.

Section 2

In this section we consider the following problem: If one colors the plane with r colors, do there always exist two points a unit distance apart of the same color? We make a formal definition of this problem:

Definition *An r-coloring of a set S is a function $\chi : S \to \{1, 2, 3, ..., r\}$. An r-coloring is proper if it is onto.*

Definition *The chromatic number of the plane, χ, is the smallest number of colors, r, so that the Euclidian plane, $\mathbb{E}^2$, can be colored with r colors so that there do not exist two points a unit distance apart of the same color.*

The best bounds on χ to date are:

$$3 < \chi \leq 7,$$

and no improvement on these bounds has been made in more than 40 years (for a complete historical discussion of this problem see [Soifer 1991]).

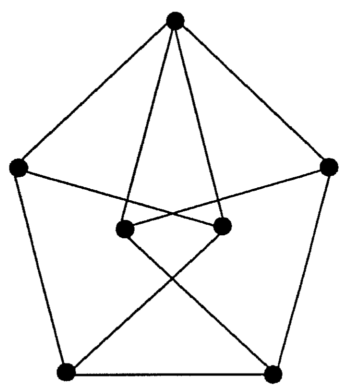


Figure 2

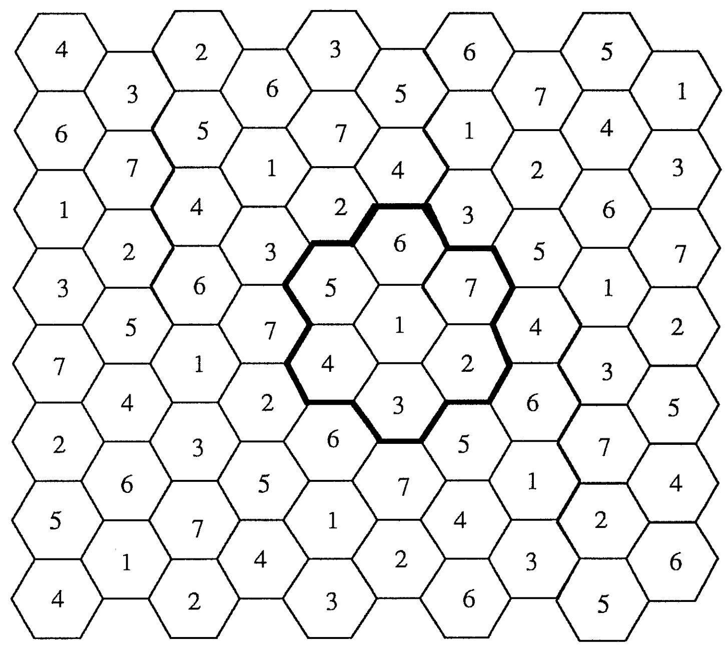


Figure 3

To show that for any two-coloring of the plane there exist two points a unit distance apart of the same color, simply consider a unit equilateral triangle. To prove the same for $r = 3$, we need a slightly more complicated figure (see fig. 2). This establishes the lower bound $3 < \chi$. To prove the upper bound, tile the plane with regular hexagons of diameter $9/10$ and color as in fig. 3. In trying to increase the lower bound, one wishes to know if, for some r there always exist two points a unit distance apart of the same color, then there is a proof of this like the ones given for $r = 2, 3$? The answer, as we shall see, is yes. To do this we need a theorem about hypergraphs:

Definition *A hypergraph, $H = (V, E)$, consists of a vertex set, V, and a set of edges, E. An edge $X \in V$ is a arbitrary collection of points $x \in V$.*

Definition *A hypergraph $H = (V, E)$ is r-colorable if the vertices, V, can be colored so that there does not exist an edge $X \in E$ that is monochromatic, i.e. all the $v \in X$ are the same color.*

Definition *The chromatic number of a hypergraph H, denoted $\chi(H)$, is the minimum number of colors, r, so that H is r-colorable. Thus if $\chi(H) = k$ then H is k-colorable but not $k-1$-colorable.*

Definition *Let $H = (V, E)$ be a hypergraph and let $W \subset V$. The restriction of H to W, denoted H_W, is defined as $H_W = (W, E')$ where*

$$E' = \{X \in E : X \subset W\}.$$

The following theorem proves very useful in Ramsey theory.

Theorem 5 *(Compactness principle) Let H be a hypergraph with every $X \in V(H)$ finite. Then if*

$$\chi(H) > r$$

there exists a finite subset $W \subset V(H)$ such that the restriction of H to W has the following property:

$$\chi(H_W) > r$$

Proof See, for example, [Graham et al. 1990]. If V is uncountable, the proof uses the Axiom of Choice.

Applying this, we see that if for every r-coloring of $\mathbb{E}^2$ there always exist two points a unit distance apart of the same color, then there is a finite subset of $\mathbb{E}^2$ on which this is also true. Indeed, take $H = (V, E)$, and let $V = \mathbb{E}^2$ and

$$E = \{(a, b) : a, b \in \mathbb{E}^2 \textit{ and } \|a - b\| = 1\}.$$

Then if for some number of colors, r, there always exist two points a unit distance apart of the same color, we are guaranteed a proof like the one for three colors. Thus the chromatic number of the plane is equal to the maximum chromatic number of a graph that can be embedded in the plane with unit length sticks for edges. From fig. 3 we see that no graph that can be embedded in the plane with unit length sticks has a chromatic number greater than 7.

Assume that for some r, there always exist two points a unit distance apart of the same color. Then what does the graph that can be used to prove this look like? First, we know that the chromatic number of the graph, χ, is greater than or equal to $r + 1$ (the chromatic number of a graph is defined in the same way as the chromatic number of the hypergraph). We need only consider $r + 1$-critical graphs, those with the property that, if one of the vertices is removed, the chromatic number drops below $r + 1$. These graphs have the following properties:

(i) Every vertex of an $r + 1$-critical graph has degree at least r.

(ii) If e is the number of edges and v is the number of vertices, then if G is a $r+1$-critical graph

$$2e \geq vr + 1$$

(iii) A connected graph is k-connected if it is necessary to remove at least k edges to break the graph into 2 components. Every $r + 1$-critical graph is $r - 1$-connected.

Since we are searching for graphs with chromatic numbers greater than five we need only consider non-planar graphs. A graph is non-planar if it cannot be embedded in the plane with no edges crossing (note that the requirement that the edges be unit length sticks is dropped here, thus the proof for $r = 3$ is planar). The four-color map theorem tells us that every planar graph is four-colorable and thus has chromatic number less than five. A subgraph of a graph $G = (V, E)$ is a subset of the vertices together with a subset of the edges. A subdivision of a graph G is created by placing new vertices in the middle of the edges of G. It is known that every non-planar graph contains a subgraph that is a subdivision of $K_{3,3}$, or a subgraph that is a subdivision of K_5. It is also known that any five-chromatic graph can be contracted to K_5 by replacing adjacent vertices with a single vertex. We note that it is possible to construct with sticks a subdivision of K_5, although not K_5 itself. In [Wormald 1973] Wormald manages to construct a graph that can be embedded in the plane with unit stick edges with a chromatic number of 4 and no cycles smaller than 5.

It is possible to turn the problem of whether for any four-coloring of the plane there always exist two points a unit distance apart of the same color into a question about more complicated properties of two-colorings.

Definition *For a four-coloring χ of $\mathbb{E}^2$, we define a standard derived two-coloring to be a χ' such that:*

$$\chi'(x) = \begin{cases} red, & if\ \chi(x) = c_1\ or\ c_2; \\ blue, & if\ \chi(x) = c_3\ or\ c_4. \end{cases}$$

where the c_k's are the four distinct colors in the domain of χ.

Definition *An odd cycle of size r is a sequence of points $x_1, x_2, ..., x_{2n+1}$ such that $||x_i - x_{i+1}|| = r$ for all $1 \le i \le 2n$, and $||x_1 - x_{2n+1}|| = r$.*

Theorem 6[Dunfield] *Let χ be a fixed four-coloring of $\mathbb{E}^2$. Then there exist two points a unit distance apart of the same color if and only if under one of the standard two-colorings of $\mathbb{E}^2$ there exists a unit monochromatic odd cycle.*

Proof. ($\Rightarrow$) Let a and b be two points a unit distance apart of the same color, and let c be a point forming an equilateral triangle with a and b. Whatever the color of c, under one of the standard two colorings the triangle (a, b, c), an odd cycle, is monochromatic.

($\Leftarrow$) Let X be a odd cycle $x_1, x_2, ..., x_{2n+1}$ that is monochromatic under one of the two-colorings. Then in the four-coloring, the color of any of the x_k's is one of two colors, say 1,2. Then if $\chi(x_1) = 1$ then, if we assume there are no two points a unit distance apart of the same color, $\chi(x_2) = 2$. Continuing, we find $\chi(x_k) = 1$ if k is odd and 2 if k is even. But $||x_1 - x_{2n+1}|| = 1$ and as both are odd, $\chi(x_1) = \chi(x_{2n+1})$. □

Concerning the problem for $r = 5, 6$ we see that we can turn those questions into the analogous question about odd cycles in three colorings. Recall Conjecture 3: the only way to two-color the plane so that there are no monochromatic equilateral triangles of unit side is to color the plane with alternating half open-bands of width $\sqrt{3}/2$ (with some freedom in coloring the boundary). If this conjecture is true, it implies the four-color problem. Consider one of the standard two-colorings. If there are no points a unit distance apart under the four-coloring there must be no unit monochromatic equilateral triangles under the two-coloring. Assuming the conjecture, the two-coloring is the pattern of stripes. We see, however that this coloring contains other unit odd cycles.

In some sense, any four-coloring which generates stripes for the standard two coloring is much too nice, and the same is true for any five-coloring, since any coloring which generates the stripes as two-colorings was made up of connected regions. As we shall see, no five-coloring that satisfies the unit distance condition can be composed of regions, and the four sets of a suitable four-coloring can have no nontrivial connected components, let alone be composed of regions.

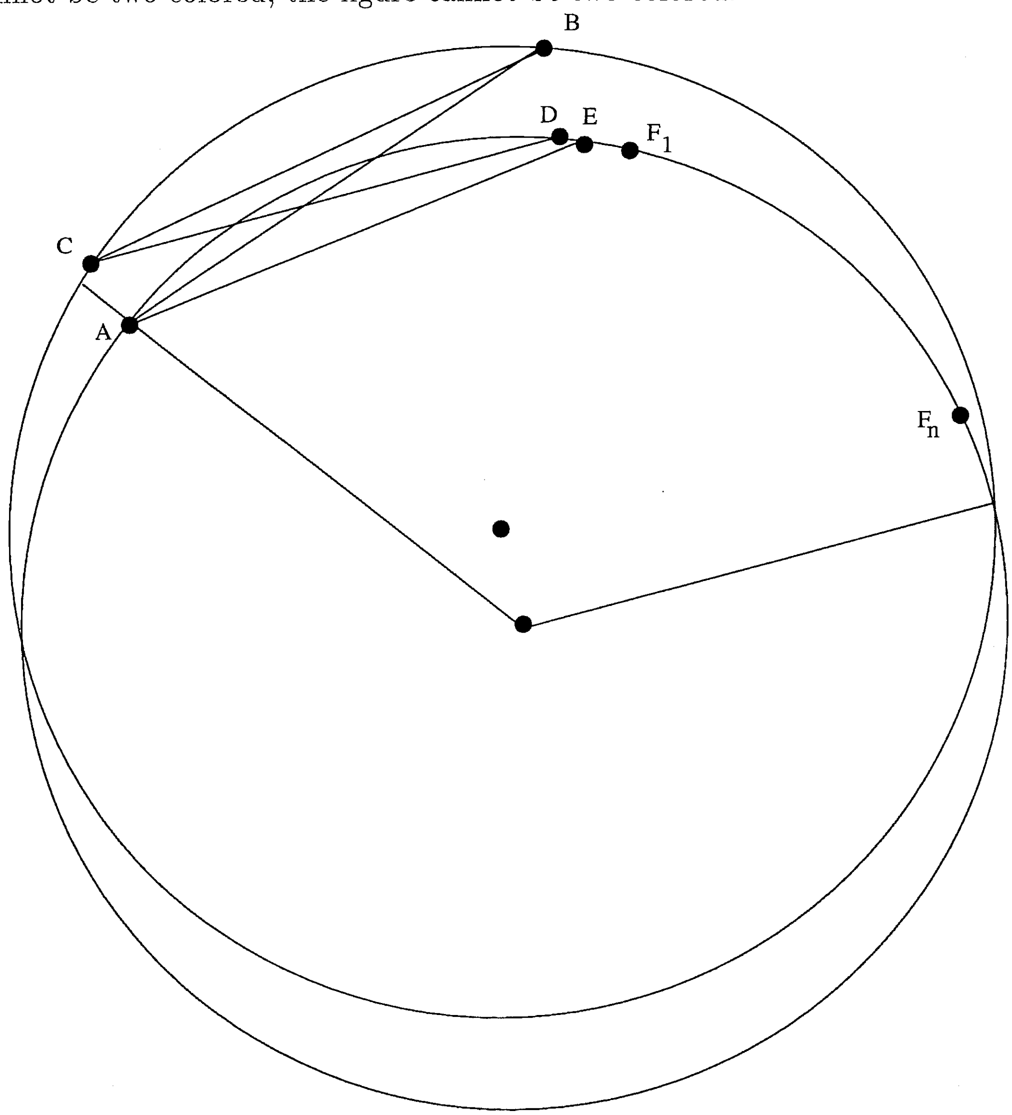
Theorem 7 [Perry]

If the plane is partitioned into four sets such that no two points distance one apart are in the same set, each of the four sets must be totally disconnected.

Lemma 1

The figure obtained by taking the union of 2 unit circles about x, y, $d(x,y) < 1/2$, and taking the intersection of that figure with a closed angle of at least $2\pi/3$, centered at one point and containing the other, such that one defining ray goes through the intersection of the two circles, cannot be two colored without 2 points the same color distance one apart.

Essentially, two intersecting arcs cannot be two colored, if the arcs are long enough. The proof consists of constructing an odd cycle that is contained in the figure, and since the odd cycle cannot be two colored, the figure cannot be two colored.



Proof

Pick point a where the angle intersects the inner arc. Pick b distance one away from a on the outer arc, c distance one from b on the outer arc, d distance one away from c on the inner arc, and e distance one from a on the inner arc. This constructs a path of length four from d to e, with e slightly to the right of d. A similar construction builds a path of length four from point e to point f_1, with f_1 slightly to the right of e. Continue the construction of points f_i until f_n is more than distance one away from point d. This constructs a path of length $4n+4$ on the circles, with first and last points more than distance one apart. Do the same construction of $4n+4$ segments for all points on the inner arc. Consider the distance between first and last points to be a function of the starting point. This function is zero when evaluated at the intersection of the arcs, and more than one when evaluated at a, so for some point, the function takes the value one, and a path of $4n+4$ exists on the circles with endpoints unit distance apart. This defines a $4n+5$ cycle, which is all we need.□

Lemma 2

If a neighborhood has at least countably many points of the same color, no two of which are within epsilon of a given angle, then the points lie along a path.

Proof

Pick any point. The set of all points at an angle within epsilon of theta from that point consists of two wedges. Neither wedge can contain a point of the same color, say blue. Repeat the construction on each of the two areas that the wedge divides the neighborhood into, and so on recursively.

As we continue the construction, choosing points somewhat close to the center of each region if possible, the area in which blue points are allowed becomes smaller, until it contracts to a path in the limiting case.□

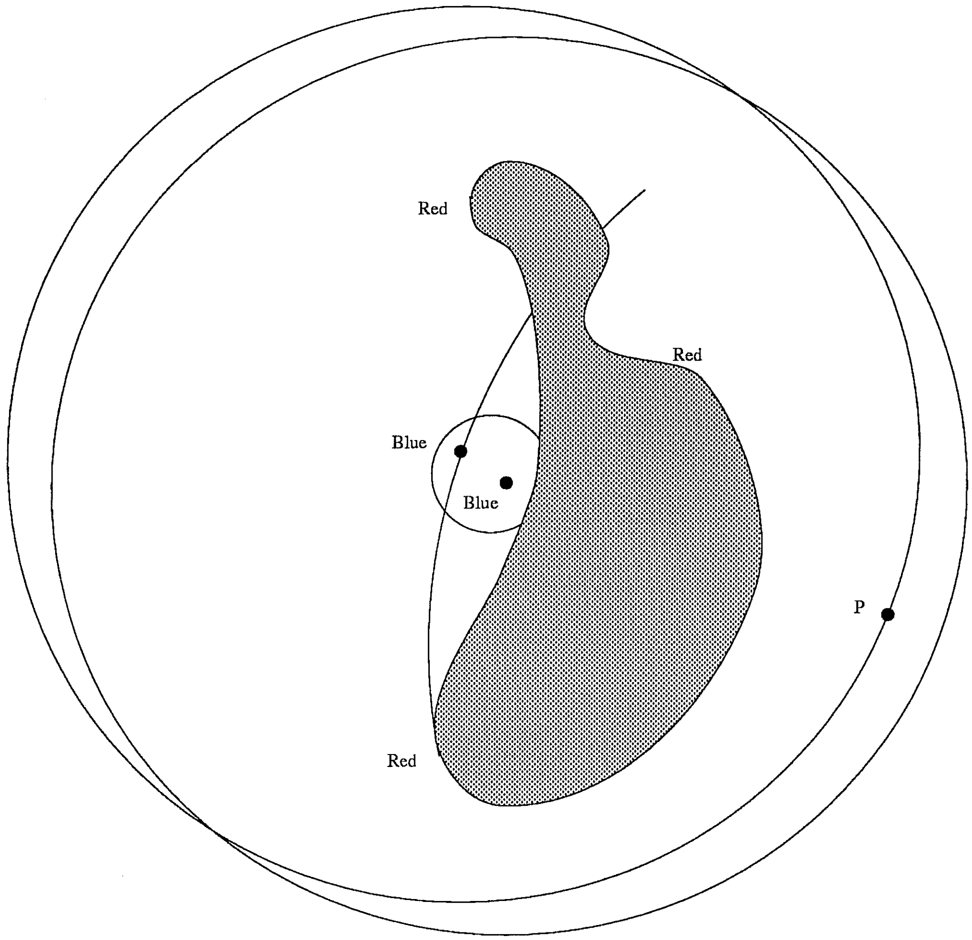
Lemma 3

In any four coloring of the plane subject to the unit distance restriction, the nontrivial connected components U of each color are subsets of convex sets Y such that $Y - U$ is a countable number of discrete points.

Proof

For any two points w, z in U, a red component, consider a neighborhood between them. For a suitable range of angles, that neighborhood contains no two points which are blue and that angle apart from vertical. If it did, the set of points distance one from either point, distance less than one from w and distance greater than one from z would form two intersecting arcs. Any point on the arcs cannot be blue, since it is one away from a blue point, and it cannot be red, since if it is not one away from a red point, the circle of radius one about the point on the arc would divide the component into two open sets, despite connectivity.

The set U can be covered by countably many such neighborhoods, except for the border. each neighborhood has non-red points only along one path. Define Y to be the intersection of all convex sets containing U. The non red points in Y are along countably many paths, each of which has slope within some bounded range.

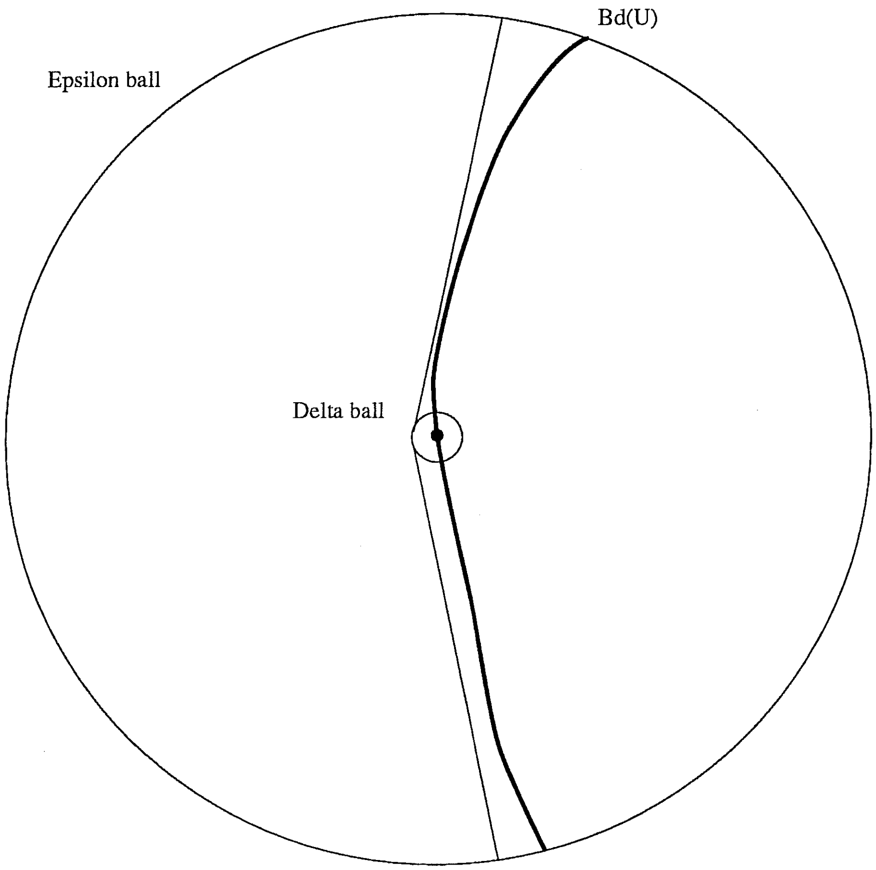


For any two points of the same color in $Y - U$ which are close together, we can pick two red points such that the two points lie in a neighborhood between them. (The planar measure of any subset of a path is zero, and measure is countably additive, so the measure of the set of non red points is zero. Two small neighborhoods can be defined such that if a red point is chosen from either neighborhood, we have two blue points at an appropriate angle in a neighborhood between two red points; which cannot be, since that defines two arcs which must be two colored. Then each neighborhood has only non red points. But each neighborhood has positive measure, and thus cannot consist entirely of non red points.)

Then, there is a small ball around each blue point with no other blue points in that ball. Countably many such balls cover Y, so Y contains only countably many blue points. the same is true for the other two non red colors, and then only countably many non red points are contained in Y. □

Proof of Theorem

Pick any neighborhood V, radius delta, about a point on the boundary of Y. That neighborhood has an open subneighborhood V' of non red points, since the set of non red points cannot be sandwiched between two red regions, due to the construction with 2 arcs. If that region lies between two red points, each epsilon away, then we can choose two non red points of the same color, at an angle which generates the figure with two arcs. Since we cannot two color this figure, V' cannot lie between two red points. Then the boundary of the red region must curve away at least one degree within a length of 2ϵ so that the subregion is not between red points.



The above construction also implies that the boundary must curve one degree within $\epsilon/2$, by doing the same argument with a $\delta/4$ ball and an $\epsilon/4$ ball. By applying the argument to smaller neighborhoods, it is shown that the boundary must curve one degree within any size neighborhood. By placing 360 such *epsilon* balls around the boundary, we can cover the boundary. By the triangle inequality, the diameter of Y is at most 360 *epsilon*, for any *epsilon*. Then the diameter of Y is not greater than zero, and Y has only a single point. □

Until now, we have been concerned with colorings of the plane, that is, dividing the plane into r disjoint sets. If we relax the restriction that the r sets be disjoint, and restrict the type of sets, some other results have been obtained. Croft shows in [Croft 1967] that if a measurable set contains no points a unit distance apart then the density $\delta \leq 2/7 \approx .2857$. Croft found a set with density $\delta \approx 0.2293$ that contains no unit distance. It has been shown by Woodall in [Woodall 1973] that if one covers the plane with five closed sets, then one of the sets contains two points a unit distance apart. Woodall also considered coverings of the plane by r sets simultaneously divisible into regions. We define a map, M, to be a graph embedded in the plane with no edges crossing. A set S is divisible into a map M if S is the union some of the regions defined by M and points on the boundry of those regions. A group of sets is simultaneously divisible into regions if there exists a map into which every set in the group is divisible. Woodall showed that if for any r simultanously divisible sets then there exist two points a unit apart in one of the sets, then that same result is also true for any covering of the plane by r closed sets. This was used to prove that if one covers the plane with five closed sets then one of the sets contains two points a unit distance apart.

The following result was originally published in 1973, using a sequence of steps to reduce the problem in terms of order of vertices, number of colors at each, presence of arcs of positive thickness, direction that each edge leaves a vertex, angles at vertices, etc. The following proof uses fewer restrictions on the sets, and I spent enough time on it that i thought i ought to include it anyway.

Theorem 8 [Woodall, 1973, this proof by Perry]

The plane cannot be divided into five sets, each of which is simultaneously divisible into regions with map M, such that no two points are in the same set and distance one apart.

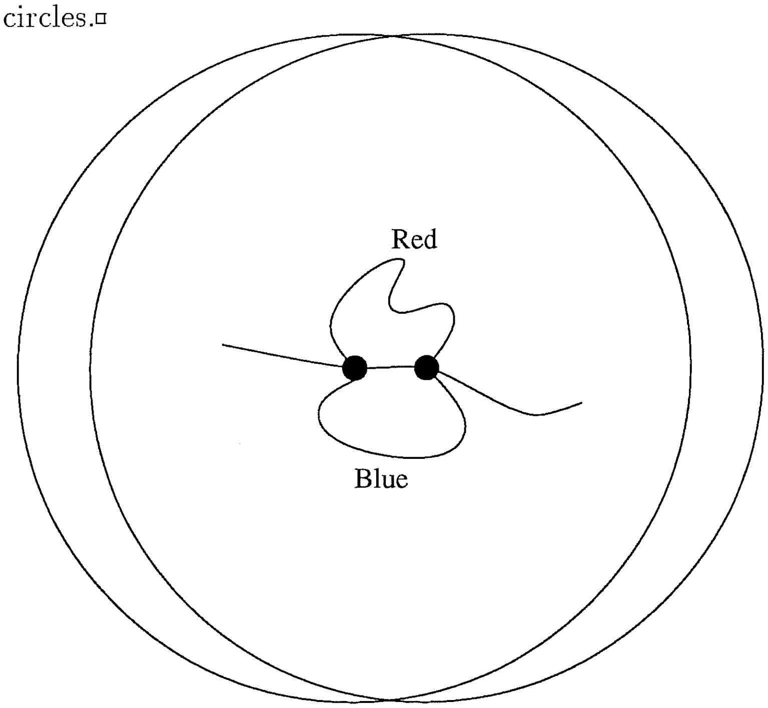
Without loss of generality, we may assume that the sets are disjoint, since any five sets which cover the pane by intersecting can be made into five which do not intersect by ordering the five sets and removing the intersection of two sets from the set which is ordered lower. The problem is now equivalent to the problem of coloring each of the regions defined by a planar map, subject to the rstriction that no two points distance one apart are the same color.

Lemma

In any coloring of the regions defined by a planar map, two regions of different colors which share a common edge define a set that contains no points of either color.

Proof

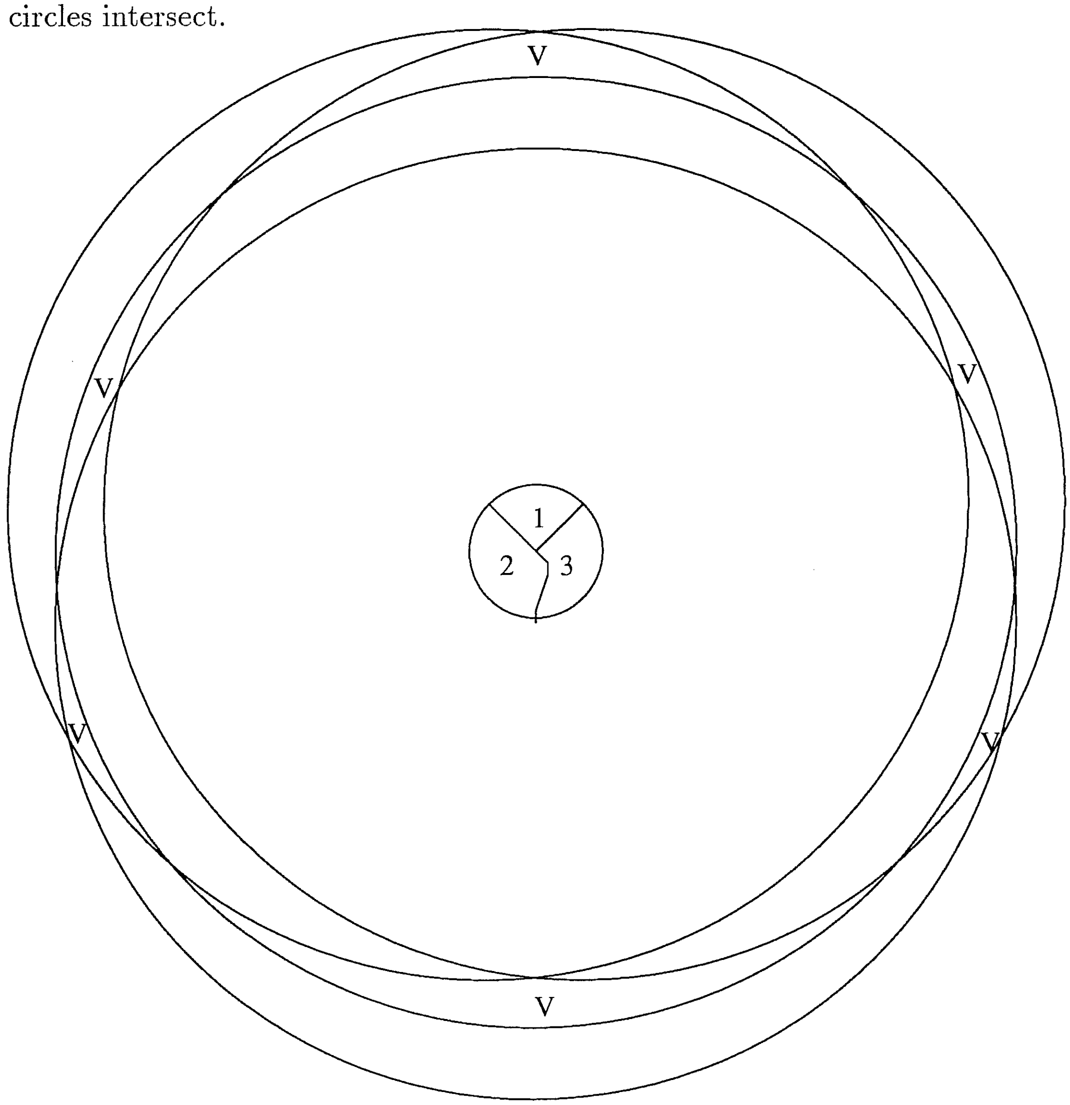
Consider two points on the shared edge, a, b. draw two paths between a and b, each contained entirely within one of the regions. Any point distance less than one away from a and distance more than one away from b is distance one away from a point on each path, by continuity. The same is true if we consider any point more than one from a and less than one from b. The set of all such points is the ring of two crescents that lies between two intersecting unit circles.□



Proof of Theorem

There exists some vertex of degree at least three, bordering on regions of at least 3 distinct colors. (The proof is similar, and easier for more than 3 colors, so we consider only the case of 3 colors and 3 regions.) Draw a small ball around the vertex, so that each edge intersects the border of the ball. The 3 edges define 3 rings of 2 crescents, such that one ring has no points of color 1 or 2, one ring has none of 2 or 3, and one ring has none of 1 or 3. Each ring borders on the circle of radius one about the original vertex.

The intersection of any two rings cannot contain colors 1, 2, or 3. The set V defined by the union of all pairwise intersections of crescent rings thus cannot contain any of the three colors either. Since each crescent is either inside or outside the circle at a given point on the circle (except where the circles cross), either two crescents are inside or two crescents are outside at all points, and V extends completely around the circle, excluding the six points where circles intersect.



Any figure contained in V must then be colored with only the two remaining colors. Since odd cycles have chromatic number three, it is sufficient to show that an odd cycle with unit length edges is contained in V. Since that odd cycle cannot be two colored, V cannot be two colored, and the plane cannot be five colored under the division into regions.

Pick a point in V. Pick another point one unit away, in v, going clockwise. Keep doing this until you have gone about 2π radians, that is seven points or six edges. By choosing points to be close to the circle about the vertex when inside, and far from the circle when outside, it can be guaranteed that less than 2π radians have been covered. Since V is open, we can rotate the entire graph to ensure that we avoid the six points where circles intersect and V has gaps. Continue the construction around the circle until the first and last points are at least distance one apart. (If we the construction converges to a smaller distance, then none of the points outside the circle can be pulled out very far, so all three must be near the intersections of circles. Then the angles between the points of intersection must all be $2\pi/3$, and we can avoid that case by moving one of the points on the border to move one of the crescents so that the angles are more convenient.)

Once we have a path of length 6n, with endpoints at least one apart, we can move the last six points toward the circle about the original vertex, keeping unit distances, until the distance between the first and last point is one. This completes a 6n+1 cycle, which completes the proof, assuming a vertex of degree three.

If there is a vertex of degree four or five, with exactly three differently colored regions, we can construct a figure that contains V, by considering the same construction on the four or five edges. If the vertex has degree five, with four colors, either two colors border, in which case we can consider it as though it were four regions, or no two colors border, and the colors are in the order ABCAD. That give us borders of AB, BC, and AC as required. If four colors and four regions, then pick a point on each border, and the vertex, and join these points by the twelve monochromatic paths possible (two each along each edge, and four going from one edge to another). Applying the crescent construction to each path generates at least one region that has none of three colors, and is large enough to contain two intersecting arcs, which then must be two colored. This is shown to be impossible in the proof that a four coloring must be totally disconnected. Then no vertices of degree 3, 4, or 5 exist, which is a contradiction since the graph is planar. □

If would be nice to show that for a fixed four-coloring there is at most one value r so that there do not exist two points r distance apart of the same color. By theorem 4, we see that there are an uncountable number of values of r for which there exist two points r distance apart of the same color. Consider Conjecture 1: For any two-coloring of the plane there is at most one equilateral triangle that does not occur monochromatically. By using theorem 6, we see that this would imply that, for a fixed four-coloring there is at most one r for which there do not exist two points a unit distance apart of the same color. The authors of [Erdös et al. 1973 B] were not successful in proving this conjecture (it remains an open problem). They were, however, able to prove that for a fixed two-coloring of the plane, the set in $\mathbb{R}^3$ of triangles (associate with each point $(a, b, c) \in \mathbb{R}^3$ the triangle with sides a, b, and c, provided that such a triangle exists) that do not occur monochromatically in the two-coloring is totally disconnected. In view of theorem 6 it seemed likely that this was true for the set of distances such that there were no points r distance apart of the same color under a fixed four-coloring. This is indeed the case. Indeed, this result is true for a five-coloring.

Theorem 9[Dunfield] *Let χ be a fixed five-coloring of $\mathbb{E}^2$ and the set of all values of r*

such that there exist no points r distance apart of the same color is,

$$S = \{x \in \mathbb{R} : If\ \|a - b\| = x\ then\ \chi(a) \neq \chi(b)\}$$

Then S is totally disconnected.

Proof. Assume S had a connected component K and we prove by contradiction. Then K contains a non-trivial interval $[\alpha, \beta]$. Let $r = (\alpha + \beta)/2$ and $\epsilon = (\beta - \alpha)/4$. Create a closed covering of the plane by five sets as follows: Tile the plane with closed squares of diameter ϵ (they will overlap at the boundary). Arbitarily select a point from each square, and color the whole square the same color as the point in the original coloring. By theorem 8 we see that there exist two points, a and b, distance r apart of the same color, say red. Then a and b are in two red squares, say A and B respectively. Then there exists a point x in A and y in B that were both red under the original five coloring. Let $B_\epsilon(a)$ and $V_\epsilon(b)$ be closed balls of radius ϵ centered at a and b. Then we see that $x \in B_\epsilon(a)$ and $y \in V_\epsilon(b)$. Then

$$\alpha \leq r - 2\epsilon \leq \|y - x\| \leq r + 2\epsilon \leq \beta \quad \forall\ x \in B_\epsilon(c)\ and\ \forall\ y \in\ V_\epsilon(b)$$

Then x and y are two points that are distance $d \in [\alpha, \beta]$ apart of the same color in the original five-coloring, a contradiction. Thus S is totally disconnected. □

The problem is has also been generalized to higher dimensional Euclidian space. There we are concerned with the chromatic number of $\mathbb{E}^n$, $\chi(\mathbb{E}^n)$. In working on the four-color problem on the plane the following result was obtained: $\chi(\mathbb{E}^3) \geq 5$. To show this, combine theorm 6 with the result shown in [Erdös et al. 1973 A], that for any two-coloring of $\mathbb{E}^3$ there exists a unit monochromatic equilateral triangle. Frankl and Wilson in [Frankl and Wilson] proved a conjecture of Erdös that $\chi(\mathbb{E}^n)$ grows exponentially with n. The best known bounds on $\chi(\mathbb{E}^n)$ are:

$$(1 + o(1))(1.2)^n \leq \chi(\mathbb{E}^n) \leq (3 + o(1))^n.$$

In [Erdös 1985] Erdös proposes the following question: Let G be the graph formed by connecting all the pairs of points (a, b) in $\mathbb{E}^2$ where the distance between a and b is between α and β, where $\alpha < 1 < \beta$. Then what is the chromatic number of G, denoted $h(2, \alpha, \beta)$. We see from the proof of theorem 9 that $h(2, \alpha, \beta) \geq 6$ for all α, β. We see also from the fig. 3 that for some α, β, $h(2, \alpha, \beta) \leq 7$.

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