Upper Bounds for the Stick Number of Torus Knots

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Introduction

In 1991, Seiya Negami showed that the stick number of a knot is less than or equal to twice the crossing number of the knot. This paper shows that the upper bound can be significantly lowered in the case of torus knots, and the the lower bound of the stick number does not hold a linear relation to the crossing number.

First, some definitions are necessary. The *stick number* of a knot (denoted $s(k)$) is the least number of straight lines necessary to represent the knot. The *crossing number* $c(k)$ of a knot is simply the minimum number of crossings in any representation of the knot. A *torus knot* is simply a knot that is embedded on the surface of a torus. A (p, q) torus knot can be created as follows [Ad]:

1. Mark p points on the outside equator of the torus and p points on the inside equator.
2. Attach the points marked on the outside equator to the corresponding points on the inside equator using strands running directly across the bottom of the torus.
3. Attach each point on the outside equator to the point on the inside equator which is a $\frac{q}{p}$ turn clockwise from the outside point using a strand running across the top of the torus.

A link is called *splittable* if the components of the link can be pulled apart and separated without cutting and pasting. The *Hopf link* can be recognized as two linked triangles. (See figure 1.) The *connected sum factor* of a knot is the knot resulting from connecting the strands of two arcs in two smaller knots, neither of which is the unknot.

Some preliminary theorems are also helpful in understanding the nature of the problem.

Theorem 1. *If a link or knot k is not equivalent to the unknot* $s(k) \geq \frac{5+\sqrt{25+8(c(k)-2)}}{2}$. *[Ne]*

Proof: Suppose that k admits a polygonal representation Q with n edges and that there is an edge e_0 which does not lie on a triangle component of Q. Then we move the polygon Q so that e_0 stands vertically (or parallel to the z-axis) and the others do not. Consider the orthogonal projection of Q to the xy-plane and count the number of crossings in $p(Q)$, where $p : \mathbf{R}^3 \to \mathbf{R}^2$ is the projection map. Since the edge e_0 projects to a single point, $p(e_0)$ contains no crossing and $p(Q)$ consists of $n-1$ edges. Each edge of $p(Q)$ contains at most $n-4$ crossings since any three consecutive edges do not cross on another. Thus $p(Q)$ contains at most $\frac{(n-1)(n-4)}{2}$ crossings and we have the inequality

$$\frac{(n-1)(n-4)}{2} \geq c(k).$$

Solving this, we get the first inequality in the theorem if Q has a component with at least four edges. When Q consists of only t triangles, clearly $s(k) = 3t$ and any two components have at most two crossings in common. Thus

$$c(k) \leq t(t-1).$$

In this case the inequality of $s(k) \geq \frac{5+\sqrt{25+8(c(k)-2)}}{2}$ still holds if $t \geq 2$ and k is equivalent to the trivial knot if $t = 1$.

Theorem 2. *If a link or knot k has neither the Hopf link as a connected sum factor nor a splittable trivial component, then $s(k) \leq 2c(k)$. [Ne]*

Proof: Let G be the four-regular graph on the xy-plane obtained from a minimum-crossing projection of k regarding each crossing as a vertex of degree four and let G' be the graph obtained from G by replacing each pair of multiple edges with a single edge. Then G' is a simple graph with vertices of degree three and four. (If k had a splittable trivial component, then G would have a part with no vertex and would no longer be a graph.)

It is well known that every graph embedded in the plane can be represented as a graph with each edge a straight line. G' is ambient isotopic to a linear embedding G'' in the plane. Then split each vertex of G'' into an upper and a lower point in $\mathbf{R}^3$ and join them by straight lines naturally corresponding to edges of G. The resulting figure forms a union of polygons Q in $\mathbf{R}^3$ which projects precisely to G''.

If Q has self-intersection, then such a crossing point lies on two edges contained in one vertical plane and these edges project to an edge of G'' which corresponds to a pair of multiple edges of G. We can, however, push them off slightly so as to eliminate the crossing and get a piecewise linear representation of k as a slightly modified Q. (Notice that no vertical plane contains three edges of Q. Otherwise, the part of k corresponding to these three edges would form a connected sum factor equivalent to the Hopf link.)

Now we count the number of edges in Q which coincinde with those in G. Since G is a 4-regular graph, we have $2|E(G)| = 4|V(G)|$ and $|V(G)| = c(k)$ by the assumption of G. Thus, k can be constructed as a union of broken lines with $2c(k)$ edges and hence $s(k) \leq 2c(k)$.

Theorem 3. *The crossing number of a (p, q) torus knot where $q \leq p$ is $p(q-1)$. (A (q,p) torus knot is equivalent to a (p, q) torus knot.) [Mu]*

Estimation of the stick number of torus knots

Theorem 4. *For all torus knots of the form $(p, 2)$ and $(2, p)$ with the exception of the trefoil knot*

$$s(k) \leq \lfloor \frac{p}{2+1} \rfloor + \lfloor \frac{p}{2} \rfloor.$$

Proof: With these knots, it is possible to draw the first $\lfloor \frac{p}{2} \rfloor$ sticks without encountering any crossings with existing sticks in the following manner.

1. Draw a stick.
2. Attach another stick of equal length to one end of that stick at an angle of $\frac{2\pi}{p+1}$ clockwise.
3. Repeat until $\lfloor \frac{p}{2} \rfloor$ sticks have been drawn.

Draw a stick attached to the last stick drawn that crosses over the first stick drawn. This gives us $\lfloor \frac{p}{2} + 1 \rfloor$ sticks.

From $n = 1$ to $n = \lfloor \frac{p}{2} - 1 \rfloor$ do the following:

1. Draw a stick attached to the previous stick drawn that crosses under the nth stick drawn. End the stick on the inside of the figure.
2. Draw a stick attached to the previous stick drawn that crosses over the $(n+1)$st stick drawn. This stick should end on the outside of the figure.

Now draw a stick attached to the end of the last stick drawn that crosses under the $\lfloor \frac{p}{2} \rfloor$ stick drawn. As in the first step, end this stick on the inside of the figure. Draw another stick attached to the end of this stick that passes over the $\lfloor \frac{p}{2} + 1 \rfloor$st stick drawn and attaches itself to the open end of the first stick drawn. This part of the algorithm uses $2\lfloor \frac{p}{2} \rfloor$ sticks, giving us a total of $\lfloor \frac{p}{2} + 1 \rfloor + 2\lfloor \frac{p}{2} \rfloor$ sticks, and it creates a $(p, 2)$ torus knot. (See figure 2.)

This formula does not work for the trefoil knot because the algorithm requires that the first stick be able to cross the second stick, which is clearly impossible.

Theorem 5. *The stick number of the trefoil knot is 6. [Ad]*

Proof: Three sticks are not enough; they would simply form a triangle that lies in a plane. If we looked down at the plane, we would see a projection of the knot with no crossings, so it would have to be the unknot.

If we view four sticks from any direction, we will see a projection of the corresponding knot. If two of the sticks are attached to each other at their ends, they cannot cross each other in the projection, as two straight lines can cross at most once. Thus each of the sticks can only cross the one stick which is not attached to either one of its ends. Therefore, there can be at most two crossings in the projection. But the only knot with a projection of two or fewer crossings is the unknot, which is not the trefoil knot.

Try five sticks. View the knot so that we are looking straight down one of the sticks. In the projection of the knot that we see, we will only be able to see four of the sticks, since the fifth stick is vertical. For the same reason as previously given, the four sticks that we see can have at most four crossings, and so the knot must be the unknot.

Therefore, it must take at least six sticks to make a knot. In fact, it is possible to make a trefoil with six sticks. Let the vertices labelled P lie in the $x-y$ plane, the vertices labelled L lie beneath the $x-y$ plane, and the vertices labelled H lie above the $x-y$ plane. Therefore the stick number of the trefoil knot is 6. (See figure 3.)

Theorem 6. *For (p,q) torus knots where $q \leq \lfloor \frac{p}{2} \rfloor$,*

$$s(k) \leq 2p + 1 - \lfloor \frac{p}{q} \rfloor.$$

Proof: By construction. Draw a circle, and draw points on it every $\frac{2\pi}{p}$ radians. Pick a point on the circle, and call it point n. Let $n = 1$. Consider the plane as viewed from above. From $n = 1$ to $n = p$ do the following:

1. Place the point n on the plane P.
2. Draw the portion of the line tangent to the circle that lies clockwise of the circle.
3. Consider the point that lies $q\frac{2\pi}{p}$ radians clockwise from point n. Call it point $n+1$.
4. Draw the portion of the line tangent to the circle at point $n+1$ that lies counterclockwise of that point.
5. Place a vertex where the two tangent lines just created intersect, call it v_n, and place it on the plane P.
6. Let $n = n + 1$. Repeat.

This algorithm gives us many intersections, which are not allowed because one stick cannot possibly go through another. Therefore we must move some of the vertices around a bit in order for our construction to work. Vertices 1 through $\lfloor \frac{p}{q} \rfloor$ may remain on the plane, and vertices v_1 through $v_{\lfloor \frac{p}{q} \rfloor}$ may also remain on the plane. From $n = \lfloor \frac{p}{q} \rfloor + 1$ to $n = p$ do the following:

1. Raise point n to a plane $U_{n-(\lfloor \frac{p}{q} \rfloor + 1)}$ lying above plane P so that the line between v_{n-1} and n crosses over all lines it would intersect that were drawn previous to that line.
2. Lower point v_n to a plane $L_{n-(\lfloor \frac{p}{q} \rfloor + 1)}$ lying below plane P so that the line between n and v_n crosses under all lines it would intersect that were drawn previous to that line.
3. Let $n = n + 1$. Repeat.

This gives us a problem because the line from vertex v_p to 1 would not cross over two other tangent lines as required. It would cross under them instead. Therefore, we need to place another vertex on this line. This vertex would lie above the plane P, preferably before the first crossing or between the first and second crossings that stick encounters. This construction leaves us with two sticks for every vertex lying on the circle except the last one, for which we have three, giving us $2p + 1$ vertices. (See figure 4.)

Looking at our construction, however, it is obvious that we have created some unnecessary sticks, namely in situations where a vertex on the circle and the two vertices at the end of its tangent line are all on plane P. Whenever this situation occurs, we can simply eliminate the vertex that lies on the circle. Since this happens in $\lfloor \frac{p}{q} \rfloor$ places, doing so will lower the stick number to $2p + 1 - \lfloor \frac{p}{q} \rfloor$.

Theorem 7. *For (p, q) torus knots where $q > \lfloor \frac{p}{2} \rfloor$,*

$$s(k) \leq 3p + 1.$$

Proof: Again by construction. Draw a circle, and place points on it every $\frac{2\pi}{p}$ radians. Draw another, larger, circle around the eisting circle. Consider these circles as viewed from above. Pick a point on the inner circle and call it point n. Let $n = 1$. For $n = 1$ to $n = p$ do the following:

1. Place point n on the plane P.
2. Draw the portion of the line tangent to the inner circle that lies clockwise of the inner circle at point n. End this line when it reaches the outer circle. Create a vertex at this point, call it vertex n', and place it on plane P.
3. Draw the portion of the line tangent to the outer circle that lies clockwise to the outer circle at this point.
4. Consider the point that lies $q\frac{2\pi}{p}$ radians clockwise from point n on the inner circle. Call it point $n + 1$.
5. Draw the portion of the line tangent to the inner circle that lies counterclockwise of the inner circle at point $n + 1$. End this line when it reaches the outer circle. Create a vertex at this point, call it vertex n'', and place it on plane P.
6. Draw the portion of the line tangent to the outer circle that lies counterclockwise to the outer circle at this point.
7. Place a vertex at the point where the two lines tangent to the outer circle intersect. Call this vertex v_n, and place it on plane P.
8. Let $n = n + 1$. Repeat.

Again, we have an algorithm that gives us many intersections, so our construction cannot be used. Once again, however, this can be remedied by moving vertices around. Vertices 1, $1'$, v_1, $1''$, 2, $2'$, and v_2 should all be fine on the plane. Raise vertex $2''$ to a plane U_3 that lies above P, so that it crosses over the lines connecting vertex $1'$ to vertex v_1. For $n = 3$ to $n = p$ do the following:

1. Lower vertex n' to a plane D_n so that the line connecting vertex n to vertex n' crosses below all lines it would intersect that were drawn before it.
2. Lower vertex v_n to a plane D_n so that the line connecting vertex n' to vertex v_n crosses below all lines it would intersect that were drawn before it.
3. Raise vertex n'' to a plane U_{n+1} so that the line connecting vertex v_n to vertex n'' crosses above all lines it would intersect that were drawn before it and the line connecting vertex n'' to $n + 1$ also crosses above all lines it would intersect that were drawn before it.
4. Let $n = n + 1$. Repeat.

The above algorithm gives us a stick number of $4p$, but can be run in a manner such that n'' to $n + 1'$ is a straight line intersecting plane P at point n for all n such that $n \neq p$. For $n = p$ this is not possible, as vertex $1'$ is on plane P. If we consider these as straight lines and eliminate vertices 2 through p, then we will be able to lower the stick number of our representation by $p-1$ sticks, therefore giving us a stick number of $3p+1$. (See figure 5.)

Theorem 8. *The lower bound of $s(k)$ is not linear in relation to $c(k)$.*

Proof: If $s(k)$ was linear in relation to $c(k)$, then for some a, b, we would have $s(k) \geq ac(k)+b$. This would work for all knots, including torus knots. Consider torus knots where $\lfloor \frac{p}{2} \rfloor \leq q < p$. In this case, $c(k) = p(q-1)$. For these knots, $s(k) = 3p+1$. Consider $c(k)$ in relation to $s(k)$. As $p \rightarrow \infty$ and $q \rightarrow p$, the added constant in $s(k)$ will have a minimal effect, so we can leave it out.

$$\frac{s(k)}{c(k)} = \frac{3p}{p(q-1)}$$

$$= \frac{3}{q-1} - 0 \quad \text{as} \quad q \rightarrow p \rightarrow \infty.$$

If $s(k)$ was linear in relation to $c(k)$, $\frac{s(k)}{c(k)}$ would approach a constant as $p, q - \infty$. Therefore the lower bound of $s(k)$ is not linear in relation to $c(k)$.

References

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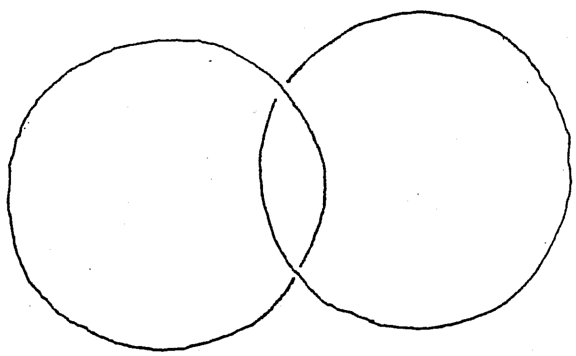

Figure 1.
The Hopf link.

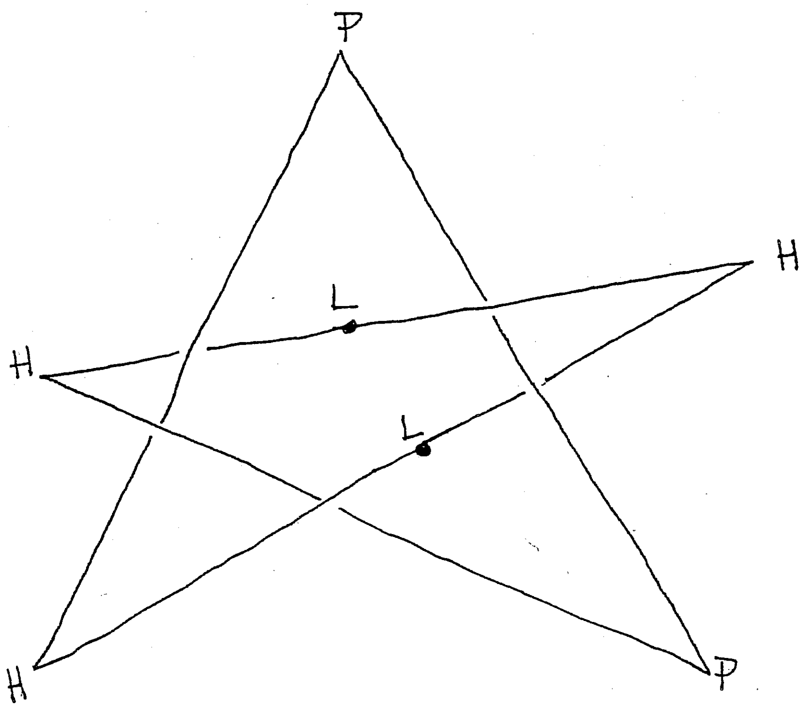

Figure 2.
Stick representation of a (5,2) knot.

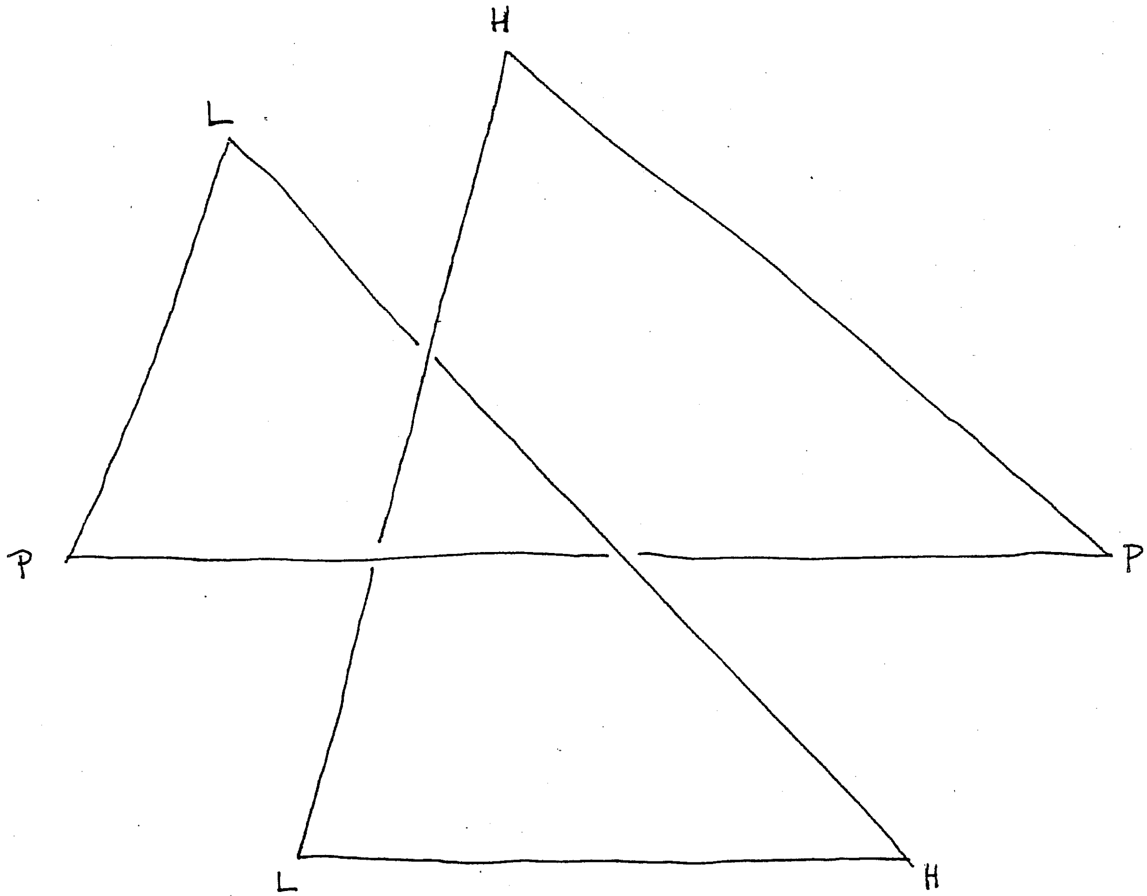

Figure 3.
A trefoil knot.

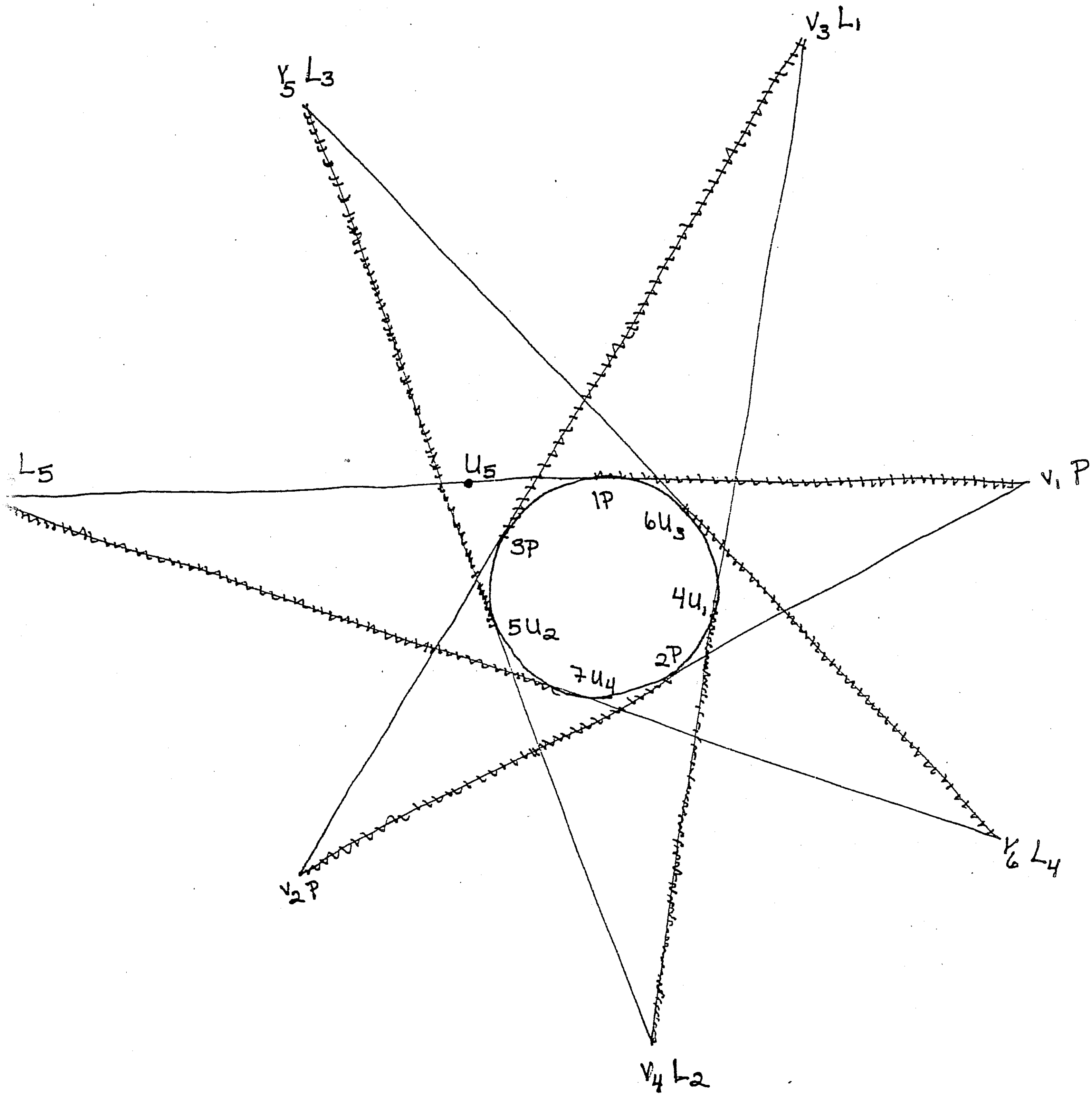

Figure 4.
Stick representation of a (7,3) knot.

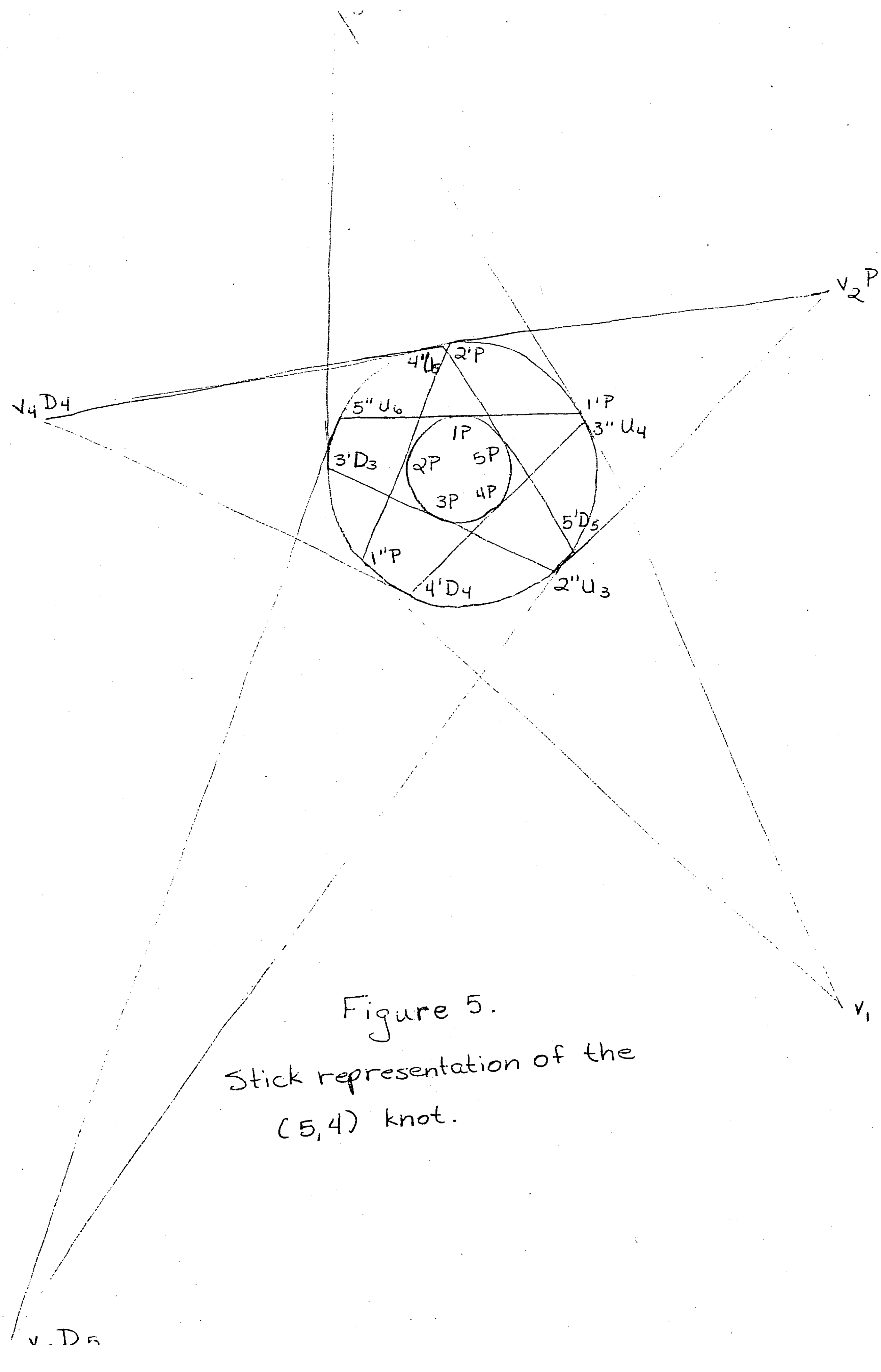

Figure 5.

Stick representation of the (5,4) knot.