R. E. U. SUMMER PROGRAM

PUZZLES: What Patterns are Possible?

Rubik's Cube & Slide Puzzles

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Introduction

The challenge of the puzzles as the Rubik's cube and the so-called 15 puzzle is to restore the given configuration back to the original configuration. This may or may not be possible. It depends on the given configuration. The purpose of this paper is to answer the question: how can we determine whether the given configuration can be restored back to the original configuration. In both the cube and 15 puzzle, the parity of permutation carried out between the given configuration and the original configuration plays an important role to answer this question. 15 puzzle can be generalized to (mxn) - 1 puzzle, and the proof is given for case $m \geq 3$ and $n \geq 4$.

(n x m) - 1 puzzle

(n x m) - 1 puzzle is made of (n x m) - 1 square tiles with the lower right hand corner blank, arranged in a nxm array. (See Figure 1.)

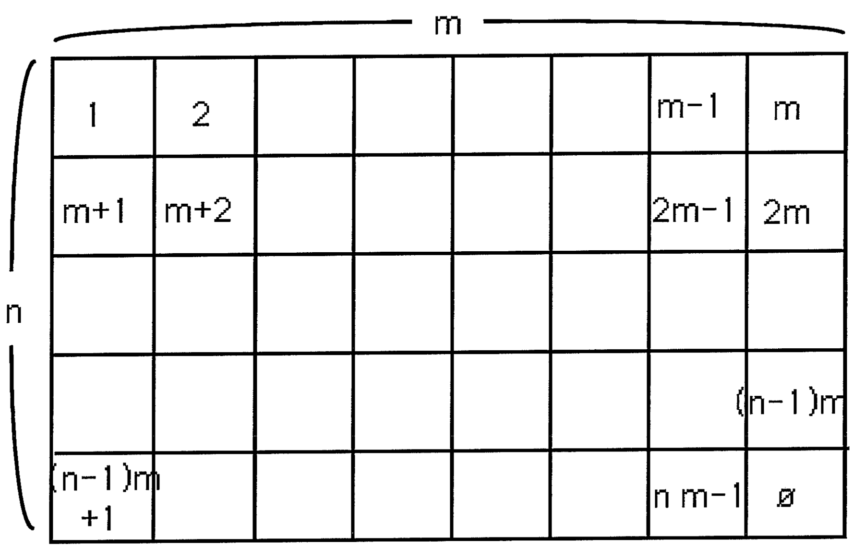


Fig. 1.

The puzzle is constructed so that tiles can be slid vertically and horizontally, such moves being possible because of the presence of the blank.

The basic rule

The performances of a sequence of slides are only allowed in such a way, at end, the lower right-hand corner is again blank. Call this new configuration *achievable*.

<u>Question</u>: What are all achievable configuration?

We denote START as a configuration shown in Figure 1.

The following fact is given in [5].

Fact 1: Let X be an achievable configuration. Then, permutation carried out between X and START must be even.

In fact, the following result is known:

"*Every even permutation is achievable.*"

We give the proof for this in the case <u>$m \geq 3$ end $n \geq 4$</u>. Before we prove this, we introduce two important results which are also used to attack the problem of Rubik's cube.

The next Lemma is from [1].

Lemma 1: Let $N = \{1, 2, ..., n\}$. Let T be a subgroup of sym (N). Suppose T contains 3-cycle for three elements a, b, $c \in N$, and if for <u>any</u> $x \neq a, b, c$, $x \in N$, there exists $p \in T$ such that

$xp = a$, $bp = b$, and $cp = c$. --- (1)

Then, T contains all 3-cycles of the elements of N.

Note: 3-cycle (abc) moves a, b, c, as a->b->c->a but fixes $i \in N$, $i \neq a, b, c$.

The idea of the proof is from [1].

Proof: Suppose T contains (abc). Then by condition (1), we obtain

$$(xbc) = p(abc)p^{-1},$$
$$(axc) = (abc)^{-1}(xbc)(abc),$$
$$(abx) = (abc)(xbc)(abc)^{-1}.$$

From these we get <u>all</u> 3-cycles

$$(xyc) = (aby)^{-1}(xbc)(aby),$$
$$(xby) = (axc)^{-1}(aby)(axc), \qquad y, z \neq a, b, c$$
$$(axy) = (ybc)^{-1}(axc)(ybc),$$

and
(xyz) = $(zbc)^{-1}(xyc)(zbc)$.

Since (abc) $\in$ T and (xbc) = $p(abc)p^{-1} \in$ T generate 3-cycles for any three elements of N, T contains 3-cycles for any three elements of N. (Remeber x is arbitrary element of N such that x $\neq$ a, b, c.) There are two possible 3-cycles for u, v, w $\in$ N, (uvw) and (uwv). Since (uwv) = $(uvw)^2$ and (uvw) = $(uwv)^2$, if one of them is in T, the other is also in T. Hence, T contains all 3-cycles of the elements of N.

The next Lemma is from [4].

Lemma 2: For each n$\geq$ 3, let A_n be alternating group of degree n. Suppose r, s to be distinct elements of {1, 2, ..., n}. Then A_n is generated by the 3-cycles {(rsk)| 1 $\leq$ k $\leq$ n, k $\neq$ r, s }.

<u>Basic slides of the puzzle</u> (3-cycle S_0)

Suppose a, b, c occupy the positions as below and let ø denote blank.

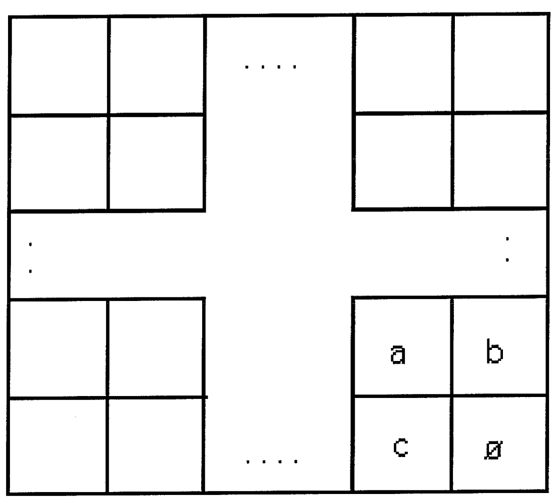
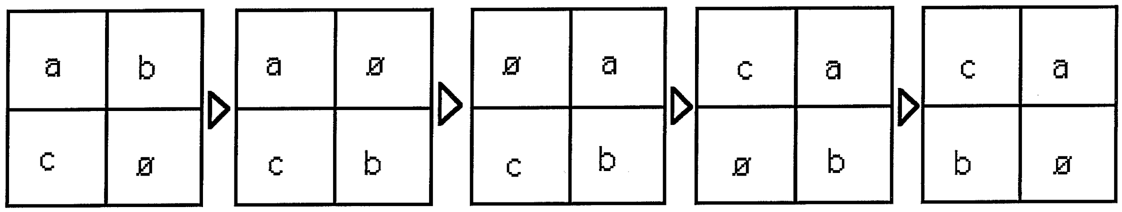


Fig. 2.

We call the following sequence of slides as <u>S_0.</u>



Then *So* produces 3-cycle (abc) where *So* doesn't change the tile positions other than a, b, c. Then we get the following result.

Claim 1: Assume we have (nxm) - 1 puzzle, where m ≥ 3 and n ≥ 4. Suppose tiles a, b, c occupy the positions as Figure 3. Then, for any 3-cycle of the tile positions, there exists a correspoinding sequence of slides to produce it, say *s*, such that *s* doesn't affect the positions of tiles other than u, v, w and fixes the blank at lower-right-hand corner.

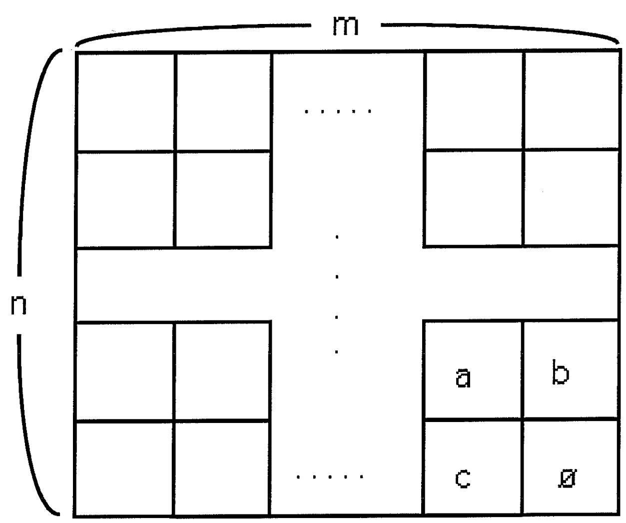


Fig. 3.

Proof: Let x be any tile where x ≠ a, b, c. We first show that there exists a sequence of slides, denote *q*, such that

q moves x to a, and *q* doesn't change the positions of b, c, and ø.

To produce *q*, consider the following steps.

Step 1. Transpose c and ø.
Step 2. Move x to a, and make ø occupy the same position as in Step 1.

Step 3. Transpose c and ø again.

We will show that there exists a sequence of slides to achieve Step 2 for any $x \neq a, b, c$. Figure 4 is the configuration after Step 1. suppose x occupies the position as in Figure 4.

Label ø and tiles as $p_0, p_1, p_2, p_3, \ldots, p_{11}, p_{12}$ as in Figure 5 so that a path $(p_0, p_1, p_2, \ldots, p_{12} = p_0)$ is a simple path in which each p_i is distinct, each $p_i \neq b, c$, and $p_j = x$ for some j. Then consider the following sequence of slides

$(p_0p_1)(p_1p_2)(p_2p_3) \ldots (p_{11}p_{12})$.

(i.e., This moves $ø = p_0 \to p_1 \to p_2 \ldots \to p_{12} = p_0$.)

After this move, ø ends up with at the same position and every tile in the path $(p_1, p_2, \ldots, p_{11})$ is shifted to the next position, but all other tiles outside of the path are not moved. (Figure 6)

Repeat this process, and at some point, we will obtain that x is moved to a, and ø occupies x_0 position. It is clear that this process works no matter what position x occupies where $x \neq a, b, c$. (Notice: We assumed $m \geq 3$ and $n \geq 4$.) Thus, by following Step 1, 2, and 3, we can produce a sequence of slides which satisfies the condition (1).

By Lemma 1, it follows thatS_0 and qS_0q^{-1} generateall 3-cycles of the tile positions. Since S_0 and qS_0q^{-1} both fix blank position, the moves generated by S_0 and qS_0q^{-1} also fix blank position.

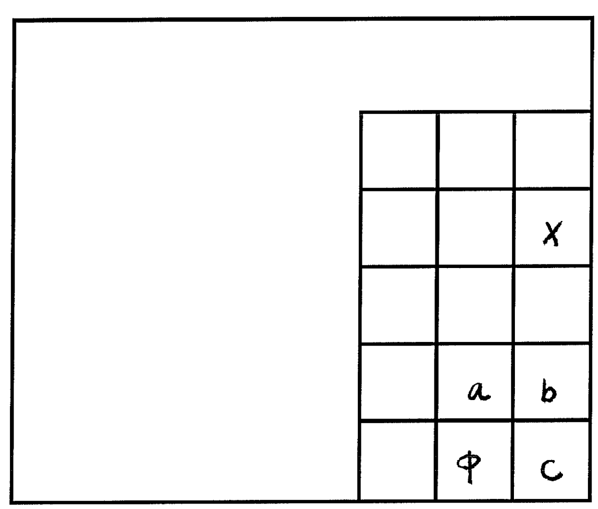


Fig.4.

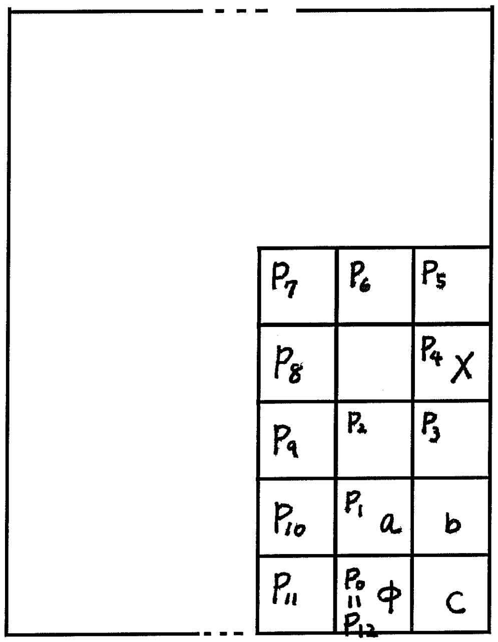


Fig. 5.

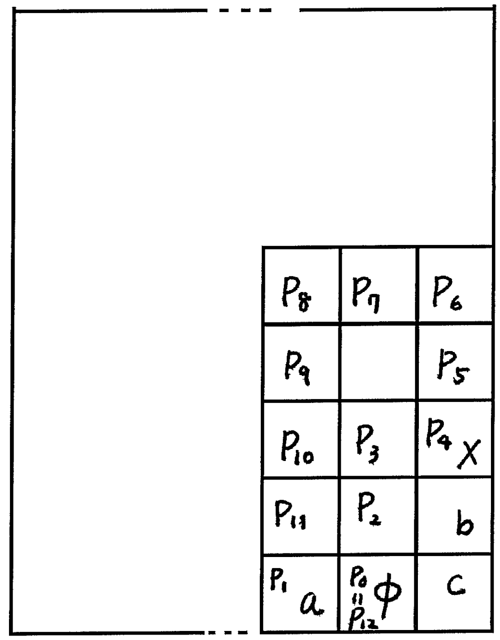


Fig. 6.

Then, the next result follows immediately.

Claim 2: Every even permutation is achievable.

Proof: By Lemma 2, any even permutation can be expressed by a product of 3 cycles. By Claim 1, for any 3 cycle of the tile positions, there exist a corresponding sequence of slides which also fixes the blank position. Thus, for any even permutation, there exists a corresponding sequence of slides which also fix blank position.

The 3x3x3 Rubik's cube

The 3x3x3 Rubik's cube consists of six different colored faces where each face is divided into nine squares with colored stickers of the same color attached. Each face of the cube rotates freely, and a few random moves of the faces soon scramble the colored squares.

The basic mathematical problem is to restore the cube from any random pattern back to its original position, namely START configuration, with each face having just a single color.

Question: Which patterns(configurations) are possible?

To obtain a random pattern, consider following procedure.

1. Obtain START configuration at beginning.
2. Disassemble the cube into pieces.
3. Reassemble the cube with all pieces randomly in place to obtain a random configuration.

It may be or may not be possible to restore the cube from that configuration back to START configuration without disassembling the cube again. The purpose of this article is to give the answer to the following question.

Question: Suppose we are given a random configuration of the cube. Then how can we determine whether the given configuration can be returned to START configuration by a sequence of basic moves of the cube? (i.e., Without disassembling the cube.)

Notation

The following terminology is from [2]. We label the six faces of the cube, as Front, Back, Right, Left, Up, Down, and we abbreviate these designations to the first letter. See Figure 7.

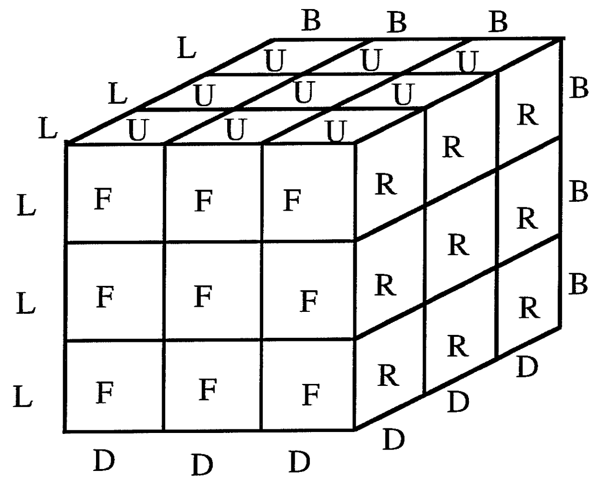


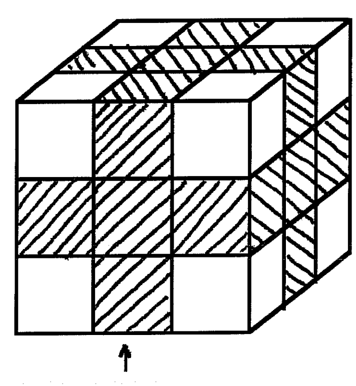
Fig. 7.

We denote the clockwise 90° turn of the R face by R, and likewise for all the other faces, and use R^2 to denote a 180° turn and R^3 or R^{-1} to denote 270° turn (i.e., 90° counter clockwise turn). Finally, we shall remark that there are divided in three classes known respectively as *corner pieces* (8 of them), *edge pieces* (12 of them), and *center face pieces* (6 of them).

The structure of the cube group

There exist nine movable regions in the cube which are six faces and three middle layers. See Figure 8.

Fig. 8.



First consider the middle layer A. Then we realize that the rotation of the middle layer A can be produced by the rotations of two face turns, R and L. For example, 90° clockwise rotation of middle layer A can be produced by 90° counter clockwise turn of face R and 90° clockwise turn of face L. Thus, we have

$$(90^\circ \text{ clockwise turn of the middle layer A}) = R^{-1}L = R^3L$$

The same thing can be said for the other middle layers. Thus, the following result is obtained.

Fact 2: *All moves of the cube are generated by six face turns* F, B, R, L, U, D.

One more remark to be made is that each face turn satisfies the following properties.

1. The center face piece remains in place.
2. The corner pieces are moved to corner place.
3. The edge pieces are moved to edge place.

Thus, the cube group consists of following two.

1. The eight corner pieces are permuted their positions among themselves and if possible some of them are <u>twisted</u>.
2. The twelve edge pieces are permuted their positions among themselves and if possible some of them are <u>flipped</u>.

We denote 1 and 2 as *corner piece group* and *edge piece group*, respectively.

<u>Wreath product</u>

It is known that the interaction between the permutations of positions and the changes of orientations is an example of a wreath product of groups.

Thus, the cube group is an example of a wreath product of groups. The definition of wreath product is from [1].

Definition: Let G and H be permutation groups that act on N = {1, 2, ..., n} and K = {1, 2, ..., n}, respectively. Then the wreath product of H by G, written H wr G, is the subgroup of sym(NxK) generated by permutations of the following two types:

$\pi(g) : (i, j) \rightarrow (ig, j)$ for $g \in G$,
and $\phi(h_1, ..., h_n) : (i, j,) \rightarrow (i, jh_i)$ for $h_1, ..., h_n \in H$

In the applications to be considered for the cube group, G will be S_n = Sym {1, ..., n} and H will be Z_k where Z_k is the group generated by an K-cycle or the group of rotations of a regular K-gon. To visualize "the cube group" is an example of a wreath product of groups, we first look at the corner piece actions of the cube.

Consider a corner piece of the cube. Since each corner piece of the cube has three distinct faces, it can have three different orientation in a certain position. (Figure 9)

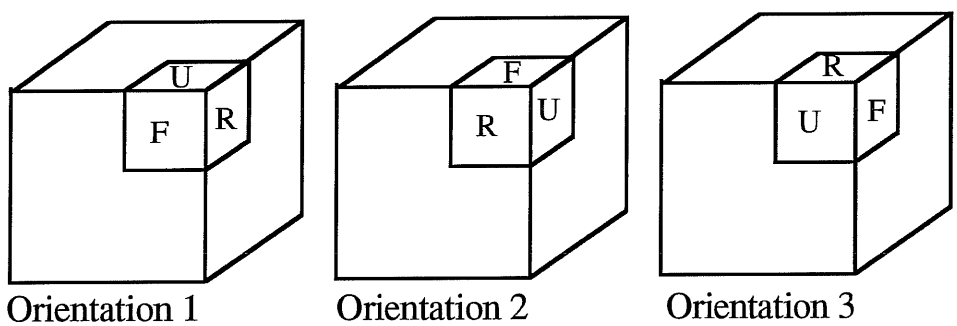


Fig. 9. Three different orientation of a corner piece

We can actually visualize this corner piece as equilateral triangle whose vertices are labeled by three different letters U, F, R so that the orientation of the triangle is distinguishable. Thus, each orientation of Figure 10 can be described as follows.

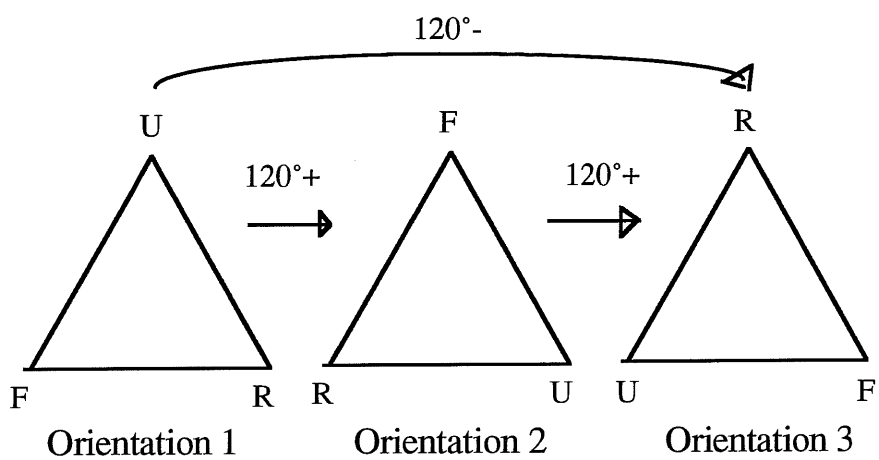


Fig. 10.

We realize, orientation 2 and orientation 3 can be obtained by 120° clockwise rotation and 240° clockwise rotation (or 120° counter clockwise rotation) of the original triangle (orientation 1), respectively.

In terms of the corner piece, we describe that the orientation 2 and orientation 3 can be obtained by 120° clockwise "twist" and 120° counter clockwise "twist" of the original corner piece (orientation), respectively. Since there exist eight corner pieces in the cube and each corner piece has three different orientations, we can conclude the following result.

Fact 3: *The corner piece group is a subgroup of* Z_3 *wr* S_8.

Since there exist twelve edge pieces in the cube and each edge piece has two different orientations, we have the following result.

Fact 4: *The edge piece group is a subgroup of* Z_2 *wr* S_{12}.

We will discuss some general properties of Z_k wr S_n.

Remark 1: Z_k can be considered as the group of K integers mod k.

The following results are from [1].

Theorem 1: Any element £ of Z_k wr S_n has a unique representation of the form £ = π(£')ø(£"(1), £"(2), ..., £"(n)) where £' ∈ S_n and £" is a function from N = {1, ..., n} to Z_k.

From Remark 1, £"(i) can take values 0, 1, ..., k-1 for i = 1, ..., n.

It is known that 3-*cycles* play special roles in both the corner piece group and the edge piece group. We will define k-cycle first.

Definition: An element £ ∈ Z_k wr S_n is called a k-cycle if £' is a k-cycle and £ has order K. (i.e., $£^k$ = identity.)

Before we state the next result, we define Act'(£).

Act'(£) = { i∈N : i£' ≠ i or £"(i) ≠ 0 }

Next result shows how to produce 3-cycle.

Theorem 2: Let a, b, c ∈ N. If £, ß, ø ∈ Z_k wr S_n satisfies

(i) Act'(£) ∩ Act'(ß) = {b}, where a£' = b = cß'

(ii) Act'(ø) ∩ {a, b, c} = {c}, where cø' = c and ø"(c) = 1

Then,

$ø^J[£,ß]ø^{-J}$ is 3-cycle for j = 0, 1, ..., k-1

where $[£,ß] = £ß£^{-1}ß^{-1}$

proof is given in [1].

Let T_0 = [£,ß] and $T_j = ø^j[£, ß]ø^{-j}$

Then T_j satisfies follows

T_j' = (abc) (i.e., T_j moves a, b, c as a->b->c->a.)

and

$T''_j(a) = T''_0(a)+j$, $T''_j(b) = T''_0(b)$, $T''_j(c) = T''_0(c)-j$

for j = 0, ..., k-1

We call this set of 3-cycles $\{T_0, T_1, ..., T_{k-1}\}$ a complete set of 3-cycles.

Example 1

Suppose we have the corner pieces labeled a, b, c as in the Figure 5. Then £ can be produced by U, ß can be produced by $R^{-1}DR$, and ø can be produced by LD.

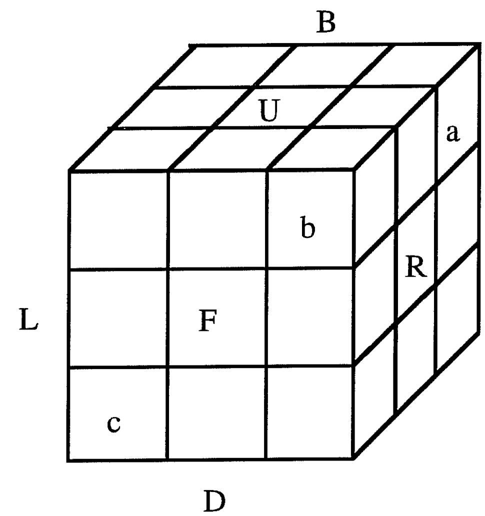


Fig. 11.

Remark 2: £ = U and ß = $R^{-1}DR$ satisfy Act'(£)∩Act'(ß) = {b}, and ø = LD satisfies cø' = c and ø"(c) = 1 (i.e., ø=LD doesn't change the position of a, b, c but "twists" the corner piece c 120° clockwise). So $T_j = (LD)^j[U, R^{-1}DR](LD)^{-J}$ produces 3-cycle on a, b, c. (i.e., T_j moves a, b, c, as a->b->c->a and T_j^3 is identity.) where j = 0, 1, -1. (Note: -1 is equivalent to 2 in Z_k.)

Remark 3: $T_j = (LD)^j[U, R^{-1}DR](LD)^{-j}$ doesn't change the positions and the orientations of the pieces in the cube other than the corner pieces a, b, c.

Example 2

Suppose we have the edge pieces labeled 1, 2, 3 as in Figure 12.

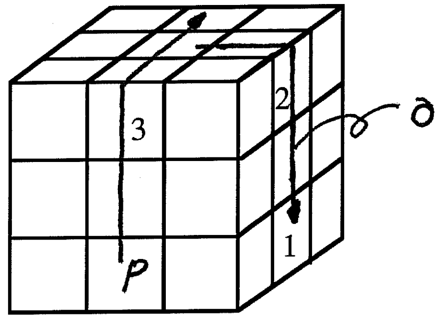


Fig. 12..

According to theorem 2, we find

$£ = \partial^{-1}$ where $\partial = F^{-1}B$
$ß = R^{-1}FR$
and $\phi = P^{-1}F^2$ where $P = R^{-1}L$

Hence, $(P^{-1}F^2)^j[\phi^{-1}, R^{-1}FR](P^{-1}F)^{-j}$ produces 3-cycle on 1, 2, 3.

We introduce some special move of the cube, next.

Define $\mu_{ab} \in Z_k$ wr S_n such that
$(\mu_{ab})' =$ identity
and
$(\mu_{ab})''(a) = 1$, $(\mu_{ab})''(b) = -1$, and $(\mu_{ab})''(i) = 0$ for $i \neq a, b$
where $a, b, i \in N$.

Then under the assumption of Theorem 2, we have the following result.

Theorem 3: Under the assumption of Theorem 2, we have $\mu_{cb} = [\phi,[£, ß]]$.

Remark 4: We take a look at Example 1 again. Since we have $£ = U$, $ß = R^{-1}DR$, and $\phi = LD$ under the assumption of Theorem 2, we have $\mu_{cb} = [LD, [U, R^{-1}DR]]$. We can verify this basic move of the cube doesn't change the piece positions at all but twists only two corner pieces b, c where c is twisted 120° clockwise and b is twisted 120° counter clockwise.

We can easily expand this idea for any two corner pieces.

Fact 5: Suppose the corner pieces a, b, c occupy the same positions as in Example 1. Let x, y be any two corner pieces. Then, there exists a sequence of basic moves of the cube, say u, such that u doesn't change the piece positions in the cube at all but changes the orientations of only two corner pieces x, y where x is twisted 120° clockwise and y is twisted 120° counter clockwise.

Proof: For any x, y, suppose we agree that there exists a basic move of the cube, say r, such that.

r moves x to c and y to b simultaneously.

By Remark 4, $u = r[LD, [U, R^{-1}D]]r^{-1}$ satisfies the condition.

By theorem 3 and the similar argument as above, we get the following result.

Fact 5': Let p, q be any two edge pieces. Then there exists a sequence of the basic moves of the cube, say ß, such that ß doesn't change the piece positions in the cube at all but changes the orientation of only two edge pieces p, q where p, q are both flipped.

Two subproblems of Rubik's cube

In the Rubik's cube, there are various pieces which can occupy certain positions with certain orientation. Thus, studying possible configurations of the cube involves the following two problems.

1. What positionings of the pieces in the cube are possible?
2. What orientations of the pieces in the cube are possible?

We deal with these two subproblem separately.

The positioning problem

When we view the basic moves of the cube as acting on positions of the pieces in the cube, we will obtain "permutation of the corner piece positions" and "permutation of the edge piece positions." Thus, it is reasonable to denote "overall permutation of the piece positions in the cube" as total permutation of the corner piece positions and the edge piece positions. Then we have the next result.

Claim 3: Overall permutation of the piece positions in the cube which is produced by a sequence of basic moves of the cube must be even.

Proof: Since all moves of the cube is generated by six face turns, F, B, R, L, U, D (by Fact 2), it suffices to show each face turn produces even permutation of the piece positions. Consider the face turn F. Label four corner pieces in face F as 1, 2, 3, 4, and four edge pieces as a, b, c, d as Figure 13.

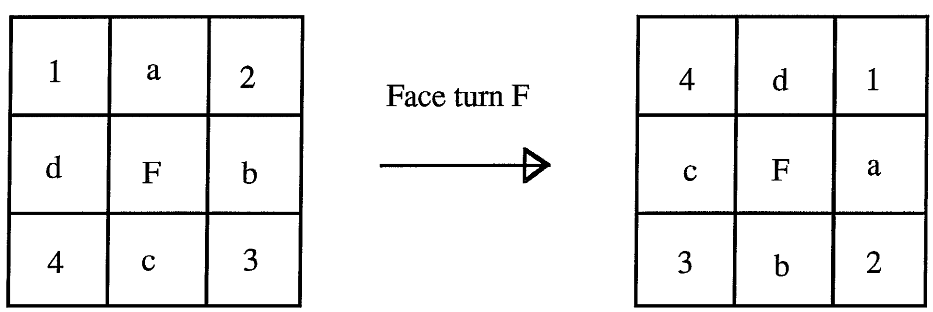


Fig. 13

Then face turn F produces the following two permutation of the piece positions.

(1234) on the corner pieces. (i.e., 1->2->3->4->1.)

and

(abcd) on the edge pieces. (i.e., a->b->c->d->a.)

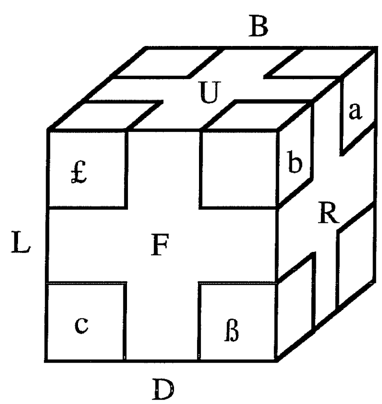
Since both (1234) and (abcd) are odd permutations, total permutation of the piece positions produced by face turn F is even, and likewise for all the other face turns.

Claim 4: Suppose the corner pieces a, b, c occupy the same positions as in Example 1. Then, for any 3-cycle of the corner piece positions, say (uvw), there exists a corresponding sequence of basic moves of the cube, denote s, to produce it such that s doesn't change the positions of the pieces in the cube other than u, v, w. (For any three edge pieces, we have exactly the same result.)

Proof: Let x be any corner piece where x $\neq$ a, b, c. We first show that there exists a sequence of basic moves of the cube, denote *P*, such that

P moves x to a, and *P* doesn't change the positions of b, c, at all.
--- (1)

1. If x is in the face B, it is easy to move x to a by turning face B.
2. If x is at £, apply the move $L^{-1}B^{-1}L$ which moves x to a and fixes b and c.
3. If x is at ß, apply the move DBD^{-1} which moves x to a and fixes b and c.



Thus, for any corner piece x $\neq$ a, b, c, there exists a sequence of basic moves of the cube which satisfies (1). From Example 1,

$$T_j = (LD)^j[U, R^{-1}DR](LD)^{-j} \quad (j = 0, \pm 1)$$

produces 3-cycle (abc) for the corner piece positions a, b, c. Hence, by Lemma 1, it follows that T_j and PT_jP^{-1} generate all 3-cycles of the corner piece positions. Last thing to be checked is 3-cycles of the corner piece positions produced by T_j and PT_jP^{-1} don't affect the edge piece positions. This is clear since both T_j and PT_jP^{-1}don't change the edge piece positions, the moves generated by T_j and PT_jP^{-1} don't change the edge piece positions.

Then we obtain the following result.

Claim 5: Every even permutation of the corner piece positions can be produced by a sequence of basic moves of the cube which doesn't affect the positions of the edge pieces.

Proof : By Lemma 2, any even permutation of the corner piece positions can be expressed by a product of 3-cycles of the corner piece positions. By Claim 4, for any 3-cycle of the corner piece positions, there exist a corresponding sequence of basic moves of the cube which doesn't affect the positions of the edge pieces. Thus, the result follows immediately.

Exactly, the same thing can be said for the edge pieces.

Claim 5': Every even permutation of the edge piece positions can be produce by a sequence of basic moves of the cube which doesn't affect the positions of the corner pieces.

The next result gives the answer to the positioning problem.

Claim 6: Every even overall permutation of the piece positions in the cube can be produced by a sequence of basic moves of the cube.

Proof: Since overall permutation of the piece positions is even, we have the following two cases.

Case 1: Permutation of the corner piece positions and permutation of the edge piece positions are both even.

Case 2: Permutation of the corner piece positions and permutation of the edge piece positions are both odd.

If we had the condition of Case 1, the result follows immediately from Claim 5 and Claim 5'. Now consider Case 2. From the proof of Claim 3, we know each 90° clockwise face turn F, B, R, L, U, D produces odd permutations on both the corner piece positions and the edge piece positions. Thus, if we had Case 2, apply one of the face turn to obtain the conditions of Case 1, then the result follows immediately.

The Orientation problem

Let Y_o denote some configuration of the cube in which each piece of Y_o occupies the same position as START configuration, but some of the pieces have different orientations.

To begin with, we introduce two strong conjectures.

Conjecture 1: Suppose Y_o is a configuration of the cube such that the only difference between Y_o and START configuration is that a single corner piece of Y_o is twisted (120° clockwise or 120° counter clockwise) compared to that of START. Then there is no sequence of basic moves of the cube which restores Y_o back to START.

Conjecture 2: Suppose Y_o is a configuration of the cube such that the only difference between Y_o and START configuration is that a single edge piece of Y_o is flipped compared to that of START. Then there is no sequence of basic moves of the cube which restores Y_o back to START.

We first consider the corner pieces.

Let Y_c to denote a configuration of the cube in which each corner piece of Y_c occupies the same position as START configuration, but some of the corner pieces (more than two) are twisted compared to START. Then, we have following two results.

Claim 7: There exists a sequence of basic moves of the cube which restores Y_c back to START configuration if total twists of the corner pieces of Y_c are multiples of 360°. And there is no such sequence of basic moves of the cube if total twists of the corner pieces of Y_c are <u>not</u> multiples of 360°.

Proof: Suppose i corner pieces are twisted compared to START configuration, where i=2, 3, ..., 8. Name these i corner pieces as C_1, C_2, ..., C_i. Choose any two corner pieces among C_1, C_2, ..., C_i, then twist one of them 120° clockwise and the other 120° counter clockwise (i.e., 240° clockwise) so that one of them ends up with no twist. By Fact 5, there exists a sequence of basic moves of the cube to accomplish this. At this point, we have at most (i-1) twisted corner pieces and total twists of those corner pieces are again multiples of 360° since we applied total 360° twists on them. Repeat this process inductively, and finally we end up with the last single corner piece has multiples of 360° twist, but it is equivalent to no twist. This proves the first part of Claim 7. By the same argument as above and Conjecture 1, the second part of Claim 7 can be proved.

In the same manner, we obtain similar result for edge pieces

Let Y_e to denote a configuration of the cube in which each edge piece of Y_e occupies the same position as START configuration, but some of the edge pieces (more than two) are flipped compared to START configuration.

Claim 7': There exists a sequence of basic moves of the cube which restores Y_e back to START configuration if total number of flips of the edge pieces in Y_e is even. And there is no such sequence of basic moves of the cube if total number of flips of the edge pieces in Y_e is odd.

The proof is analogous to Claim 7.

Conclusion

In order to determine whether a given configuration, denote X, can be restored to START configuration by a sequence of basic moves of the cube.

1. Check the parity of overall permutation of the piece positions which is carried out between X and START configuration.

If it is odd, there is no sequence of basic moves of the cube which restores X to START (Claim 3). Otherwise, there exist a sequence of basic moves of the cube which restores X to a configuration X', where each piece of X' occupies the same position as START configuration. (Claim 6)

Assume we have found such a sequence of basic moves of the cube and obtained the configuration X' so that we could observe how each piece of X' is oriented comparing to START.

2. Check the orientation of each piece of X' comparing to START.

If total twists of the corner pieces of X' are multiples of 360° and total number of flips of the edge pieces are even, there exists a sequence of basic moves of the cube which restores X' back to START (Claim 7 and Claim 7'). Otherwise, there is no way to restore X' back to START without disassembling the cube. (Claim 7 and Claim 7')

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