The Towers of Hanoi: The Case of Four Towers.

Augustine P. Waters
Research Experience for Undergraduates
Oregon State University

Summer 1992

Directors:

Prof. Dennis Garrity

Prof. Paul Cull

Prof. Jim Simpson

Prof. Robby Robson

Introduction

In 1939 B.M. Stewart[1] proposed the following problem;

> "Given a block in which are fixed k pegs and a set of n washers, no two alike in size, and arranged on one peg so that no washer is above a smaller washer. What is the minimum number of moves in which the n washers can be placed on another peg, if the washers are moved one at a time, subject always to the condition that no washer be placed above a smaller washer?
>
> For $k = 3$ this problem is called the Towers of Hanoi ...and the solution is $2^n - 1$".

A solution was presented in 1941 by J.S. Frame[2] and part of the solution follows;

> "Halfway through the process of moving the n washers, the largest washer lies alone on its original peg, and the chosen final peg is free to receive it. The other n-1 washers are distributed among the $h = k-2$ auxiliary pegs and we may assume that the n_1 largest of these washers are on the first auxiliary peg, the next n_2 on the next ..., and the n_h smallest are on the last auxiliary peg. In some cases the solution requiring the least number of moves is unique in others it is not."

J.S. Frame's assumption will be the basis for this paper. While it appears to be quite logical I have not found a proof of it's correctness, and I will attempt to prove it for the case where $k = 4$ (i.e., Pegs = 4).

Definitions

Redefinition's: From this point on,

the term **disk** will be used in lieu of washer,

the term **tower** in lieu of peg,

and the term **work tower** will be used in lieu of auxiliary tower.

Labeling: The following conventions will be used;

Towers one through four will be labeled **T1, T2, T3, T4** respectively.

Disks will be labeled in increasing size from **1** to **n.**

Definition 1: The **Source Tower** will be the tower on which the n disks are originally placed.

Definition 2: The **Destination Tower** will be the tower onto which the n disks will ultimately be transferred.

Definition 3: The **Start Configuration** will be the situation where all n disks are arranged, in order from smallest on top to largest, on the source tower and no disk has been moved.

Definition 4: A **Halfway Configuration** will be any point in the solution of the 4 Towers of Hanoi problem (4ToH) where the following is true;

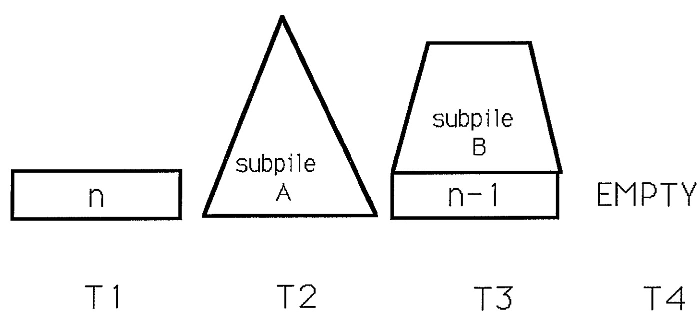
i)The largest (n^{th}) disk is isolated on the source tower

ii)The destination tower is empty.

iii)The remaining n-1 disks are distributed over the remaining two work towers.

Definition 5: A **Minimal Halfway Configuration** will be a halfway configuration which cannot be duplicated in fewer moves from the start configuration.

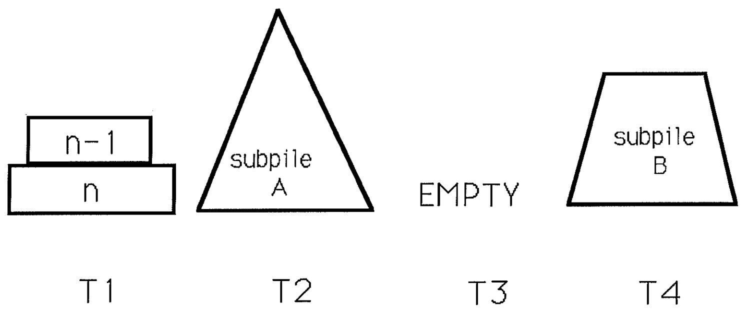
Minimal Halfway Configuration



Definition 6: A **Subpile** will be a subset of the original n disks distributed on a single tower.

Definition 7: A **Deconstruction** will be a function which takes a minimal halfway configuration for n disks and produces a halfway configuration for n-1 disks, which may or may not be minimal.

Deconstruction from Minimal Halfway Configurati



Definition 8: A **Reconstruction** will be a function which takes a halfway configuration, produced by some deconstruction which may or may not be minimal, and restores a minimal halfway configuration.

Methodology

The logical place to start any examination of this problem is at the Halfway Configuration suggested by J.S.Frame. For the four tower scenario this configuration can be represented by figure 1.

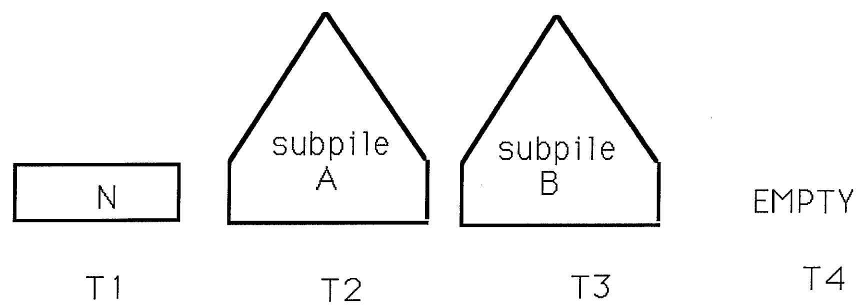


fig 1

N represents the largest disk and it is situated on T1. The destination tower is T4 and it is empty(i.e.., free of disks). The two work towers, T2 and T3 have subpiles A and B arranged on them respectively. Frame assumed that the smaller disks were arranged on one of these work towers and the larger arranged on the other. For example, subpile A might contain disks 1 through m and subpile B would contain disks m+1 through n-1.

Using this assumption I wrote a computer program that would calculate the minimal number of moves if this was indeed the case. Given a value of n it would calculate the moves required to solve the puzzle for every distribution of the n-1 disks over the two work towers abiding by Frame's assumption. This effectively divided the problem into three parts;

i) Solving the 4 tower sub problem for the subpile with the smaller disks

ii) Solving the 3 tower sub problem for the subpile with the larger disks

iii) Moving the largest disk from the source tower to the open destination tower.

If the expression T(n,k) represents the minimal solution for the Towers of Hanoi problem given n disks and k towers, then we can express Frame's assumption for the four tower problem as;

$$T(n,4) = 2T(n_1,4) + 2T(n_2,3) + 1 \qquad \text{Eq (1)}$$

where $n_1 + n_2 + 1$ and $n >= 1$

T(n,4) was calculate d from the following information;

$$T(n,3) = 2^n - 1 \qquad \text{Eq (2)}$$

$$T(0,4) = 0 \qquad \text{Eq (3)}$$

$$T(1,4) = 1 \qquad \text{Eq (3)}$$

Therefore Eq (1) can be expressed as;

$$T(n,4) = 2T(n_1,4) + 2(2^{n_2} - 1) + 1$$

$$T(n,4) = 2T(n_1, 4) + 2^{n_2+1} - 1 \qquad \text{Eq (4)}$$

By selecting the minimal value(s) generated by this function (i.e., Eq 4) I produced table 1 which follows.

n	T(n,4)
0	0
1	1
2	3
3	5
4	9
5	13
6	17
7	25
8	33
9	41
10	49
15	129
21	321
28	769

Table 1

This is just a small sample of the results obtained and they agree with those of Cull & Eklund[3] and Boardman, Garrett, & Robson[4]. Cull & Eklund derived the closed form for generalized Towers of Hanoi (ToH) problem (i.e., k >= 3) and the closed form for 4ToH follows;

$$((3s-1)/2)\,2^{s-1} + 2^{s}(n - (s+2)(s+1)/2) \qquad \text{Eq. 5}$$

Boardman et al compiled tables recording the minimum number of moves necessary to solve the generalized ToH problem when Frame's assumption is adhered to. At the very least I now had an upper bound on minimality for this problem and I could now search the solution space for a contradiction to Frame's assumption. I wrote another program which would search this solution space for small values of n, and arrived at the following conjectures.

Conjecture 1: For n-1 <= k (in this case n-1 <= 4) the disks may be arranged in a random manner at the minimal halfway configuration and still produce a minimal solution.

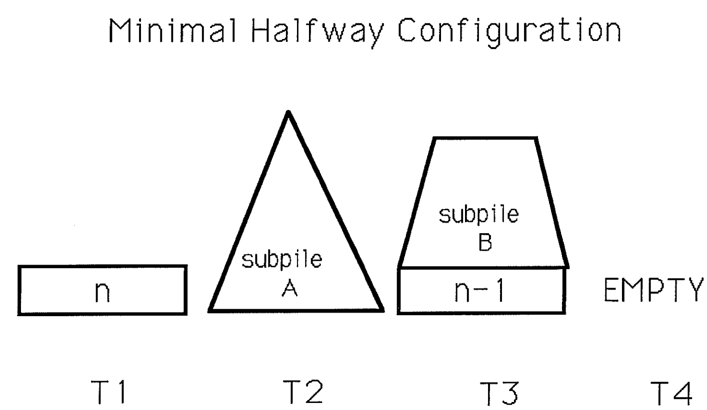
Conjecture 2: For n-1 > k The disks must be arranged in order on the two working towers at the minimal halfway configuration, with all the disks on one of the working towers being larger than those on the opposing work tower.

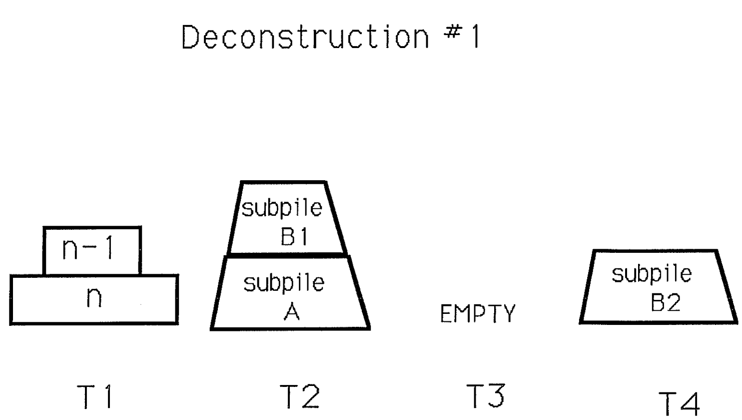
I will not prove conjecture 1 explicitly but the reader should immediately see that it is a consequence of the fact that the cost of moving two disks is independent of the number of towers when k >= 3. The second conjecture is far more involved and will be presented as a theorem and then subsequently proved.

<u>Theorem 1</u>: Every minimal solution to the four Towers of Hanoi problem must generate a minimal halfway configuration where the smallest n-1 disks are distributed over the two work towers in such a manner that all the disks on one of the work towers are smaller than the those on the opposing work tower.

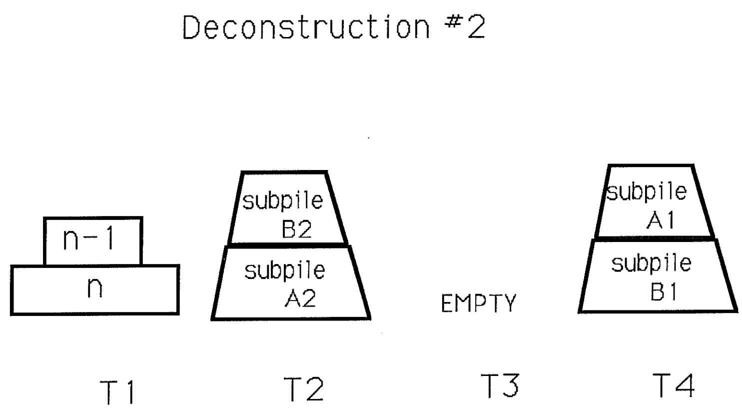
Proof of Theorem 1:

Part 1: In this section of the proof I will show that when given a minimal halfway configuration the disks of one subpile are transferred as a unit to a single tower for minimality to be preserved. This will be accomplished by contradiction. There are two possibilities to consider, the first involves moving only one subpile but dividing it up between two towers. The other distributes both subpiles over two towers. All deconstruction's will be evaluated by the cost of attaining the halfway configuration. for n-1. I will use the function C() to represent this cost symbolically. The cost of moving the n-1 disk will be omitted as it is constant throughout the proof.





If B1 is non empty then clearly it would require fewer moves to attain a minimal halfway configuration by simply moving B2 onto tower 3 after the n-1 disk had been returned to tower 3. However this would contradict our assumption that the original configuration was minimal, therefore in this instance B1 must be empty for our assumption to hold.



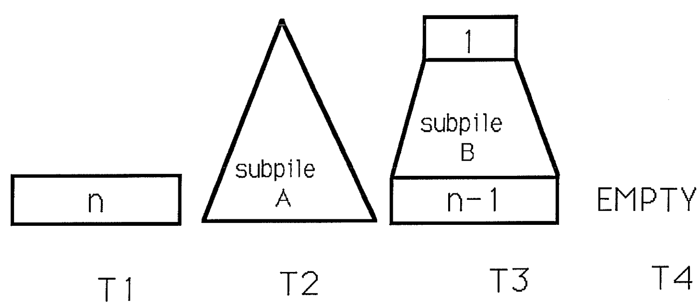
In Deconstruction #2 Subpiles A and B are divided into subpiles A1, A2 and subpiles B1,B2 respectively such that B1 < B2, and A1 < A2. By this I mean all the disks in A1 are smaller than those in B1 etc. There are six possible arrangements of the n-1 disks on the two work towers;

B2 contains the largest disks:	(1)	A1 < A2 < B1 < B2
	(2)	A1 < B1 < A2 < B2
	(3)	B1 < A1 < A2 < B2
A2 contains the largest disks:	(4)	B1 < B2 < A1 < A2
	(5)	B1 < A1 < B2 < A2
	(6)	A1 < B1 < B2 < A2

The only time the elements of subpiles A and B are actually "mixed" is when A1 < B1 < B2 < A2, otherwise the deconstruction generated is equivalent to deconstruction #1. The cost of attaining this configuration is C(B1) +C(A1) +C(B2). However the cost of reconstructing a minimal halfway configuration would only be C(A1) + C(B1) which is clearly less than the deconstruction cost unless B2 is empty. If that is the case then deconstruction #1 applies and we still have a contradiction of our original assumption. Therefore only one subpile may be transferred to a single tower for minimality to be preserved when deconstructing a minimal halfway configuration.

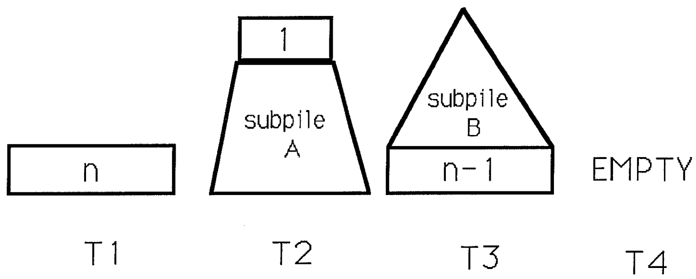
Part 2: Given a minimal halfway configuration (fig 1) there are two cases that can occur involving the smallest (1) and second largest disk (n-1). Case #1 occurs when the smallest and the n-1 disk are on the same work tower;

CASE #1



and Case #2 occurs when the smallest and the n-1 disk are on opposite work towers;

CASE #2

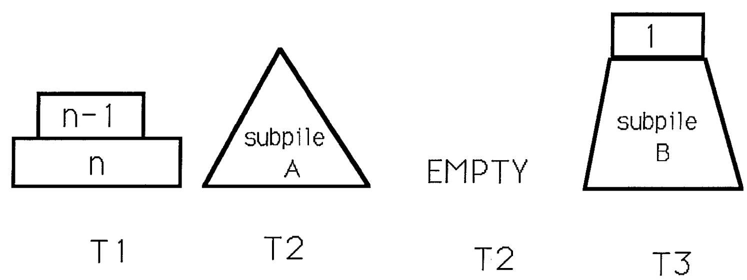


What I will prove now is that Case #2 must hold for the halfway configuration to indeed be minimal. I will use an exhaustive proof by contradiction to show that case #1 cannot hold. This will be accomplished by deconstructing and reconstructing the case #1 configuration in such a manner as to require fewer moves thereby contradicting the original assumption that case #1 configuration is minimal.

There are a total of two possible deconstruction's and each will be listed with their reconstruction's immediately following. It is assumed that both are done in a minimal fashion. All reconstruction's will be evaluated by the cost of restoring the minimal halfway configuration. I will use the function C() to represent this cost symbolically.

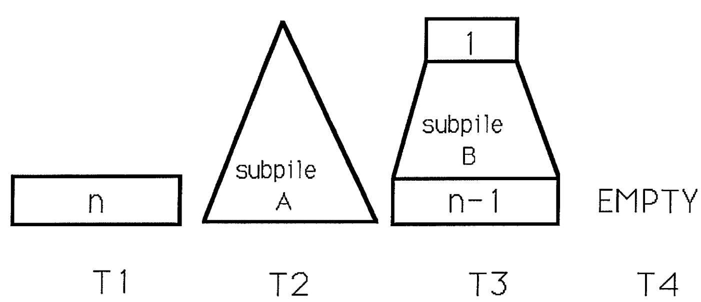
Deconstruction #3 is accomplished by moving subpile B and the smallest disk as a unit to the free tower then returning the n-1 disk to the source tower.

Deconstruction #3

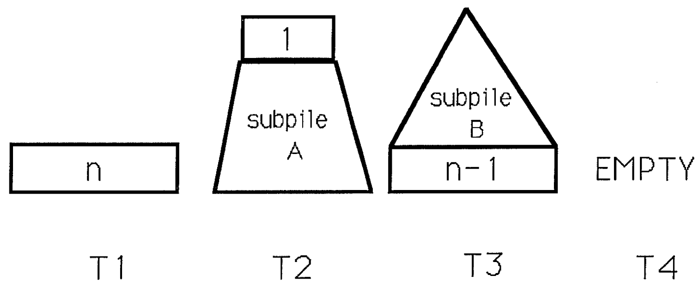


Since it is assumed that the deconstruction is performed in a minimal manner there are two possible reconstruction's.

Reconstruction #3a



Reconstruction #3b



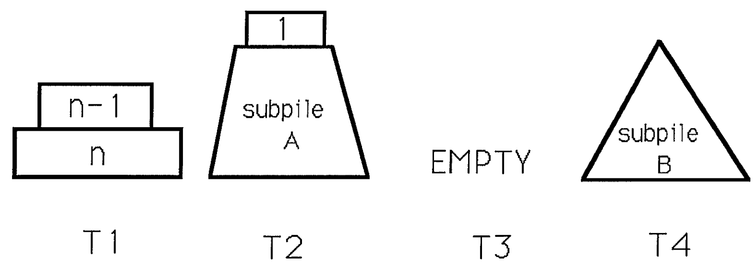
The cost of Reconstruction #3a is C(B+1)

The cost of Reconstruction #3b is C(B)

Clearly #1b requires fewer moves since C(B) < C(B+1). This contradicts the assumption that the original halfway configuration was minimal.

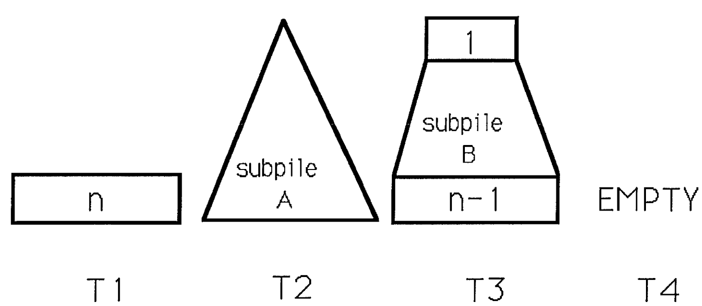
Deconstruction #4 is accomplished by moving the smallest disk onto subpile A and then moving subpile B as a whole to the free tower.

Deconstruction #4

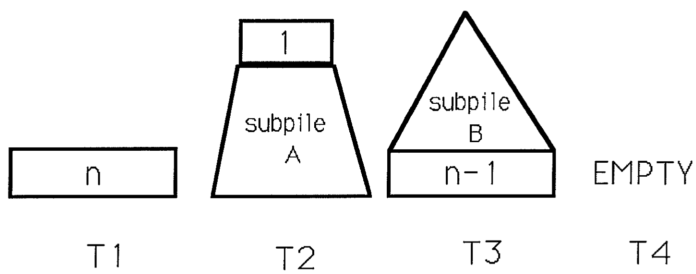


Again there are two reconstructions.

Reconstruction #4a



Reconstruction #4b

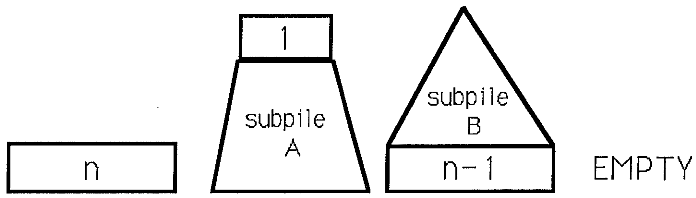


Cost of Reconstruction #4a is	C(B)+1
Cost of Reconstruction #4b is	C(B)

Again Case #1 is non minimal. Therefore since every deconstruction and reconstruction of Case #1 produces a contradiction of the assumption of minimality for the halfway configuration Case #2 must hold.

Part 3: What I would like to show now is that the disks on the two work towers are arranged in order and that all the disks on one of these towers must be smaller than those on the opposing work tower. This is a direct consequence of Parts 1 and 2 of this proof. It must be emphasized that a minimal solution for n disks is not necessarily generated from a configuration of disks for the n-1 disks' minimal solution. Therefore a minimal halfway configuration for n disks when deconstructed may not produce the minimal halfway configuration for n-1 disks, however this is still the minimum solution path for n disks. This situation occurs when subpile B is empty and the disks from subpile A must be distributed over the two empty towers after the n-1 disk has been transferred to the source tower.

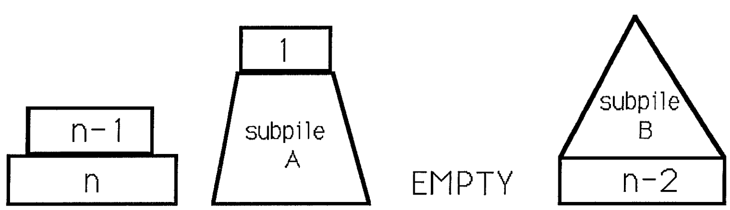
Minimal Halfway Configuration



When B is non empty, for the next deconstruction to be minimal the smallest and n-2 disks must be on separate work towers as well.

Halfway Configuration for n-1

(May be nonminimal)



Conclusions: While I am fairly certain of the results in parts 1 & 2 of my proof, part 3 appears to have some holes which need to be filled and I will attempt to correct this outside of the scope of this program. It would also be desirable to generalize this solution for $k >= 3$ towers.

References

1. B.M. Stewart, *American Mathematical Monthly* , Vol. 46, June-July 1939.

2. J.S. Frame, *American Mathematical Monthly,* Vol. 48, March 1941.

3. Paul Cull and E.F. Eklund, *Congressus Numerantium,* Vol. 35 December, 1982.

4. J.T. Boardman, C. Garrett, G.C.A. Robson, *The Computer Journal,* Vol. 29, No. 2, 1986.