A Construction of the Universal Menger Curve

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The goal of this paper is to construct the Universal Menger Curve, μ_1 from copies of the dyadic solenoid. Dranishnikov proved in 1986 that on every compact Menger manifold there exists a free action of the p-adic integers, and what is more, of any zero-dimensional compact group [3]. So, μ_1 will always have a group action but this example will differ from some others in that there is an effective group action at every stage of the inverse limit. This paper contains two preliminary constructions, **X** and **Y**, and the final construction, **Z**, which is homeomorphic to μ_1. In 1984 Bestvina [1] proved that:

A space X is homeomorphic to μ_1 if and only if it is:
(1) compact and 1-dimensional
(2) path connected
(3) locally path connected
(4) satisfies the disjoint arcs property

Some definitions:

A space S is locally path connected, if for any point s in S and any neighborhood U of s, there is a path connected neighborhood V of s contained in U.

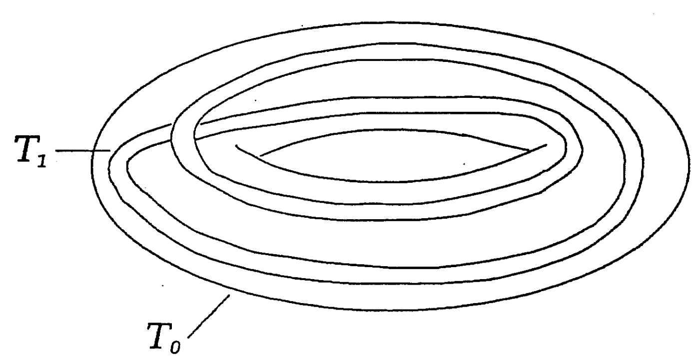
A space S satisfies the disjoint arcs property if given any ϵ and any path g in S, there exists a g^ϵ in S such that $g \bigcup g^\epsilon = \emptyset$ and $d(g, g^\epsilon) < \epsilon$.

The function $f_n : X_{N+1} \to X_n$ is a covering map if for each point in X_n, there exists a connected open neighborhood U, such that each component of $f_n^{-1}(U)$ is mapped homeomorphically on to U by f_n.

The group of dyadic integers, G, is the inverse limit $Z_2 \leftarrow Z_4 \leftarrow Z_8 \leftarrow \cdots$ (i.e. each element is an infinite sequence, $g_1, g_2, ...$ where $g_{i+1} \bmod 2^i = g_i$.

Dyadic Solenoid:

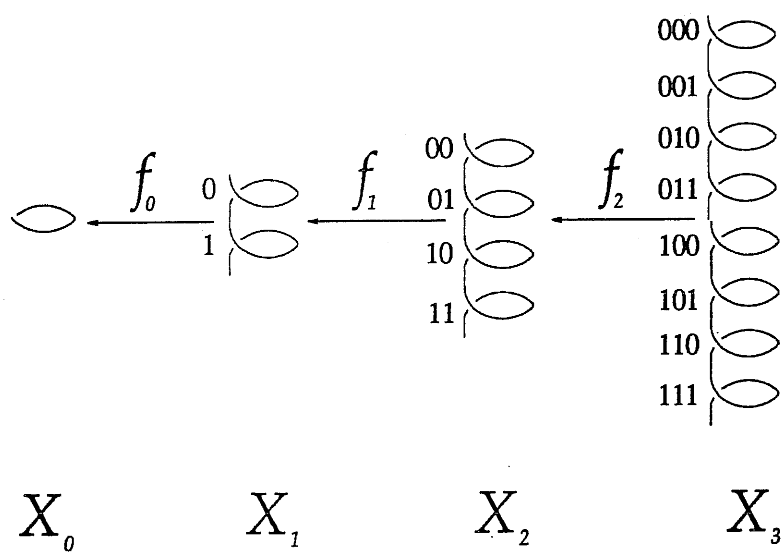
The dyadic solenoid is the infinite intersection of nested tori where the $(n+1)$st torus wraps twice around inside n-th one.



This intersection can be represented as the inverse limit of the system $(T_n, g_n, \mathbf{N})$ where the bonding maps are inclusions. It can also be represented by the inverse system circles where the $(n+1)$st circle is the double cover of the n-th one [2]. The second form of the inverse limit is the one that will be used throughout this paper. Only intuitive references will be made to the first form. The dyadic solenoid will be represented by the inverse system $(X_n, f_n, \mathbf{N})$ where each X_n is the union of loops of length one indexed by the sequence $x = \{0,1\}^n$ and joined together in a spiral by the following identifications:

$$[x, 1] \sim [x+1, 0]$$

where in X_n, $x = (x_n x_{n-1} ... x_1)$ (this sequence will be referred to as the y sequence.) and $x + 1 \equiv$ a binary addition mod2^n.
Note: The symbol + will be used both for regular addition and binary addition.



X_{n+1} is the double cover of X_n. Notice that points in X_{n+1} on the first half (first loop) of the cover have $x_{n+1} = 0$ while those in the second half have $x_{n+1} = 1$.

The bonding map $f_n : X_{n+1} \to X_n$ is defined:

$$f_n([x,t])_{n+1} = [\overline{x}, t]_n$$

where $\overline{x} = x \bmod 2^n$ or simply, truncation of the leftmost digit in the y sequence.
f_n is well defined: where one loop is joined to another, i.e. when $[x,1]_{n+1} \sim [x+1,0]_{n+1}$, then $f([x,1]_{n+1}) = [\overline{x},1]_n$ and $f([x+1,0]_{n+1}) = [\overline{x+1},0]_n$
$\overline{x+1} = (x+1) \bmod 2^n = (x \bmod 2^n) + 1 = \overline{x} + 1$, so

$$f([x,1]_{n+1}) = [\overline{x},1]_n \sim [\overline{x}+1,0]_n = [f([x+1,0]_{n+1})$$

That the f_n's are continuous follows from standard results on quotient spaces and the details will not be given. Continuity will also be assumed for all other maps in this paper.

A point in the inverse limit, X, is also of form $[x, t]$ where the loop number x is an infinite sequence of 0's and 1's and $t \in \mathcal{I}$ denotes the position on the loop. The loop numbers of $\mathbf{X}$ are, in fact, a binary representation of G so there is a transparent effective group action of G on X given by:

$$g([x, t]) = [g + x, t] \text{ where } g \in G$$

Similarly, there is an effective group action on every X_n by the binary numbers $\bmod 2^n$.

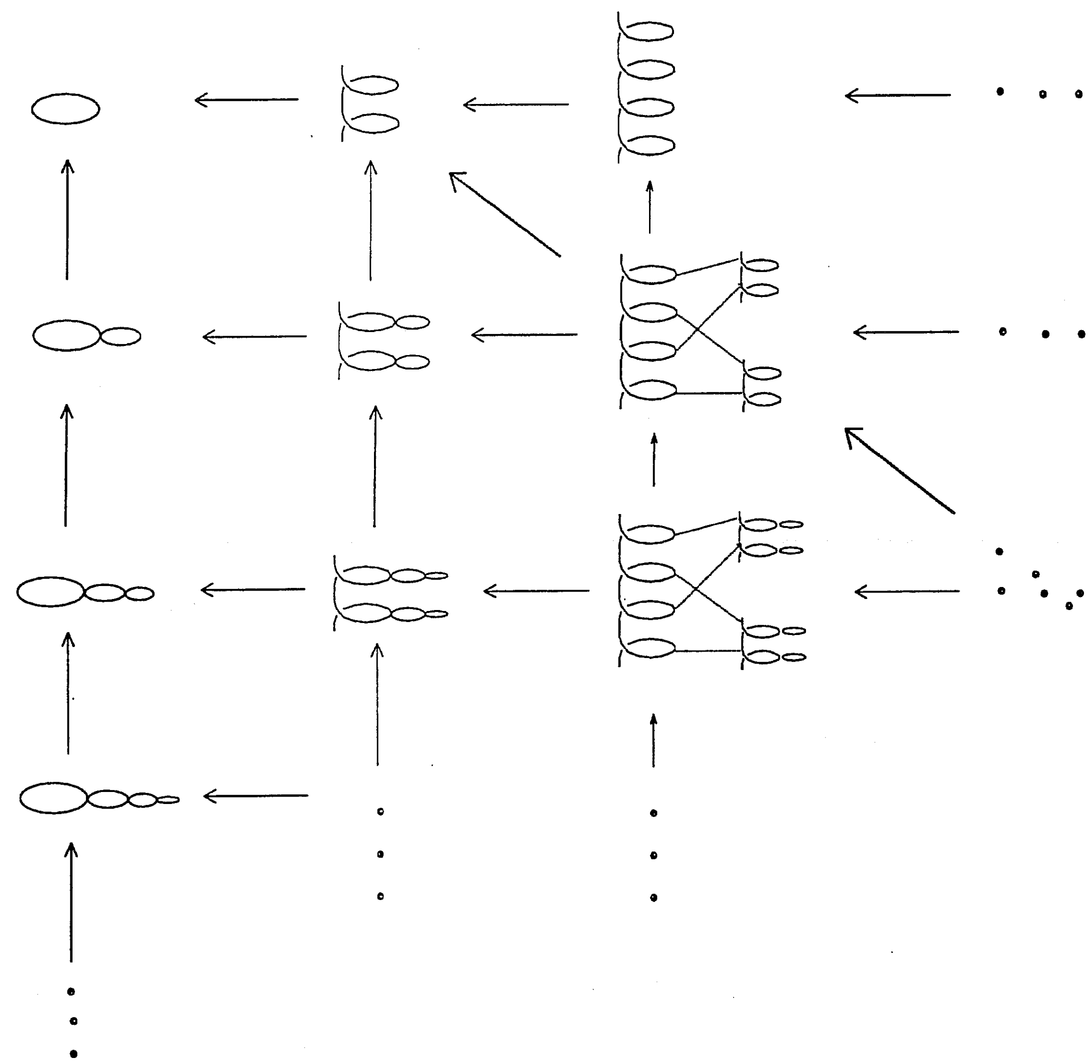
1. COMPACT 1-DIMENSIONAL

The inverse limit of the product of compact spaces is compact, and the inverse limit of the product of 1-dimensional spaces is 1-dimensional, so all constructions will be compact and 1-dimensional.

2. PATH CONNECTED

The dyadic solenoid is not path connected. The proof is omitted.

We will now construct the preliminary path connected space $\mathbf{Y}$. A simple way to make a path connection would be to identify all points $[x, 0]$ to each other for all possible x. But, this would ruin the effective group action. For any group element g, the action $g([x, t]) \rightarrow [x + g, t]$ would map all points of form $[x, 0]$ to themselves. In an effort to keep the group action transparent, we wish to construct $\mathbf{Y}$ from unadulterated copies of $\mathbf{X}$. The following construction was proposed by R. Edwards and D. Garity.



The horizontal maps are the familiar double covers. In the vertical direction the $(n+1)$st space is a copy of the nth space with a loop added to the point with $t=\frac{1}{2}$ of each loop in the nth column. The vertical map simply maps these loops back to the points with $t=\frac{1}{2}$ of the loop to which they are attached. All other points are mapped bijectively. $\mathbf{Y}$ is the diagonal inverse limit where the diagonal map is simply the composition of the corresponding vertical and horizontal maps. This gives a structure where Y_{n+1} is a double cover of Y_n with additional sprouts of X_1.

In the new space, Y_n is the set of all equivalence classes

$$[k,q,x,t]_n$$

where

$k\in\{0,1,...,n-1\}$ is the column number,

$q\in\{0,1\}^k$ where $\{0,1\}^0$ gives a blank is the coil number,

$x \in \{0,1\}^{n-k}$ is the loop number,

$t \in I$ is the position of a point on a loop.

It will be useful to let $q = (y_1...y_k)$ and $x = (y_n...y_{k+1})$ because both k and q of a point in Y_n are determined by the sequence $(y_1...y_n)$ with the distribution of digits between x and q dependent on the column number k.

There are two types of identifications:

Type **(1)** connects the loops to form a coil:

$$[k,q,x,1]_n \sim [k,q,x+1,0]_n$$

where $y+1$ is a binary addition $\mathrm{mod}2^{n-k}$

At this stage we have a disjoint union of coils with 2^k coils of length 2^{n-k} in each column k.

Type **(2)** identification joins the point $t = \frac{1}{2}$ of each loop in column k to the point $t = 0$ of a loop in column $k+1$ with identical y sequence:

$$\left[k,q,y,\frac{1}{2}\right]_n \sim [k+1, qy_{k+1}, \underline{x}, 0]_n$$

where $qy_{k+1} \equiv (y_1...y_k y_{k+1})$ and $\underline{x} \equiv [\frac{x}{2}]$ or truncation of the rightmost digit.

Now define the bonding map $f : Y_{n+1} \to Y_n$

$$f_n([k,q,x,t]_{n+1}) = \begin{cases} [k,q,\overline{x},t]_n & \text{if } k < n \\ [k-1,\underline{q},y_n,\frac{1}{2}]_n & \text{if } k = n \end{cases}$$

where $\overline{x}$ is again left truncation of x and $\underline{q}$ is right truncation of q.

There are several cases to check to insure that f_n is well defined.

Type (1) identification: $[k,q,x,1]_{n+1} \sim [k,q,x+1,0]_n$

case 1: $k < n$

$$f([k,q,x,1]_{n+1}) = [k,q,\overline{x},1]_n, \quad f([k,q,x+1,0]_{n+1}) = [k,q,\overline{x+1},0]_n,$$

$\overline{x+1} = x+1 \bmod 2^{n-k} = x \bmod 2^{n-k} = 1 = \overline{x}+1$ (stipulating that in X_n, the x entry is always $\mathrm{mod}2^{n-k}$), so

$$f([k,q,x,1]_{n+1}) \sim f([k,q,x+1,0]_{n+1})$$

by type (1) connection.

case 2: $k = n$

$$f([k,q,x,1]_{n+1}) = [k-1,\overline{q},q_k,\frac{1}{2}]_n, \quad f([k,q,x+1,0]_{n+1}) = [k-1,\overline{q},q_k,\frac{1}{2}]_n,$$

and these are identical points.

Type (2) identification: $\left[k,q,y,\frac{1}{2}\right]_n \sim [k+1, qy_{k+1}, \underline{x}, 0]_n$

case 1: $k+1 < n$

$$A = f([k,q,x,\frac{1}{2}]_{n+1}) = [k,q,\overline{x}\frac{1}{2}]_n, \quad B = f([k+1,qx_1,\underline{x},0]_{n+1}) = [k+1,qx_1,\overline{x^*},0]_n$$

if $x = x_{n+1-k}...x_1$, then $\overline{x} = x_{n-k}...x_1$, and $\overline{x^*} = x_{n-k}...x_2$ so $A \sim B$ by type (2) identification.

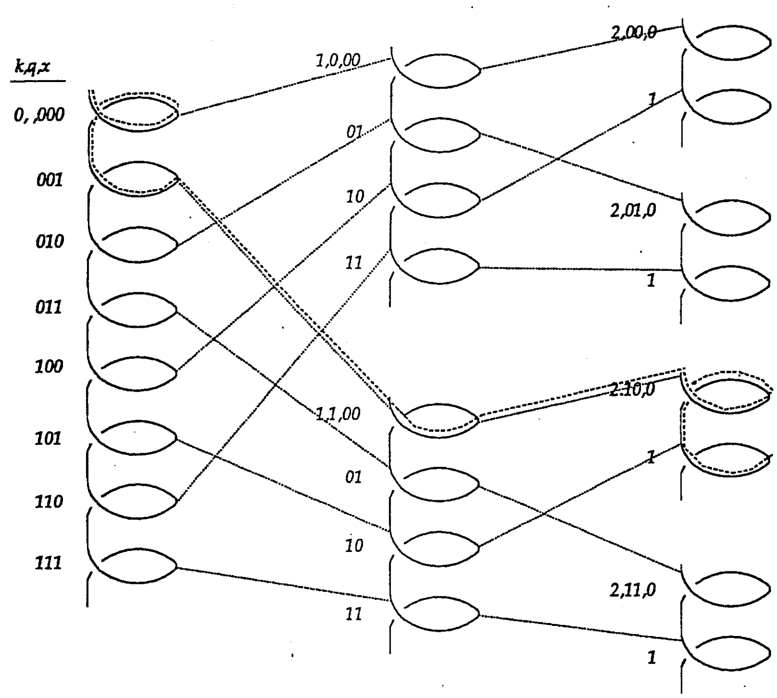
case 2: $k+1 = n$

$$A = f([k,q,x,\frac{1}{2}]_{n+1}) = [k,q,\overline{x},\frac{1}{2}]_n, \quad B = f([k+1, qx_1, \underline{x}, 0]_{n+1}) = [k, \underline{qx_1}, x_1, \frac{1}{2}]_n$$

Notice $\underline{qx_1} = q$ and x has $n+1-k$ digits, or $(k+1)+1-k = 2$ digits, so $x = x_2x_1$ and $\overline{x} = x_1$ so $A = B$. A point in the inverse limit, Y, can be either column finite or column infinite. A column finite point has form $[k, y_1...y_k, ...y_{k+1}, t]$ where k is finite. A column infinite point is the inverse limit of points $[n-1, y_1...y_{n-1}, y_n, \frac{1}{2}]_n$ for all n and has no finite column number. All points in Y have an infinite y sequence. Claim: $\mathbf{Y}$ is path connected.

This is equivalent to the statement: There exists a path from the point $\mathcal{O} = [0, \_, 0..., 0]$ to any other point $\mathcal{T}$ in $\mathbf{Y}$. (A path between any two points of $\mathbf{Y}$ is then just the path from $\mathcal{O}$ to the first taken in reverse, followed by the path from $\mathcal{O}$ to the second.) A path from $\mathcal{O}$ to $\mathcal{T}$ will also be broken into two. The first part, $I : \mathcal{I} \to \mathbf{Y}$, is from $\mathcal{O}$ to the point $V_{\mathcal{T}}$ where $V_{\mathcal{T}}$ is the column infinite point with the same y sequence as $\mathcal{T}$. The second part is a path $J : \mathcal{I} \to \mathbf{Y}$ from $\mathcal{T}$ to $V_{\mathcal{T}}$.

Below is an example of a path in Y_3 from $p_3(\mathcal{O}) = [0, , 000, 0]$ to $p_3(V_{\mathcal{T}}) = [2, 10, 1, \frac{1}{2}]$ where p_n is the projection from $\mathbf{Y}$ onto Y_n. Notice that $y_{\mathcal{O}} = 000...$ and $y_{\mathcal{T}} = 101...$ Starting at $\mathcal{O}$, the path travels down one loop in the 0 column to get $y_1 = 1$, remains in the same loop in column 1 to get $y_2 = 0$, and travels down one in column 2 to get $y_3 = 1$. The path then remains at $p_3(V_{\mathcal{T}})$ on $[\frac{1}{2}, 1]$.



I is the inverse limit of paths $I_n : \mathcal{I} \to Y_n$ connecting $p_n(\mathcal{O})$ to $p_n(V_\mathcal{T})$. By inverse limit we mean $p_n(I) = I_n$ and $f_n(I_{n+1}) = I_n$. The existence of the I_n's will be shown by induction.

1. There exists a path $I_1 : \mathcal{I} \to Y_1$ from $p_1(\mathcal{O})$ to $p_1(V_\mathcal{T})$ with $I_1\left[\frac{1}{2}, 1\right] = p_1(\mathcal{T})$.
define:

$$g_1(w) = \begin{cases} [0,, 0, w)]_1 & \text{if } y_1 = 0,\ w \in [0, \frac{1}{2}) \\ [0,, 0, 4w]_1 & \text{if } y_1 = 1,\ w \in [0, \frac{1}{4}] \\ [0,, 1, 2(w - \frac{1}{4})]_1 & \text{if } y_1 = 1,\ w \in [\frac{1}{4}, \frac{1}{2}) \end{cases}$$

If $y_1 = 0$ then the path goes from 0 to $\frac{1}{2}$ on the top loop of Y_1. If $y_1 = 1$ the path goes along the entire top loop and goes from 0 to $\frac{1}{2}$ on the bottom loop.

$$h_1(w) = \left[0, \_, y_1, \frac{1}{2}\right] \text{ for } w \in \left[\frac{1}{2}, 1\right]$$

$$I_1(w) = \begin{cases} g_1(w) & \text{if } w \in [0, \frac{1}{2}) \\ h_1(w) & \text{if } w \in [\frac{1}{2}, 1] \end{cases}$$

2. Assume there exists a path I_n in Y_n from $p_n(\mathcal{O})$ to $p_n(V_\mathcal{T})$ with $I_n\left[\frac{2^n-1}{2^n}, 1\right] = p_n(V_\mathcal{T})$.
Claim: There exists a path in Y_{n+1} from $p_{n+1}(\mathcal{O})$ to $p_{n+1}(V_\mathcal{T})$ with $I_{n+1}\left[\frac{2^{n+1}-1}{2^{n+1}}, 1\right] = p_{n+1}(V_\mathcal{T}$ and $f_n(I_{n+1}) = I_n$.
define:

$$g_{n+1}(w) = \begin{cases} \mathcal{F}_n^{-1}(g_n) & \text{if } w \in \left[0, \frac{2^n-1}{2^n}\right) \\ \left[n, y_1 ... y_n, 0, 2^n\left(w - \frac{2^n-1}{2^n}\right)\right] & \text{if } y_{n+1} = 0,\ w \in \left[\frac{2^n-1}{2^n}, \frac{2^{n+1}-1}{2^{n+1}}\right) \\ \left[n, y_1 ... y_n, 0, 2^{n+2}\left(w - \frac{2^n-1}{2^n}\right)\right] & \text{if } y_{n+1} = 1,\ w \in \left[\frac{2^n-1}{2^n}, \frac{2^{n+2}-3}{2^{n+2}}\right] \\ \left[n, y_1 ... y_n, 1, 2^{n+1}\left(w - \frac{2^{n+2}-3}{2^{n+2}}\right)\right] & \text{if } y_{n+1} = 1,\ w \in \left[\frac{2^{n+2}-3}{2^{n+2}}, \frac{2^{n+1}-1}{2^{n+1}}\right) \end{cases}$$

$$h_{n+1}(w) = \left[n, y_1 ... y_n, y_{n+1}, \frac{1}{2}\right] \text{ for } w \in \left[\frac{2^{n+1}-1}{2^{n+1}}, 1\right]$$

where $\mathcal{F}_n^{-1}$ is the lift of g_n path starting at $p_{n+1}(\mathcal{O})$. (Because f_n is a double cover, $\mathcal{F}_n^{-1}(g_n)$ gives two paths in Y_{n+1}.) Technically this can be written: $\mathcal{F}_n^{-1}([k, q, x, t]) = [k, q, 0x, t]$ where $0x = 0y_n ... y_{k+1}$ (y_{n+1} is set to 0)

$$I_{n+1}(w) = \begin{cases} g_{n+1}(w) & \text{for } w \in \left[0, \frac{2^{n+1}-1}{2^{n+1}}\right) \\ h_{n+1}(w) & \text{for } w \in \left[\frac{2^{n+1}-1}{2^{n+1}}, 1\right] \end{cases}$$

This is continuous because,

$$\lim_{w \to \frac{2^n-1}{2^n}} \mathcal{F}_n^{-1}(g_n) = \left[n-1, y_1 ... y_{n-1}, 0y_n, \frac{1}{2}\right] \sim [n, y_1 ... y_n, 0, 0] = g_{n+1}\left(\frac{2^n-1}{2^n}\right)$$

and

$$\lim_{w \to \frac{2^{n+1}-1}{2^{n+1}}} g_{n+1}(w) = \left[n, y_1 ... y_n, y_{n+1}, \frac{1}{2}\right] = h_{n+1}\left(\frac{2^{n+1}-1}{2^{n+1}}\right)$$

Also,
$f_n(I_{n+1}) = I_n$:
$f_n(\mathcal{F}_n^{-1}(g_n)) = g_n$ because $\mathcal{F}_n^{-1}(g_n) \subset vf_n^{-1}(g_n)$
and $f_n(I_{n+1} \backslash \mathcal{F}_n^{-1}(g_n)) = h_n$

If T is the inverse limit of points of form $[n-1, y_1 ... y_{n-1}, y_n, \frac{1}{2}]$ for all n, then $\mathcal{T} = V_\mathcal{T}$ and the path is complete. For any other $\mathcal{T}$, the second part of the path (J) from $\mathcal{T}$ to $V_\mathcal{T}$ is also necessary. $\mathcal{T}$ can now be written $[R, q, x, t]$ where R is some finite column number. Because the y sequences of $\mathcal{T}$ and $V_\mathcal{T}$ are the same, and any loop in column k is joined to the loop with identical y sequence in column $k-1$, the path J goes from $\frac{1}{2}$ to 0 on each loop with y sequence identical to $\mathcal{T}$.

J is the inverse limit of paths $J_n : \mathcal{I} \to Y_n$ connecting $p_n(\mathcal{T})$ to $p_n(V_\mathcal{T})$. The existence of the J_n's will also be shown by induction. For n where $R \geq n$, $p_n(\mathcal{T}) = p_n(V_\mathcal{T})$ and J_n is trivial. So the path and first inductive step will start with Y_{R+1}.

1. There exists a path $J_{R+1} : \mathcal{I} \to Y_{R+1}$ from $p_{R+1}(\mathcal{T})$ to $p_{R+1}(V_\mathcal{T})$ with $JR+1\left[\frac{1}{2}, 1\right] = p_{R+1}(V_\mathcal{T})$. Note: $p_{R+1}(\mathcal{T}) = [R, y_1 ... y_R, y + R + 1, t]$

define

$$g_{R+1}(w) = [R, y_1 ... y_R, y_{R+1}, w(1-2t) + t] \text{ for } w \in \left[0, \frac{1}{2}\right)$$

$$h_{R+1}(w) = \left[R, y_1 ... y_R, y_{R+1}, \frac{1}{2}\right] \text{ for } w \in \left[\frac{1}{2}, 1\right]$$

$$J_{R+1}(w) = \begin{cases} g_{R+1}(w) & \text{for } w \in \left[0, \frac{1}{2}\right) \\ h_{R+1}(w) & \text{for } w \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Assume for $m > R+1$ there exists a path J_u in Y_u from $p_u(\mathcal{T})$ to $p_{u+1}(V_\mathcal{T})$ with $J_u\left[\frac{2^{u-R}-1}{2^{u-R}}\right] = p_u(\mathcal{T})$.

Claim: There exists a path J_{u+1} in Y_{u+1} from $p_{u+1}(\mathcal{T}$ to $p_{u+1}(V_\mathcal{T})$ such that $f_u(J_{u+1}) = J_u$.

define

$$g_{u+1}(w) = \begin{cases} \mathcal{K}_u^{-1}(g_u) & \text{for } w \in \left[0, \frac{2^{u-R}-1}{2^{u-R}}\right) \\ \left[u, y_1 ... y_u, y_{u+1}, 2^{u-R}\left(w - \frac{2^{u-R}-1}{2^{u-R}}\right)\right] & \text{for } w \in \left[\frac{2^{u-R}-1}{2^{u-R}}, \frac{2^{u-R+1}-1}{2^{u-R+1}}\right) \end{cases}$$

$$h_{u+1}(w) = \left[u, y_1 ... y_u, y_{u+1}, \frac{1}{2}\right] \text{ for } w \in \left[\frac{2^{u-R+1}-1}{2^{u-R+1}}, 1\right]$$

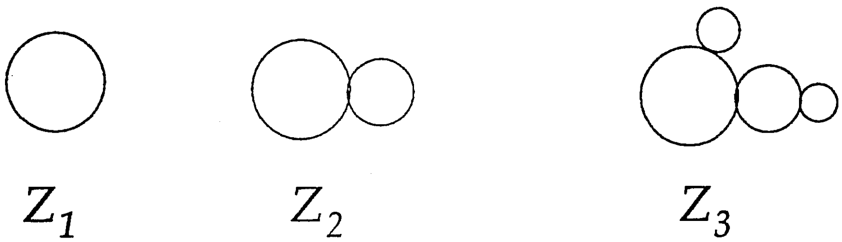
where $\mathcal{K}_u^{-1}$ is the lift of g_u which is path connected to $p_{u+1}(\mathcal{T})$. Technically : $\mathcal{K}_u^{-1}[k, q, x, t] = [k, q, x_{u+1}x, t]$

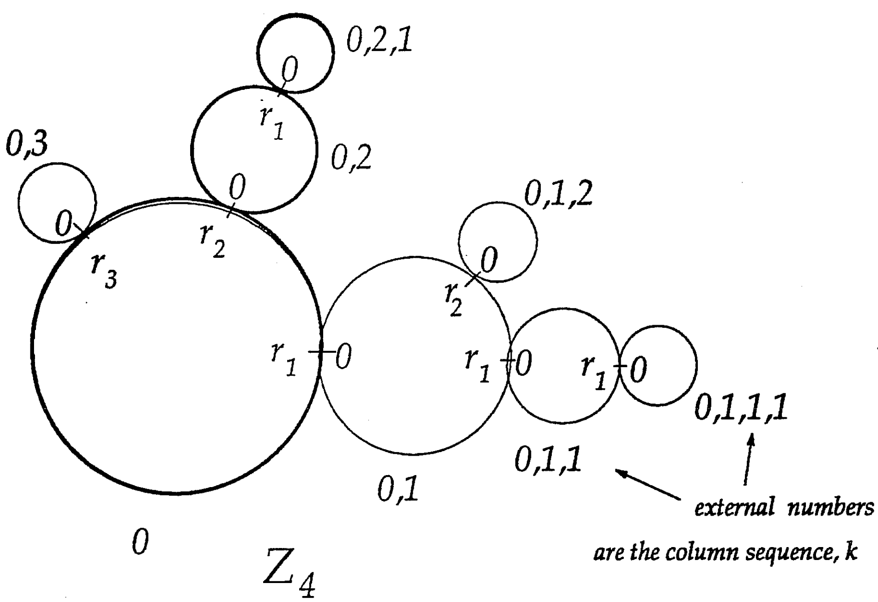
$$J_{u+1}(w) = \begin{cases} g_{u+1}(w) & \text{for } w \in \left[0, \frac{2^{u-R+1}-1}{2^{u-R+1}}\right] \\ h_{u+1}(w) & \text{for } w \in \left[\frac{2^{u-R+1}-1}{2^{u-R+1}}, 1\right] \end{cases}$$

That J_{u+1} is continuous and that $f_u(J_{u+1}) = J_u$ follow by similar arguments to those for I.

3. LOCALLY PATH CONNECTED

Let $\mathcal{V} = \{p_n^{-1}(V) | n \in \mathcal{N}$ with V an open set in $X_n\}$, then $\mathcal{V}$ is a basis for **X**. This is true of any inverse system [C]. In **Y** the basis elements are generally not path connected. So now we will construct the final space Z which will be locally path connected. First, order the rationals with the function $r : \mathcal{Q} \to \mathcal{N}$. Where Y_{n+1} was a double cover of Y_n with 2^n coils of length two added, Z_{n+1} is a double cover of Z_n with 2^n coils of length two added to each existing column at the next free rational. The following picture is a top view with each circle representing a column.





In this space Z_n will again be the set of all equivalence classes:

$$[k, q, x, t]_n$$

where now

$k = k_0, k_1, ...k_l$ with $l \in \{0, ..., n\}$, $k_0 = 0$ and $\sum_{i=0}^{l} k_i = s_l$ where $s_l \leq n - 1$
$q \in \{0,1\}^{s_l}$
$x \in \{0,1\}^{n-s_l}$
$t \in \mathcal{I}$

The only changes in the notation are that the k that appeared in **Y** as a column number is now a sequence and where k appeared in **Y** as an index, it is now replaced by s_l, the summation of the new sequence k. Again there are two types of connections.

Type **(1)** connections (the connecting of loops) hold exactly as before:

$$[k, q, x, 1]_n \sim [k, q, x + 1, 0]_n$$

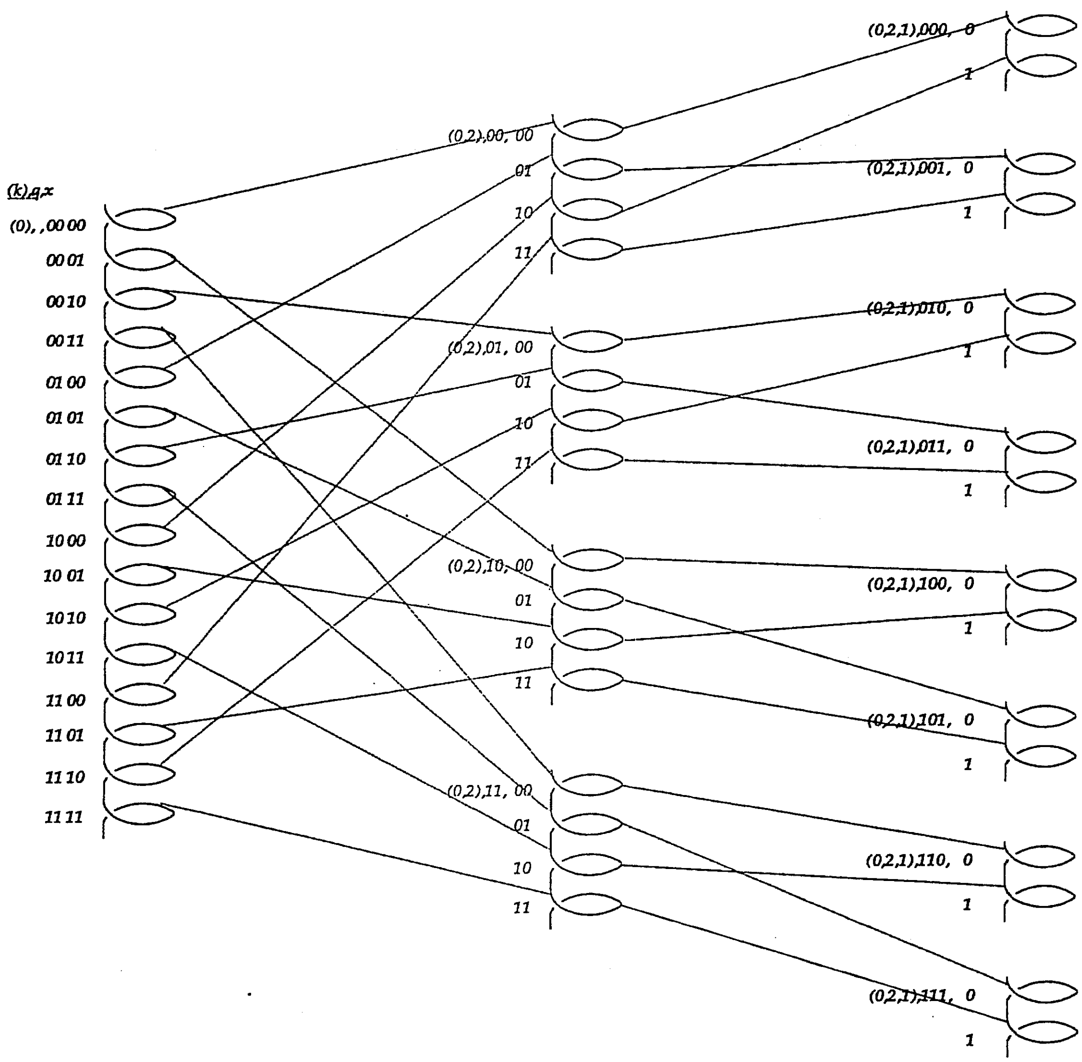
where $x + 1$ is binary addition mod2^{n-s}.

Type **(2)** connections are modified as follows:

$$[k, y_1...y_{s_l}, y_n...y_{s_l+1}, r(m)] \sim [km, y_1...y_{s_l+m}, y_n...y_{s_l+m+1}, 0]$$

provided that $s_l + m < n$

The following picture is a sideview of only the boldface columns of the previous picture.



The map f_n for the case $s_l < n$ is the same as for case $k < n$ in $\mathbf{Y}$.

$$f_n[k,q,x,t]_{n+1} = [k,q,\overline{x},t]_n$$

$$\overline{x} = x \text{ with left truncation.}$$

for case $s_l = n$

$$f_n[(k_0,...,k_l), y_1...y_s, y_{n+1}, t] = [(k_0,...,k_{l-1}), y_1...y_{s-k_l}, y_n...x_{s-k_l=1}, r(k_l)]$$

The next goal is to show that $\mathbf{Z}$ is locally path connected. First, it will be useful to show that $\mathbf{Z}$ is path connected. The path construction will be very similar to that for $\mathbf{Y}$. To make the similarities more visible, I will use many of the same letter names. The first path I will be from $\mathcal{O}$ to the point $V_\mathcal{T}$ where $V_\mathcal{T}$ is now defined as the inverse limit of points $p_n(V_\mathcal{T}) = [(k_0,...,k_l), y_1...y_{s_l}, y_n...y_{s_l} - 1, r(k_{l+1})]$. If the k component of $\mathcal{T}$ is finite: $(k_0,...,k_m)$ then let $k_i = 1$ for all $i > m$. Again the y sequence of $V_\mathcal{T}$ is equal to the y sequence of $\mathcal{T}$. I is the inverse limit of paths $I_n : \mathcal{I} \to Y_n$ connecting $p_n(\mathcal{O})$ and $p_n(V_\mathcal{T})$. Existence of the I_n's is shown by induction.

1. There exists a path,I_{s_1} in Z_{s_1} from $p_{s_1}(\mathcal{O})$ to $p_{s_1}(V_\mathcal{T})$ with $I_{s_1}\left[\frac{1}{2},1\right] = p_{s_1}(V_\mathcal{T})$. Note: $s_1 = k_1$

define

$$g_{s_1}(w) = \begin{cases} [0,-,0_k...0_1 + [2Rw], 2Rw - [2Rw]] & \text{for } w \in \left[0, \frac{R-1}{2R}\right] \\ [0,-,y_{k_1}...y_1, r(k_1)2R[w - \frac{R-1}{2R}] & \text{for } w \in \left[\frac{R-1}{2R}, \frac{1}{2}\right) \end{cases}$$

where $(y_k...y_1) + 1 = R$

$$h_{s_1}(w) = [0,-,y_{k_1}...y_1, r(k_1)] \text{ for } w \in \left[\frac{1}{2},1\right]$$

$$I_{s_1}(w) = \begin{cases} g_{s_1}(w) & \text{for } w \in \left[0,\frac{1}{2}\right) \\ h_{s_1}(w) & \text{for } w \in \left[\frac{1}{2},1\right] \end{cases}$$

$$f_{j-1}(I_j) = I_{j-1} \text{ for } 1 < j \leq s_1$$

2. Assume there exists a path in $Z_{s_{m-1}}$ from $p_{s_{m-1}}(\mathcal{O})$ to $p_{s_{m-1}}(V_\mathcal{T})$ with $I_{s_m-1}\left[\frac{2^{s_m-1}-1}{2^{s_m-1}}\right] = p_{s_m-1}(V_\mathcal{T})$ and $f_{j-1}(I_j) = I_{j-1}$ for $1 < j \leq s_{m-1}$. $\sum_{i=0}^{l} k_i = s$.

Claim: There exists a path in Z_{s_m} from $p_{s_m}(\mathcal{O})$ to $p_{s_m}(V_\mathcal{T})$ with $I_{s_m}\left[\frac{2^{s_m}-1}{2^{s_m}},1\right] = p_{s_m}(V_\mathcal{T}$ and $f_{j-1}(I_j) = I_{j-1}$ for $1 < j \leq s_m$. Define:

$$g_{s_m}(w) = \begin{cases} \mathcal{F}^{-1}_{s_m-1}((g_{s_m-1})(w)) & \text{for } w \in \left[0, \frac{2^{s_m-1}-1}{2^{s_m-1}}\right] \\ [k,q,(0_{s_m}...0_{s_{m-1}+1})[\mathcal{R}], R - [\mathcal{R}] & \text{for } w \in \left[\frac{2^{s_m-1}-1}{2^{s_m-1}}, \frac{(2^{s_m}-1)(R-1)}{2^{s_m}R}\right] \\ \left[k,q,y_{s_m}...y_{s_{m-1}+1}, Rr(k_m)\left(\frac{2^{s_m}}{2^{s_m}-1}\right)\left(w - \frac{(2^{s_m}-1)(R-1)}{2^{s_m}R}\right)\right] & \text{for } w \in \left[\frac{(2^{s_m}-1)(R-1)}{2^{s_m}R}, \frac{2^{s_m}-1}{2^{s_m}}\right) \end{cases}$$

where

$k = (k_0,...,k_m)$

$q = (y_1 ... y_{s_m - 1})$
$\mathcal{R} = 2^{s_m} R\left(w - \frac{2^{s_m-1}-1}{2^{s_m-1}}\right)$

$$h_{s_m}(w) = [k, q, y_{s_m} ... y_{s_{m-1}+1}, r(k_m)] \text{ for } w \in \left[\frac{2^{s_m}-1}{2^{s_m}}, 1\right]$$

$$I_{s_m}(w) = \begin{cases} g_{s_m}(w) & \text{for } w \in \left[0, \frac{2^{s_m}-1}{2^{s_m}}\right) \\ g_{s_m}(w) & \text{for } w \in \left[\frac{2^{s_m}-1}{2^{s_m}}, 1\right] \end{cases}$$

If $\mathcal{T}$ is a column infinite point, the path is complete. Otherwise a path J from $V_{\mathcal{T}}$ to $\mathcal{T}$ can also be constructed by going from $r(m)$ to 0 in each column traversed by I. Details are omitted!

Now we need to show that given any point $\alpha = [k, q, x, t]$ in $\mathbf{Z}$ and a neighborhood $\mathcal{W}$ of α, there exists a nbhd. $\mathcal{U}$ with $\alpha \in \mathcal{U}$ and $\mathcal{U} \subset \mathcal{W}$ such that $\mathcal{U}$ is path connected. Because $\mathcal{V}$ is a basis we can pick a $\mathcal{V} \subset \mathcal{W}$ containing α. By definition, $\mathcal{V} = p_m^{-1}(V)$ for some m.

case 1: If α has $t = r(b)$ and $k = k_1, ... k_l$ such that $\sum_{i=0}^{l} k_i + b < m$, then pick a $\hat{U}$ in X_m such that $\alpha \in \hat{U}, \hat{U} \subset V$ and $\hat{U}$ contains no point with $t = r(a)$ and $a < b$. Let β be the point in $\hat{U}$ with $t = r(c)$, where $r(c)$ is the next rational (i.e. c is the least number such that $c \leq b$.) Now let $w = s_l + c$ and let U be the component of $p_w(p_m^{-1}U)$ containing $p_w(\alpha)$. U is homeomorphic to an X. One arc, D, is in column k and the other is in column $k, r(b)$ (or $k_1, ..., k_l, r(b)$). The two arcs intersect at $p_w([k, q, x, r(b)]$. In general, any component of $p_n^{-1}(D)$ will intersect a component of $p_n^{-1}(E)$ in the set $\{p_n^{-1}[k, q, x, r(b)]\}$ for $n \geq w$. In Z_{w+1}, 2^w 2-coils will be added to the kth column at points with $t = r(c)$. Let Q be the 2-coil connecting the two lift components of $f_w{-}1(U)$. In Z_{w+1}, Q prevents the two lifts of U from being disjoint. $p_{w+1}^{-1}(Q) = p_w(\beta)$ is homeomorphic to the entire space $\mathbf{Z}$, so it is path connected. It connects $p_n^{-1}(D) \cup p_n^{-1}(E)$ which each have countably many sprouts homeomorphic to $\mathbf{Z}$ (i.e. $p_w^{-1}([k, q, x, r(d_i)])$ are the sprouts of $p_n^{-1}(D)$ for any $d_i \in \mathcal{N}$ such that $c < d_i$). So, $p_n{-}1(U)$ is the desired path connected nbhd. of α.

case 2: For the case of α with $t = r(b)$ where $s_l + b \geq m$ the argument is similar with $r(c) = r(b)$.

case 3: If α has an irrational t, the argument is again similar. Choose U to include a point β with $t = r(c)$ so that $s_l + c \geq m$ and there are no points in U with $t = r(a)$ such that $a < c$.

In the last two cases U (and $\hat{U}$) are homeomorphic to a single arc instead of an X.

4. DISJOINT ARCS PROPERTY

Put a metric on $\mathbf{Z}$ such that if α and β are points of $\mathbf{Z}$,

$$d(p_n(\alpha), p_n(\beta)) < 1$$

for all n and

$$D(\alpha, \beta) = sup\left\{\frac{d(p_n(\alpha), p_n(\beta))}{n}\right\}$$

Now given a path $h : \mathcal{I} \to \mathbf{Z}$ and an ϵ, $D(h, \hat{h}) < \epsilon$ if $h_m = \hat{h}_m$ for all m where $\frac{1}{m} \geq \epsilon$, and it is possible where $\frac{1}{m} < \epsilon$ for $h \neq \hat{h}$. ($h_m = p_m(h)$) Let M be the largest m such that $\frac{1}{m} \geq \epsilon$.

By the Simplicial Approximation Theorem [2], we can draw vertices on the graph of h_M and also subdivide the unit interval so that there exists a map $j_M : \mathcal{I} \to \mathbf{Z}$ such that any vertex on $\mathcal{I}$ is mapped to a vertex and any edge is mapped linearly onto an edge or onto a vertex in such a way that $d(h_M, j_M) < \epsilon_1$, and because M is finite, ϵ_1 can be chosen so that $sup\left(\frac{d(h_n,j_n)}{n}\right) < \epsilon$ for $n \leq M$. Because $j_n^\epsilon = j_n$ for $n \leq M \leq R$, $D(j^\epsilon, j) < \epsilon$. $\mathcal{I}$ is finitely subdivided into vertices and edges, denoted by $i_1, ..., i_s$. Because of this there are only a finite number of pairs such that $j_M(i_s) = j_M(i_r)$ with $s \neq r$. For $n \geq M$, define j_{n+1} to be the lift of j_n starting at $f_n^{-1}(j_n(0))$ with $y_{n+1} = 0$. If $j(i_s) \neq j(i_r)$ then there exists a u such that $j_u(i_s) \neq j_u(i_r)$ for $n \geq u$. We take $R = max(u)$ over all pairs. Define j_n^ϵ as follows: for $n \leq R$ $j_n^\epsilon = j_n$. For $n > R$ define j_{n+1}^ϵ to be the lift of j_n starting at $f_n^{-1}(j_n(0))$ with $y_{n+1} = 1$.

Claim: $j_{R+1}^\epsilon \bigcap j_{R+1} = \emptyset$, therefore $j^\epsilon \bigcap j = \emptyset$. If $j^\epsilon \bigcap j \neq \emptyset$, then there exists $j_{R+1}^\epsilon(i_s) = j_{R+1}(i_r)$.

case 1: $r = s$

We know that $j_{R+1}^\epsilon(0) \neq j_{R+1}(0)$ so the following lemma provides a contradiction.

Lemma (uniqueness of lifts): Let (Z_{R+1}, f_R) be a covering space of Z_R. Given any two continuous maps, $j_{R+1}^\epsilon, j_{R+1} : \mathcal{I} \to Z_{R+1}$ such that $j_R^\epsilon = j_R$, the set $\{b \in \mathcal{I} | j_{R+1}^\epsilon(b) = j_{R+1}(b)\}$ is either empty or all of $\mathcal{I}$. See [4] for proof.

case 2: $r \neq s$

$j_{R+1}^\epsilon(i_s) = j_{R+1}(i_r)$ so $j_R^\epsilon(i_s) = j_R(i_s) = j_R(i_r)$

$j_{R+1}(i_s) = j_{R+1}(i_r)$ follows because $j^\epsilon(i_s)$ must equal $j(i_r)$.

Substitute the first line into the third and it reduces to the first case: $j_{R+1}^\epsilon(i_s) = j_{R+1}(i_s)$

So $j^\epsilon \bigcap j = \emptyset$.

Conclusion:

Z is the Universal Menger Curve and, as promised, there is an obvious effective group action. Because the loops, x, in **Z** (or in **Y**) are indexed by the dyadic integers and each loop is connected only to the loop $x+1$ in the same column and coil, or to a loops in different columns with identical y sequence, the group action: $g([k,q,x,t]) = [k,q,x+g,t]$ of the dyadic integers will hold on the inverse limit **Z** (or **Y**). Also, the group of binary numbers mod2^n acts effectively on Z_n (or Y_n) for the same reasons.

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