Hypercube Variant Linkages

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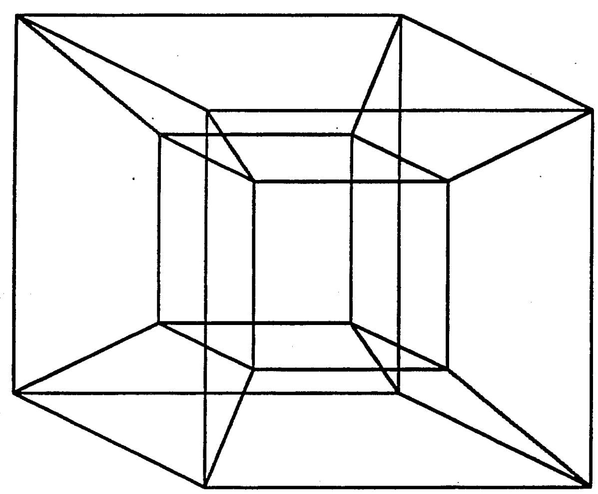


Abstract

The topic of this paper is structure systems for the linkages of computer processors to be arranged in parallel. I will explore specifically the common case where the number of processors in on the order of 2^n where n is the *dimension* of the system.

Dimension: *Dimension of a system is the minimum number of bits required to uniquely determine a binary address for each node (or vertex, or computer).*

Introduction

This paper will address some of the dynamics of some types of of these linking systems with the purpose of characterizing what makes a system that works for this type of linking. I will explore some types of examples of small networks ($n = 1, 2, 3$) and the generalized models proposed by others as efficient (minimal diameter and average distance), and try to generalize aspects that make these models work.

Diameter: *The diamter of a network is the greatest distance between any two vertices within the system. Diameter*$=$ *max* $\{\forall\, X, Y$ *vertices min* $\{\forall$ *paths connecting any vertices* X *and* $Y\}\}$

The model we are interested in is a *resource preserving variant* system of node (vertex) linkage. That is, a linear system that, using only linear manipulation of three n-by-n matrices, generates a n-dimensional structure of linked nodes. By multiplying the selector matrix by the address of a node, a binary sequence is yielded. This sequence corresponds to the columns of the base matrices to be added (mod 2) to the address to give the addresses of the neighboring nodes.

Network: *A collection of nodes (vertices) that may be connected (by edges) to each other. In this application, the nodes represent computers that should ideally be connected in parallel. In diagrams, the nodes appear as dots and are differentiated from each other by way of binary addresses. A network of dimension* n *has* 2^n *nodes of constant out-degree,* n*. A network in this context may be described as the composition of it's vertices,* V*, and it's edges,* E *as follows:*

$$V = \{Z_2\}^n \tag{1}$$

$$E = \left\{(X,Y) : X + Y = B_i^{\phi}\right\} \tag{2}$$

such that:

$\phi \in Z_2$ *and* $\phi = (AX)_i$ *for* $1 \leq \phi \leq i$.

Directed Connection: *A connection between two nodes that allows flow (of information in the computer model) strictly from one to the other (not the converse). Directed connections are represented in diagrams as vector arrows between nodes. There is a directed connection (edge) from address* X *to address* Y *if and only if the* i^{th} *column of the base matrix* B^{ϕ} *equals the difference (or sum) of the addresses of* X *and* Y *where the* i^{th} *component of the selector function operating on* X *is* ϕ*.*

Reflexive Connection: *A reflexive connection is a connection from a node directly back to itself. This type of connection is represented in diagrams as a circular vector originating and terminating at one node. A reflexive connection is formed when a column of a base matrix is selected and is equal to the zero vector.* $(X, X) \in E$ *is a reflexive connection leftrightarrow the* i^{th} *column of the* B^{ϕ} *matrix is uniformly zeros where* ϕ $=$ *the* i^{th} *element of* AX*.*

Reciprocal Connection: *A reciprocal connection is the union of two oppositely directed connection between a particular pair of node. This allows information to flow both ways between the two nodes in the computer model. Reciprocal connections are shown in diagrams as two-headed vectors connecting pairs of nodes, except in the case of reflexive connections which must also be reciprocal. There is a reciprocal connection between addresses* X *and* Y *if and only if two opposing directed connections link* X *and* Y*. If* $(X, Y), (Y, X) \in E$*, then there is a reciprocal connection between* X *and* Y*.*

Symmetric Connection: *A symmetric connection is a reciprocal connection that uses the same base column to travel both ways between addresses. That is: a symmetric connection* *exists between address X and address Y if and only if the i^{th} component of the product of the selector matrix on X equals the i^{th} component of the product of the selector matrix on Y.*

Redundant Connection: *A redundant connection is two or more directed connections running parallel, originating and terminating at the same nodes. Redundant connections are shown in diagrams as separate vectors that originate and terminate at the same nodes. Redundant connections occur when two or more identical columns are selected to be used by one particular address.*

Connected Network: *A connected network is one that for any pair of nodes, there is at least one path connecting them.*

Base Matrices: *Two-by-two binary matrices comprised of columns of numbers which may be added to an address to yield a second address; if a column from one of the base matrices is added to an address, this indicates a directed connection from the input address to the node corresponding to the sum of the address and the column. The base matrices are named B^0 and B^1 (B-zero and B-one).*

Selector Matrix: *A function that takes as its argument a binary n-bit address of a node and yields an n-bit binary number that indicates which columns from the base matrices should be added to the address to determine the directed connections from the given address. In this model, the selector function is in the form of an n-by-n binary matrix by which an address is multiplied by (the selection matrix on the left, the address on the right). In this paper, the selector function (matrix will be represented by A.*

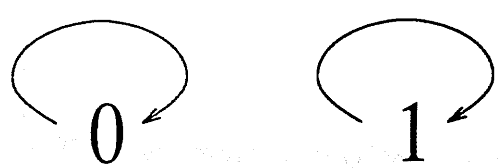
Distance: *The distance between two nodes is defined to be the smallest number of directed connections that must be followed to travel from one node to the other. That is: the minimal number of edges of the network that must be used to reach one node from the other.*

Diameter: *The diameter of the network is the maximum of all distances between all pairs of nodes in the network.*

In the simplest case, $n = 1$, implying two vertices. The corresponding addresses then, would be 0 and 1, and A, B^0, and B^1 are all one-by-one matrices. If:

$$[B^0] = [B^1] = [0] \tag{3}$$

then regardless of what A is, then the zero vector will always be added to the address and each node would be only connected to itself like this:



and the same digraph will result if:

$$[A] = [B^0] = [0] \tag{4}$$

for any (either) B^1 matrix because the selector matrix (A) would ensure that the B^1 matrix could never be selected. Notice that the connections in this digraph are both reflexive and reciprocal, but the network is not connected. If:

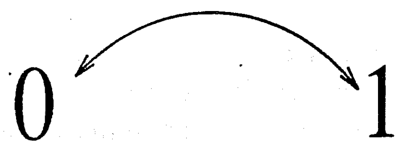
$$[A] = [1], [B^0] = [0], and [B^1] = [1], \tag{5}$$

then the system would look like this:

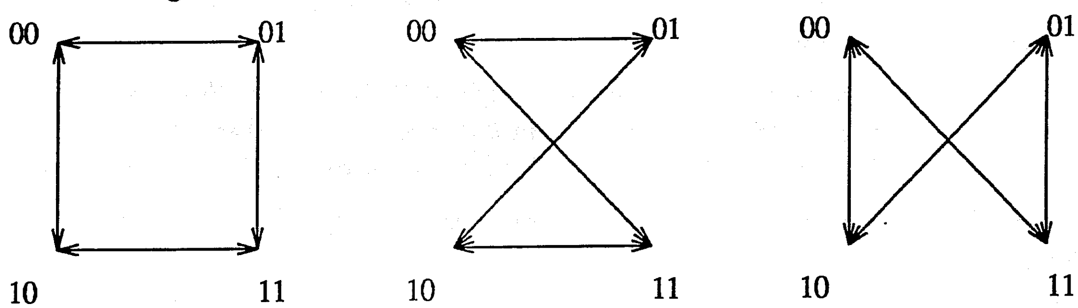
which is connected (unilaterally), and has a reflexive but no reciprocal connection. In the remaining case where:

$$[A] = [B^0] = [B^1] = [1], \tag{6}$$

the following digram results:



notice that this digram is connected, reciprocal, and has no reflexive or redundant edges.



These are the three diagrams possible for $n = 2$ such that all connections are reciprocal and the system is connected. In the left digram, either

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad and \quad B^0 \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \tag{7}$$

for any B^1 matrix (because the selector matrix precludes any column of the B^1 matrix from being used) **or:**

$$B^0 = B^1 \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \tag{8}$$

for any selector, **or:** B^1 shares exactly one column with B^0, and the selector function is one of the four possibilities (out of the sixteen total possible binary 2-by-2 matrices) that permit only that duplicated column of B^1 to be selected. For example: If

$$B^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad and \quad B^1 \in \left\{ \begin{bmatrix} 0 & X \\ 1 & Y \end{bmatrix} \right\} \tag{9}$$

for some $(X,Y) \neq (1,0)$, then:

$$A \in \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\} \tag{10}$$

or: neither column of B^1 is identical to B^0, in which case

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{11}$$

The other two cases (the center and right above) are similar to the first, varying only in the B^0 matrix (one column has exactly one 1 in it while the other is comprised of both 1's) and accordingly varying selector matrix. For each of these two remaining cases, there is a bijective transformation between the $n = 2$ hypercube (a square) and the crossed configurations. This type of relationship is called a homeomorphism.

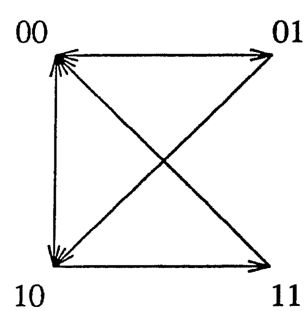
Homeomorphic: *Two digraphs are homeomorphic to each other if and only if there is a bijective continuous linear transformation from one to the other.*

Hypercube: *An n-dimensional hypercube is a network such that:*

$$E = \{(X,Y) : X,Y differ\ by\ exacly\ one\ bit\} \tag{12}$$

Now if connectedness and reciprocality are required, but the prohibition of reflexiveness is relaxed, then the verticies may be connected in series (as well as above), with selector and base matrices making each ordering possible.

If connectedness and non-reciprocality are required, but not reciprocality, other possibilities arise; example:



$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} B^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} B^1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{13}$$

The number of possibilities clearly grows as a function of freedoms allowed.

Notice that in all cases, regardless of the selector matrix, the zero vector (from the address 0 0 0...0) always uses exactly the columns of the B^0 matrix to determine it's neighbors; also notice that these three configurations are homeomorphic to the $n = 2$ hypercube (the square). You get the idea, I'm sure.

Several models of twisted cube-type structure patterns have been proposed as efficient and extendable versions for parallel computer linkage. Among them are, The Möbius Cube (both the 'zero' and 'one' versions), by Shawn Larson and Paul Cull, The M-Cube by Nitin Singhvi and Kanad Ghose, the Twisted N-Cube by P Hilbers, M. Koopman, and J. van de Snepscheut, and the Crossed Cube by Kemel Efe, and of course the basic hypercube. These systems can be represented in the resource preserving three-matrix linear system as follows (the matrices are for the $n = 8$ case, and the diagrams represent the $n = 4$ case): First the regular hypercube that employs, in any combination, exactly the columns of the n-dimensional identity matrix; the $n = 4$ diagram is as follows:

The 0-Möbius Cube:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, B^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (14)$$

The 1-Möbius Cube:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, B^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (15)$$

The M-Cube:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, B^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (16)$$

The Twisted N-Cube:

$$A = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0 \\ 1&1&1&1&1&0&0&0 \\ 1&1&1&1&1&1&0&0 \\ 1&1&1&1&1&1&1&0 \end{bmatrix}, B^0 = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}, B^1 = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&1&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix} \tag{17}$$

Our purpose is to explore the characteristics of a system like the above examples so that we may generalize and idealize them. The desireable aspects of these systems in particular were: connectedness, reciprocality of connections, non-redundantcy, non-reflexiveness and expandability. In each of the above case, note that both of the base matrices form a basis, as do three of the four selector matrices.

ul**Product Address:** *The product address is the n-by-n matrix formed by the composition of columns from the base matrices according to the product of the selector matrix, A, and the address, X.*

The condition for reflexive connections to exist in the system is for a column of the matrix formed by the conjuntion of the base matricies as a function of the selector matrix operating on the address of any vertex.

Redundant connections are formed if and only if not all columns of the product address matrix are unique for some vertex address.

Reciprocal connections may result in two ways. The first, and most simple is the case of a symmetric connection where the same column from the base matrices is employed for $E_i = (X, Y)$ and $E_j = (Y, X)$. In this case, the i^{th} element of the selector matrix operating on X, $(= AX_i)$ equals the i^{th} element of AY_i. This implies:

for $X, Y \in V$ such that: $(X, Y), (Y, X) \in E$, and $1 \leq i, j \leq n$:

$$Y = X + B_i^{AY_i} and X = Y + B_j^{AX_j} \tag{18}$$

$$\Rightarrow B_i^{AX_i} = B_j^{A\left(X + B_i^{AX_i}\right)_j} \tag{19}$$

so for $X = 000...0$,

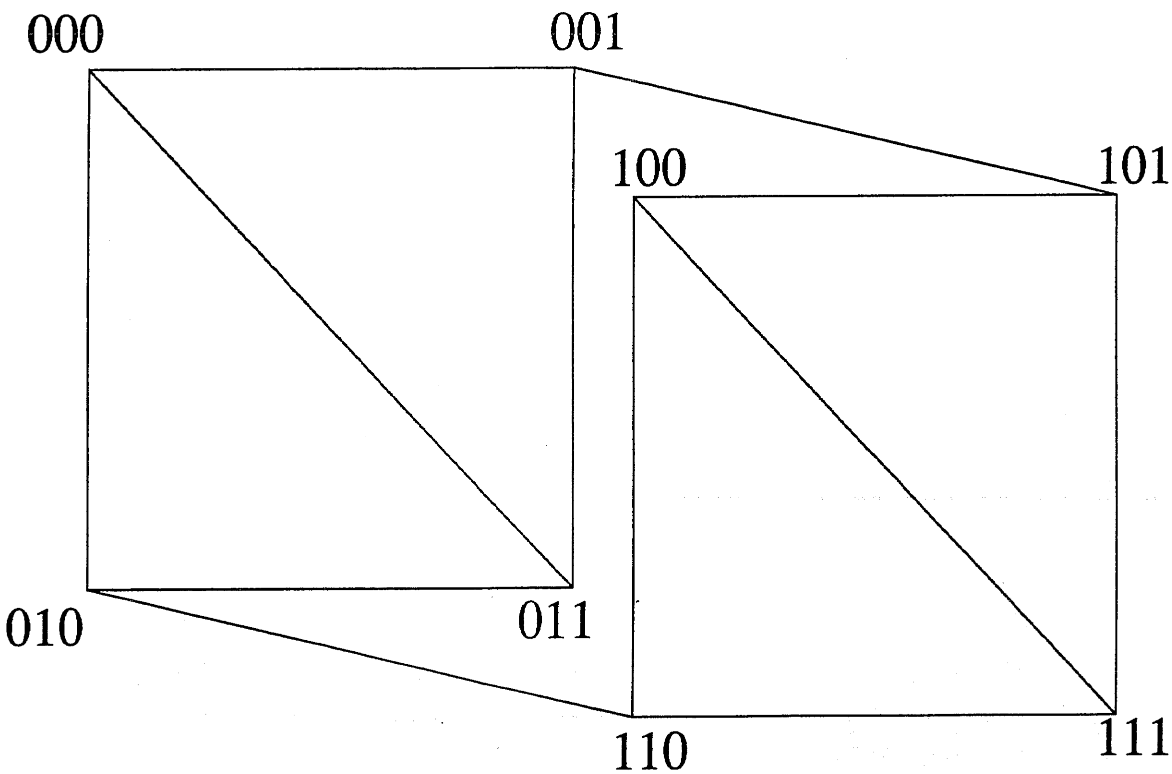
$$\Rightarrow B_i^0 = B_j^{A\left(B_i^0\right)_j} \tag{20}$$

so if the relation is symmetric, then either **case a:**

$\bullet A\left(B_i^0\right)_i$ **or case b:**

$\bullet \exists j : A\left(B_i^0\right)_j = 1 and B_i^0 = B_j^1$

Satan's Counterexample



$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad B^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad B^1 = \begin{bmatrix} \alpha & \delta & 1 \\ \beta & \epsilon & 0 \\ \gamma & \zeta & 0 \end{bmatrix} \tag{21}$$

This example (with Greek letters arbitrary) disproves the popular theory that a basis must be formed by the B^0 matrix in order that the system be connected with reciprocal connections.