KNOTS OF THE FORM $[a, b, c, d, e]$

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ABSTRACT. In this paper we develop a technique for distinguishing between knots and links which are written in Conway notation, develop a technique for determining the unknotting number of a Conway knot in its minimal projection, and provide a prime knot with an arbitrary gap.

INTRODUCTION

In 1983 S. Bleiler and Nakanishi independently found a counter exmaple to the conjecture that the unknotting number of the minimal projection of a knot was the actual unknotting number of the knot. Bernhard then expanded on this knot ($[5, 1, 4]$) to demonstrate that a family of knots of the form $[odd, even, odd]$ all have this same property[BE]. Eva Wailes expanded this research further by exmaining the properties of the entire family of knots $[a, b, c]$. Wailes's work revealed three knots whose gaps are $\geqq 1$ and at least one family of links whose gap was arbitrarily large [W]. In this paper, we have extended these ideas to the class of Conway knots of the form $[a, b, c, d, e]$. In our effort to determine the unknotting number of the two knots which we studied, we developed several things: two techniques for identifying whether a given sequence of integers written in Conway notation is actually a link or a knot, a technique for deducing the unknotting number of the minimal projection of a Conway knot, and a counter exmaple to a conjecture regarding the unknotting number of the even-continued fraction expansion of a Conway knot. Of the two 5-component families we studied, one will be shown to have a gap $\geqq 1$ and the other to have an arbitrary gap.

Notation. *Throughout this paper, when we are referring to a 5-component Conway knot, we will write the knot in the form $[a, b, c, d, e]$, where each of a, b, c, d, e are integers. When we speak of a position in this knot, we are referring to one of the five components, i.e. the fourth position refers to the integer d. Often we will desire to make changes to a sub-set of the five integers and we will denote these changes in the form $\{p, q, r, s, t\}$ where each of the variables p, q, r, s, t are integers and p is understood to signify that we are making p changes in the first position of the knot, q changes in the second position of the knot, etc.*

Definition. *We will use the notation $\equiv$ to signify that two knots are equivalent.*

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Definition. *The notation cfrac$[a, b, c, d, \dots]$ will refer to the associated contuned fraction of the knot $[a, b, c, d, \dots]$* [C].

For example, cfrac$[a, b, c]$ refers to the associated continued fraction:

$$c + \cfrac{1}{b + \cfrac{1}{a}}$$

Theorem. *Any knot written in Conway notation where all of the positions are all positive or all negative is a reduced alternating knot and is thus knotted* [K].

We will be using this fact to determine when a given knot is unknotted or not. For example, if we make a series of changes, say $\{p, q, r, s, t\}$ on a knot so that the resulting knot has a projection where all of the positions are poisitive integers then we know that this knot is still knotted.

Theorem. *Two knots written in Conway notation whose associated continued fraction are equal are neccessarily equivalent* [C].

We will use this result extensively to show that certain non-alternating projections have equivalent alternating projections and thus are still knotted. For example the knot $[1, -2, 3, 7]$ which is non-alternating has cfrac$[1, -2, 3, 7] = \frac{15}{2}$ The knot $[2, 7]$ also has cfrac$[2, 7] = \frac{15}{2}$ and thus the knots are equivalent. Since these knots are equivalent and the second is reduced alternating, then the original knot is knotted. The equivalences for the 5-component knots for the 30 permutations of positive and negative positions are computed in the Appendix.

Zero Reductions. *If a position in a knot is zero then the following is true:*

1) $[0, a, b, c, d] \equiv [b, c, d]$.
2) $[a, b, 0, c, d] \equiv [a, b + c, d]$

This can be proved using a continued fraction analysis.

Definition. *The unknotting number of a knot K is the minimal number of crossing changes required to unknot K taken across all projections of K. We label this number $U(K)$.*

Definition. *The unknotting number of the reduced alternating projection of a knot K will be called $U_{min}(K)$.*

Definition. *The Gap of a knot K is the difference between $U_{min}(K)$ and $U(K)$.*

In this paper we never compute the exact unknotting number of 5-component knot and thus never compute the actual Gap(K). Rather, we find a bound for the size of the Gap(K) by finding a non-minimal projection of K whose unknotting number is $<$ U$_{min}(K)$.

Definition. *The evenfrac of a knot K is the continued fraction of a knot which consists of either:*

1) one odd integer followed by a finite number of even integers if the Conway knot is a knot.

2) a sequence of even integers if the Conway knot is actually a link or knot Prime (Defined later).

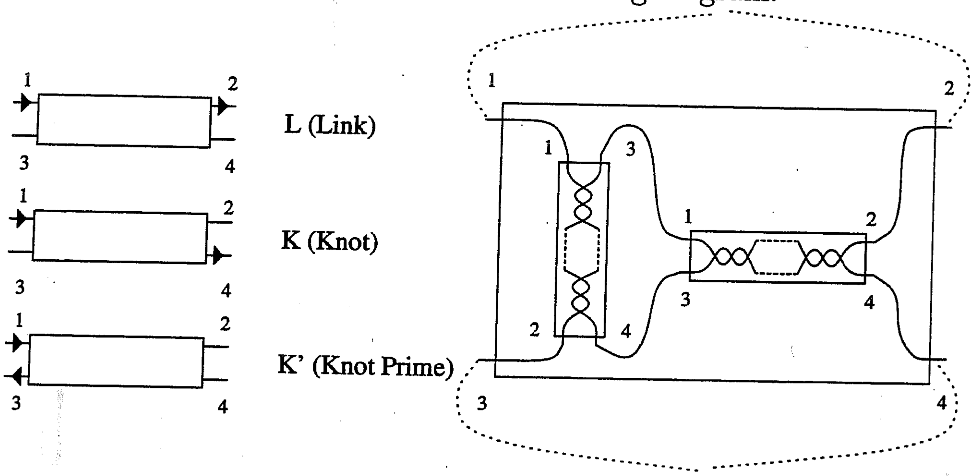
This representation of a knot is useful in that even positions can be readily reduced to zeros thus simplifying the reduction of the knot by the zero-reduction method above. Thus the even projection of a knot may possess the property that it is easier to unknot than the minimal projection. For example, the knot $[5, 1, 4]$ has an unknotting number of 3 for its minimal projection, while the evenfrac of this knot $[2, -2, 2, -2, 2, 4]$ has unknotting number of 2 which is the actual unknotting number of the knot. This property of even continued fraction is not true for every knot however, as will be demonstrated by later.

For other introductory information see previous work by Eva Wailes and Cassandra S. McGee [MW].

Map Method for Determining Conway Link Nature

In working with Conway notation, it is often convenient to be able to identify whether a given 'knot' is actually a knot or a link. In what follows we will construct a road-map which determines wheter a given 'knot' is actually a knot or a link depending only on the sequence of even and odd tangles for a given 'knot'. We will also derive a result for how many 'knots' of n-tangle length are actually knots and how many are links.

First, we need to create a set of labels which identify the behavior of a tangle or a set of rational tangles. To this end, we will think of a block, containing k tangles connected in the Conway fashion, with four nodes labelled 1,2,3,4 and and then make the observtion that there are three possible behaviors depending on the interior connections - we label these behaviors as demonstrated in the following diagram:



Block Types and a Two-Component Example

The 'knot-prime' block is named as such since the connected block will form a knot, but the internal behavior is distinct from the block which we've labeled as a 'knot'

To begin the construction, we need to consider the block consisting of one tangle. Given the nature of one-tangles, only the L and K behaviors are possible. Furthermore, it is easy to see that if the number of crossings in the tangle is odd, then node 1 will be connected to node 4 and thus the block is a knot. Likewise, if the number of crossings is even, then node 1 is connected to node 2 and the block is a link.

Now we need to consider the behavior of a two tangle block which can be formed in four ways :

$$[odd, odd]$$
$$[odd, even]$$
$$[even, odd]$$
$$[even, even]$$

Referring to the two-component example above, it can be seen that if both the components are odd, then node 1 of the block is connected to node 2 and the block is a link. If the first component is odd and the second is even, then node 1 of the block is connected to node 4 and the block behaves as a knot. Finally, if the first component is even and the second is either even or odd, then node 1 of the block is connected to node 3 and the block behaves as a knot-prime. We have, then, the following table:

$$[odd, odd] \equiv Link$$
$$[odd, even] \equiv Knot$$
$$[even, odd] \equiv KnotPrime$$
$$[even, even] \equiv KnotPrime$$

We now proceed by connecting another tangle to the blocks above, which gives us six cases to check, namely

$$[L, odd]$$
$$[L, even]$$
$$[K, odd]$$
$$[K, even]$$
$$[K', odd]$$
$$[K', even]$$

We can again refer to the two-component diagram above, treating the first internal component as one of the previous three block types. We notice that if we connect the two component link to an odd tangle then node 1 of the block will be connected to node 3 and the block will behave as a knot-prime. The same is true for adding and even tangle to a link. If we add an odd tangle to a two component knot, then node 1 of the block will be connected to node 2 and the block behaves as a link. If we add an even tangle then node 1 is connected to node 4 and the block is behaving as a knot. Finally, if we add an odd tangle to a knot Prime, then node 1 is connected to node 4 and the block behaves as a knot. And if we connect and even tangle to a knot-prime, then node 1 of the block is connected to node 2 and the block behaves as a link. The important thing to note is that the three blocks we have defined are the only types of blocks that will ever occur. Consequently, we have a closed pattern of the following form:

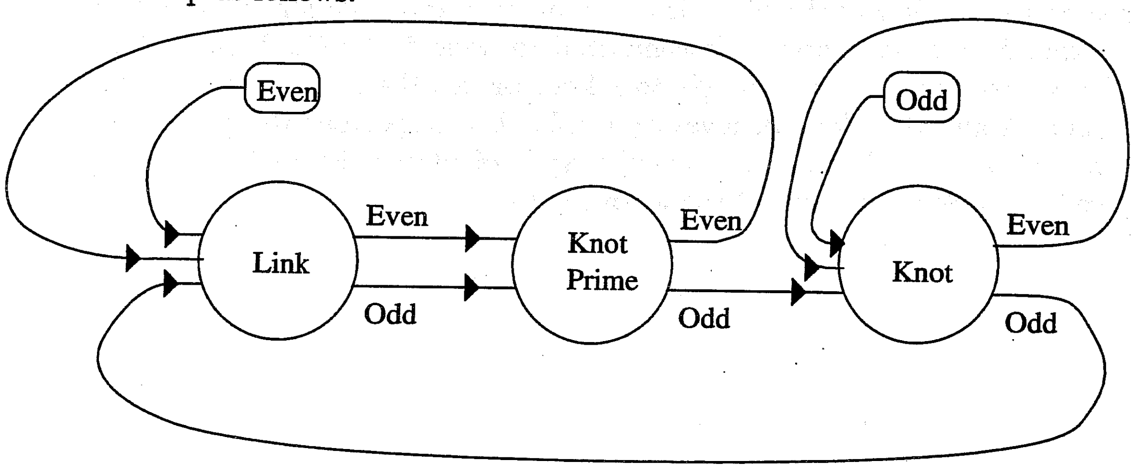
State Table

Current State	**Next Component**	**Next State**
Link	odd	Knot Prime
Link	even	Knot Prime
Knot	odd	Link
Knot	even	Knot
Knot Prime	odd	Knot
Knot Prime	even	Link

At this point we have a method of identifying a knot from a link by scanning along the components and keeping track of what the next block is. For example, the knot $[3,1,4,2,6,7]$ would be mapped as follows:

$$[3] \Rightarrow K$$
$$[K,1] \Rightarrow L$$
$$[L,4] \Rightarrow K'$$
$$[K',2] \Rightarrow L$$
$$[L,6] \Rightarrow K'$$
$$[K',7] \Rightarrow K$$

And, thus, $[3,1,4,2,6,7]$ is indeed a knot. This State table can also be visualized as a State Map as follows:



Now we would like to be able to compute that for n tangles, there are exactly m links and p knots, etc. To do this we will introduce the following notation:

$$K(n) = \text{The number of knots for n-components}$$
$$L(n) = \text{The number of links for n-components}$$
$$K'(n) = \text{The number of knot-primes for n-components}$$

Now, looking at the State Map above, we can make the following observations:

1). *Since L always goes to a knot-prime regardless of the nature of the next tangle, and since the only way for the next block to be a knot-prime is for the current block to be a link then:*

$$K'(n+1) = 2 * L(n)$$

2). *Since adding an even to an n-tangle knot will generate an n+1 tangle knot and adding an odd to an n-tangle knot-prime will generate an n+1 tangle knot and since these are the only two ways to arrive at an n+1 tangle kknot then:*

$$K(n+1) = K(n) + K'(n)$$

3). *By a similar analysis as 2), we can conclude that:*

$$L(n+1) = K'(n) + K(n)$$

Conditions 2 and 3 can be combined to give : $K(n+1) = L(n+1)$, *or with no loss of generality:*

$$L(n) = K(n)$$

4). *For any given* n *there are* 2^n *unique sequences of odd-even patterns, each of which must generate one of the three results;* L, K, K'. *Therefore:*

$$L(n) + K(n) + K'(n) = 2^n$$

The above equations can be combined and simplified to yield the difference equation:

$$L(n) = 2^{(n-1)} - L(n-1)$$

For any given n, this recursion generates a sequence such as :

$$L(n) = 2^{(n-1)} - 2^{(n-2)} + 2^{(n-3)} - \ldots.. + (-1)^{(n-1)}$$

This expression is the quotient of a polynomial division, namely, letting $x = 2$, then:

$$L(n) = x^{(n-1)} - x^{(n-2)} + x^{(n-3)} - \ldots.. + (-1)^{(n-1)}$$
$$(x+1) * L(n) = x^n + x^{(n-1)} - x^{(n-1)} - x^{(n-2)} + x^{(n-2)} + \ldots + (-1)^{(n-1)}$$

Pairwise terms cancel and the expression reduces to:

$$(x+1) * L(n) = x^n + (-1)^{(n-1)}$$

And finally:

$$L(n) = \frac{x^{(n)} + (-1)^{(n-1)}}{x+1} = \frac{2^{(n)} + (-1)^{(n-1)}}{3}$$

From this result, and the previous relations, the number of knots for n-tangles can be computed (here knots are considered to be the sum of knots and knot-primes since both behave as knots):

$$Knots(n) = \frac{2^{(n+1)} + (-1)^{(n)}}{3}$$

Continued Fraction Method For Determining Conway Link Nature

In considering whether a given knot, written in Conway notation, is actually a link, knot, or knot-prime, as defined earlier there is an easier way than the map method.

Consider the set of rationals and define the following:

$$\frac{odd}{even} = K' \qquad \text{for Knot-Prime}$$

$$\frac{even}{odd} = L \qquad \text{for Link}$$

$$\frac{odd}{odd} = K \qquad \text{for Knot}$$

Then observe that if the knot has one component, then if the component is odd we have a knot, and if it is even then we have a link. Also note that the continued fraction associated with these knots corresponds to the above definitions ($odd/1$ $even/1$).

Now we consider the four cases where we add either an even or odd component to the base even or odd component. All we are interested in is the nature of the associated continued fraction.

Let o stand for odd and e stand for even. Also notice the following properties:

$$e * e = e$$
$$e * o = e$$
$$e + 1 = o$$
$$o + 1 = e$$
$$o * o = o$$
$$o + e = o$$
$$e + e = e$$

Now consider:

$$
\begin{array}{ll}
cfrac[o,o] = o + 1/o = (oo+1)/o = e/o & \text{Which is } L \text{ by Definition} \\
cfrac[o,e] = e + 1/o = (eo+1)/o = o/o & \text{Which is } K \text{ by Definition} \\
cfrac[e,o] = o + 1/e = (eo+1)/e = o/e & \text{Which is } K' \text{ by Definition} \\
cfrac[e,e] = e + 1/e = (ee+1)/e = o/e & \text{Which is } K' \text{ by Definition}
\end{array}
$$

Now, as before, we need to consider adding an even or odd component to the above results to see the cycle:

$$
\begin{array}{rl}
cfrac[L,o] = o + 1/(e/o) = (oe+o)/e = o/e & \text{Knot-Prime} \\
cfrac[L,e] = e + 1/(e/o) = (ee+o)/e = o/e & \text{Knot-Prime} \\
cfrac[K',o] = o + 1/(o/e) = (oo+e)/o = o/o & \text{Knot} \\
cfrac[K',e] = e + 1/(o/e) = (oe+e)/o = e/o & \text{Link} \\
cfrac[K,o] = o + 1/(o/o) = (oo+o)/o = e/o & \text{Link} \\
cfrac[K,e] = e + 1/(o/o) = (oe+o)/o = o/o & \text{Knot}
\end{array}
$$

Consequently, to determine the state of a Conway knot we need merely compute the associated continued fraction and then observe which ratio of evens and odds results.

For example, the knot $[3, 4, 5, 3]$ has $cfrac = 217/68$ which is o/e so this knot is a knot-prime.

Counter-Exmaple to the Even Continued Fraction Conjecture

S. Bleiler conjectured that the unknotting number of the even continued fraction representation of a knot was the actual unknotting number of the knot [BL]. The knot $[2, 2, 1, 1, 2]$ written in Conway notation is a counter-example to this conjecture.

First, notice that the Conway continued fraction of the knot $[2, 2, 1, 1, 2]$ is :

$$
2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}} = \frac{31}{12}
$$

Now, notice that the continued fraction of the even knot $[2, -4, 2, 2]$ is also $31/12$. We know that since the continued fractions are equal, then $[2, 2, 1, 1, 2] \equiv [2, -4, 2, 2]$ [C].

Clearly, $[2, 2, 1, 1, 2]$ is knotted. Furthermore, we can unknot this knot with one crossing change, namely:

$$\{0, 0, 1, 0, 0\}$$

Which gives us the following knot and its reductions:

$$\begin{aligned}
&\equiv [2,2,-1,1,2]\\
&\equiv [2,0,1,-1,2]\\
&\equiv [3,-1,2]\\
&\equiv [1,1,0]\\
&\equiv [1] \equiv \text{UNKNOT}.
\end{aligned}$$

Now we want to show that the even knot [2,-4,2,2] can not be unknotted in one change - we need to consider only four cases, namely one change in each of the four possible positions::

Case 1: $\{1,0,0,0\}$ **on** $[2,-4,2,2]$.

$$\begin{aligned}
&\equiv [0,-4,2,2]\\
&\equiv [2,2] \equiv \text{KNOTTED}.
\end{aligned}$$

Case 2: $\{0,1,0,0\}$ **on** $[2,-4,2,2]$.

$$\begin{aligned}
&\equiv [2,-2,2,2]\\
&\equiv [3,1,2] \equiv \text{KNOTTED}.
\end{aligned}$$

Case 3: $\{0,0,1,0\}$ **on** $[2,-4,2,2]$.

$$\begin{aligned}
&\equiv [2,-4,0,2]\\
&\equiv [2,-2]\\
&\equiv [-3] \equiv \text{KNOTTED}.
\end{aligned}$$

Case 4: $\{0,0,0,1\}$ **on** $[2,-4,2,2]$.

$$\begin{aligned}
&\equiv [2,-4,2,0]\\
&\equiv [2,-4]\\
&\equiv [-2,-3] \equiv \text{KNOTTED}.
\end{aligned}$$

Therefore, the even knot can not be unknotted in one crossing change. Furthermore, since the actual unknotting number of $[2,2,1,1,2]$ must be less than or equal to the unknotting number of this projection and since this projection is knotted with an unknotting number of 1 then the unknotting number is EXACTLY 1, i.e.

$$0 < U(k) \leqq Umin(k) = 1 \rightarrow U(k) = 1$$

Consequently, the unknotting number of the even projection of $[2,2,1,1,2]$ does not realize the unknotting number.

Unknotting The Minimal Projection

To prove the unknotting number of the minimal projection of a Conway knot $[a, b, c, d, e]$ we will use a procedure which progressed from earlier methods of unknotting a three-tangle Conway knot. In earlier work, the proof would proceed by induction adding a position at each level in the proof. So, for example, the first part may prove by induction a certain unknotting number for the knot $[2k+1, 1, 4]$, and then use this result to prove unknotting number for the next knot in the sequence $[2k+1, 2j+1, 4]$. For the three position Conway knots this method is involved but managable. But, for the five position Conway knots this method is too tedious.

An alternative method was developed which cut the number of cases that needed to be considered down to 30 for the five Conway knots. Most of the time the number of cases can be reduced further down to around 20 cases that must be analyzed. The general idea of the procedure is that we make an arbitrary change $\{p, q, r, s, t\}$ on our knot, where any of the $p, q, r, sort$ can be equal to zero. Obviously, this change will cover all possible ways to make crossing changes on the knot. In general a guess on the unknotting number is needed so that an upper bound can be placed on $p+q+r+s+t$. We then need to check when any one, two, three, or four positions of the knot become negative as a result of our changes. To simplify matters we first check what occurs when any of the even positions are changed to zero. When one of the even positions are reduced to zero, the knot simplifies to a Conway three tangle knot for which we already know the unknotting number. Once we eliminate the possibities of a zero in any positions, each position needs only to be checked for when it is strictly positive or negative.

The notation $\{+-+-+\}$ is used to label which case is presently being worked on. Here, the $+-+-+$ signifies that we have made changes on the knot so that the first position will still be positive, the second position will become negative, and so on. When we make these arbitrary changes on the knot we assume that each position will be able to obtain any value except zero. For each of these cases we need to show that the resulting knot is either still knotted or requires more changes to unknot than our target unknotting number.

Knots of the form $[2i+1, 2j+1, 2k, 2l+1, 2m]$ with $j \geqq k$ and $j \geqq 1$ and $l \geqq 1$.

We will be considering the unknotting number of the Conway knot $[2i+1, 2j+1, 2k, 2l+1, 2m]$ in its minimal Conway projection. We will show that the unknotting number of knots of this form is $i+k+m$ in their minimal projection. To prove that the unknotting number is $i+k+m$, we show that it cannot be unknotted in $i+k+m-1$ changes and that it can be unknotted with $i+k+m$ crossing changes. To see that $[2i+1, 2j+1, 2k, 2l+1, 2m]$ can be unknotted in $i+k+m$ changes, we make i changes in the first position, k changes in then 3rd position and m changes in the last position. The resulting knot is $[1, 2j+1, 0, 2l+1, 0]$. After simplifying and unwinding we see that we have the trivial knot. Hence, $[2i+1, 2j+1, 2k, 2l+1, 2m]$ can be unknotted in $i+k+m$ crossing changes.

To prove that $[2i+1, 2j+1, 2k, 2l+1, 2m]$ with $j \geqq k$ and $j > 1$ and $l > 1$ can not be unknotted in $i+k+m-1$ changes, we will consider the crossing change $\{a, b, c, d, e\}$ where $a, b, c, d,$ or e are all greater than or equal to zero. After making these crossing changes the resulting knot is $[2i+1-2a, 2j+1-2b, 2k-2c, 2l+1-2d, 2m-2e]$ where

$a+b+c+d+e \leqq i+k+m-1$.

First notice that if we change either the third or fifth position to zero then we will have a three component knot. From Eva Wailes's work we know the unknotting numbers of the three component knots and links. So, if we make k changes in the third position then we get the knot $[2i+1, 2j+2l+2, 2m]$ which takes $i+m$ changes to unknot [W]. The total number crossing changes $(i+m+k)$ is above the target number allowed. If we make m changes in the fifth position then we get the knot $[2i+1, 2j+1, 2k]$ which takes $i+k$ changes [W]. Here we have used the $i+k+m$ changes. Since we have shown that changing any of the positions to zero will result in an unknotting number greater then $i+k+m-1$, we can ignore those changes where the third or fifth position become zero. More specifically we have covered the cases where $c=k$ or $e=m$.

In order to cover all of the possible ways to unknot our knot with the change $\{a, b, c, d, e\}$, we now consider the cases where one, two, three, and four positions are changed to negative.

Part I. *One Position is changed to negative.*

$\{-++++\}$ Assume $2i+1-2a \leqq -1$, $2j+1-2b \geqq 1$, $2k-2c \geqq 2$, $2l+1-2d \geqq 1$, and $2m-2e \geqq 2$. From equivalence 1 we get the knot $[2a-2i-2, 1, 2j-2b, 2k-2c, 2l+1-2d, 2m-2e]$. The last three positions are all positive so the only terms to be wary of are if $2a-2i-2=0$ or $2j-2b=0$. But even if both of these positions are equal to zero we still get the knot $[2l+1-2d, 2m-2e]$ which will be knotted from our assumtions.

$\{+-+++\}$ Assume $2i+1-2a \geqq 1$, $2j+1-2b \leqq -1$, $2k-2c \geqq 2$, $2l+1-2d \geqq 1$, and $2m-2e \geqq 2$. From equivalence 2 we get the knot $[2i-2a, 1, 2b-2j-3, 1, 2k-2c-1, 2l+1-2d, 2m-2e]$. The last three terms are always positive by our hypothesis. The only possible non-positive terms occur when $2i-2a=0$ or $2b-2j-3=-1$. If $2i-2a=0$ and $2b-2j-3 \geqq 1$ then we reduce the knot to $[2b-2j-3, 1, 2k-2c-1, 2l+1-2d, 2m-2e]$ which is still non trivial. If $2i-2a \geqq 2$ and $2b-2j-3=-1$ then by equivalence 32 we get the knot $[2i-2a-1, 1, 2k-2c-2, 2l+1-2d, 2m-2e]$ which is knotted even when $2k-2c-2=0$. If $2i-2a=0$ and $2b-2j-3=-1$ then we get the knot $[2l+1-2d, 2m-2e]$ which is also non trivial.

$\{++-++\}$ Assume that $2i+1-2a \geqq 1$, $2j+1-2b \geqq 1$, $2k-2c \leqq -2$, $2l+1-2d \geqq 1$, and $2m-2e \geqq 2$. From equivalence 3 we get the knot $[2i+1-2a, 2j-2b, 1, 2c-2k-2, 1, 2l-2d, 2m-2e]$. The last and first terms are always positive by our hypothesis. We have three non-positive possibilities to consider: $2j-2b=0, 2c-2k-2=0$, and $2l-2d=0$. Even if all three are zero, we get the knot $[2i-2a+2m-2e+3]$ which is knotted since $2i-2a+2m-2e+3>1$ from the hypothesis.

$\{+++-+\}$ Assume that $2i+1-2a \geqq 1$, $2j+1-2b \geqq 1$, $2k-2c \geqq 2$, $2l+1-2d \leqq -1$, and $2m-2e \geqq 2$. From equivalence 4 we get the knot $[2i+1-$

$2a, 2j+1-2b, 2k-2c-1, 1, 2d-2l-3, 1, 2m-2e-1]$. The first three and the last terms are always positive by the hypothesis. The only non-positive possibility is when $2d-2l-3=-1$, which when true gives us the knot $[2i+1-2a, 2j+1-2b, 2k-2c-2, 1, 2m-2e-2]$ by equivalence 33. This gives us two non-positive possibilities, namely if $2k-2c-2=0$ and $2m-2e-2=0$. However, even if both positions are zero then we still have the knot $[2i+1-2a]$. Notice that we have made at least $l+k+m-1$ changes. So, we have at most $i-l$ changes left. Since, $l \geqq 1$ we know that $a \leqq i-1$. Hence $2i+1-2a > 1$ and therefore is non trivial.

$\{+++++-\}$ Assume that $2i+1-2a \geqq 1$, $2j+1-2b \geqq 1$, $2k-2c \geqq 2$, $2l+1-2d \geqq 1$, and $2m-2e \leqq -2$. From equivalence 5 we get the knot $-[2i+1-2a, 2j+1-2b, 2k-2c, 2l-2d, 1, 2e-2m-1]$. The first three and the last terms are positive by our hypothesis. The only non-positive possibility is if $2l-2d=0$. Even if $2l-2d=0$ we get the knot $-[2i+1-2a, 2j+1-2b, 2k-2c+1, 2e-2m-1]$ which is also non trivial.

Part II. *Two Positions are changed to negative.*

$\{-,-,+,+,+\}$ Assume that $2i+1-2a \leqq -1$, $2j+1-2b \leqq -1$, $2k-2c \geqq 2$, $2l+1-2d \geqq 1$, and $2m-2e \geqq 2$. From equivalence 6 we get the knot $[2a-2i-1, 2b-2j-2, 1, 2k-2c-1, 2l+1-2d, 2m-2e]$. The last three and the first positions are positive by our hypothesis. The only non-positive possibility is if $2b-2j-2=0$. When $2b-2j-2=0$ we get the knot $[2a-2i, 2k-2c-1, 2l+1-2d, 2m-2e]$ which is non trivial.

$\{-+-++\}$ Assume that $2i+1-2a \leqq -1$, $2j+1-2b \geqq 1$, $2k-2c \leqq -2$, $2l+1-2d \geqq 1$, and $2m-2e \geqq 2$. From equivalence 7 we get the knot $[2a-2i-2, 1, 2j-2b-1, 1, 2c-2k-2, 1, 2l-2d, 2m-2e]$. The last position is the only one which is always positive. The other four positions can be non-positive when $2a-2i-2=0, 2j-2b-1=-1, 2c-2k-2=0$, or $2l-2d=0$. Notice we have made at least $i+k+2$ changes, so we have at most $m-3$ changes left. So we know that $2m-2e>5$.

First, assume that $2j-2b-1 \neq -1$, then it is easy to see that if the other positions are zero we will still minimally have the non trivial knot $[2j-2b-1, 2m-2e+2]$. Otherwise, if $2j-2b-1=-1$ then we need to consider combinations of when $2a-2i-2=0, 2c-2k-2=0$, and $2l-2d-2=0$.

If $2a-2i-2=0$ then we get $[1, 2l-2d, 2m-2e]$ which is knotted. Then even when $2c-2k-2=0$ or $2l-2d=0$ or both we get the knot $[2m-2e+1]$ which will also be knotted since 2m-2e+1 ¿ 6.

If $2c-2k-2=0$ and $2l-2d>0$ then we get the knot $[2a-2i-4,1,2l-2d-1,2m-2e]$ which is non trivial even if $2a-2i-4=0$. Notice that we can assume $2a-2i-2>0$ since it was covered in the previous case. Then if $2c-2k-2=0$ and $2l-2d=0$ we get the knot $[2m-2e+2i-2a+3]$. Notice that $b+c+d \geqq j+k+l=1$ which then implies that $a+e \leqq i+m-l-k$. Therefore, we know that $2m-2e+2i-2a+3>1$.

If $2l-2d=0$, $2a-2i-2>0$, and $2c-2k-2>0$ we get the knot $[2a-2i-3,1,2c-2k-1,2m-2e-1]$ which is non trivial.

$\{-++-+\}$ Assume $2i+1-2a \leqq -1$, $2j+1-2b \geqq 1$, $2k-2c \geqq 2$, $2l+1-2d \leqq -1$, and $2m-2e \geqq 2$. From equivalence 8 we get the knot $[2a-2i-2,1,2j-2b,2k-2c-1,1,2d-2l-3,1,2m-2e-1]$. The non positive possibilities are if $2a-2i-2=0, 2j-2b=0$, or $2d-2l-3=-1$. Notice that we have made at least i+2+l changes. We have at most k+m-l-3 changes left.

If only $2a-2i-2=0$ then the equivalent knot is $[2j-2b,2k-2c-1,1,2d-2l-3,1,2m-2e-1]$ which is non trivial. If $2j-2b=0$ and $2d-2l-3>-1$ then our knot becomes $[1,2d-2l-3,1,2m-2e-1]$ which is still knotted. If $2d-2l-3=-1$ then our knot becomes $[2m-2e-1]$. We know that $2m-2e-1>1$ since we have made at least $i+j+l$ changes and have $k+m-l-j-1$ or less than $m-2$ changes left.

If $2j-2b=0$ and $2d-2l-3>-1$ then our knot becomes $[2a-2i-2,2k-2c,1,2d-2l-3,1,2m-2e-1]$ which is knotted. Then if $2d-2l-3=-1$ we will get the knot $[2a-2i-2,2k-2c-1,2m-2e-2]$. We can assume that $2a-2i-2>0$ since convered in previous case and $2m-2e-2>1$ since we have made at least $i+j+l$ changes and have at most $m-l-1$ changes left.

If only $2d-2l-3=-1$ then we get the knot $[2a-2i-2,1,2j-2b,2k-2c-2,2m-2e-2]$. We can assume that $2j-2b>0$ and that $2a-2i-2>0$ since covered in two immediate previous cases. Even if $2k-2c-2=0$ or $2m-2e-2=0$ or both we notice that we at least get the knot $[2a-2i-2]$ which is non trivial.

$\{-+++-\}$ Assume $2i+1-2a \leqq -1$, $2j+1-2b \geqq 1$, $2k-2c \geqq 2$, $2l+1-2d \geqq 1$, and $2m-2e \leqq -2$. From equivalence 9 we get the knot $-[2a-2i-2,1,2j-2b,2k-2c,2l-2d,1,2e-2m-1]$. The only non-positive possibilities for the positions are if $2a-2i-2=0$, $2j-2b=0$, or if $2l-2d=0$. Even when all three are zero we get the knot $-[2e-2m-1]$ which is still knotted.

$\{+--++\}$ Assume $2i+1-2a \geqq 1$, $2j+1-2b \leqq -1$, $2k-2c \leqq -2$, $2l+1-2d \geqq 1$, and $2m-2e \geqq 2$. From equivalence 10 we get the knot

$[2i-2a, 1, 2b-2j-2, 2c-2k-1, 1, 2l-2d, 2m-2e]$. The only non-positive possibilities are if $2i-2a=0$, $2b-2j-2=0$, or $2l-2d=0$. Even when all three of these positions are zero we still get the knot $[2m-2e+1]$ where $2m-2e+1 \geqq 3$ and therefore knotted.

$\{+-+-+\}$

Assume $2i+1-2a \geqq 1$, $2j+1-2b \leqq -1$, $2k-2c \geqq 2$, $2l+1-2d \leqq -1$, and $2m-2e \geqq 2$. From equivalence 11 we get the knot $[2i-2a, 1, 2b-2j-3, 1, 2k-2c-2, 1, 2d-2l-3, 1, 2m-2e-1]$. The last position is always positive. The non-positive possibilities are if $2i-2a=0$, $2b-2j-3=-1$, $2k-2c-2=0$, or $2d-2l-3=-1$. We need to consider all of these combinations. Notice that we don't need to consider when $2i-2a=0$ and $2k-2c=0$ are zero separately, but can assume that they are both zero or both greater then zero.

If $2i-2a=0$, $2k-2c-2=0$, $2b-2j-3>-1$, and $2d-2l-3>-1$ then we get the knot $[2b-2j-3, 2, 2d-2l-3, 1, 2m-2e-1]$. Now if $2b-2j-3=-1$ then the resulting knot is $[1, 2d-2l-3, 1, 2m-2e-1]$. If only $2d-2l-3=-1$ then our knot is $[2b-2j-4, 1, 2m-2e-3]$. Notice we have made at least $i+k+j+l+3$, and we have at most $m-j-l-4$ changes left. So, we will always have a knot since $2m-2e-3>1$. And, if both $2b-2j-3=-1$ and $2d-2l-3=-1$ then our knot becomes $[2m-2e-1]$ which will also be knotted since $2m-2e-1>1$.

We now assume that both $2i-2a>0$ and $2k-2c-2>0$. If $2b-2j-3=-1$ then our resulting knot is $[2i-2a-1, 1, 2k-2c-3, 1, 2d-2l-3, 1, 2m-2e-1]$ which is all positive. If only $2d-2l-3=-1$ then we get the knot $[2i-2a-1, 1, 2k-2c-4, 1, 2m-2e-2]$ which will be knotted since we do not have enough crossing changes left to make $2m-2e-2=0$. If both $2b-2j-3=-1$ and $2d-2l-3=-1$ then the knot becomes $[2i-2a-1, 1, 2k-2c-4, 1, 2m-2e-2]$. which will be knotted since $2m-2e-2>0$.

$\{+-++-\}$

Assume $2i+1-2a \geqq 1$, $2j+1-2b \leqq -1$, $2k-2c \geqq 2$, $2l+1-2d \geqq 1$, and $2m-2e \leqq -2$. From equivalence 12 we get the knot $-[2i-2a, 1, 2b-2j-3, 1, 2k-2c-1, 2l-2d, 1, 2e-2m-1]$. The only non-positive terms are if $2b-2j-3=-1$ or $2l-2d=0$. Note that $2i-2a>0$ since $a \leqq i-3$. If $2l-2d=0$ and $2b-2j-3>-1$ then we still have a non trivial knot. If $2b-2j-3=-1$ and $2l-2d=0$ then our knot becomes $-[2i-2a-1, 1, 2k-2c-1, 2e-2m-1]$ which is also knotted.

$\{++--+\}$

Assume $2i+1-2a \geqq 1$, $2j+1-2b \geqq 1$, $2k-2c \leqq -2$, $2l+1-2d \leqq -1$, and $2m-2e \geqq 2$. From equivalence 13 we get the knot $[2i+1-2a, 2j-2b, 1, 2c-2k-1, 2d-2l-2, 1, 2m-2e-1]$. The only possible non positive positions are when $2j-2b=0$ or $2d-2l-2=0$. Even when both are zero we still get $[2i+2-2a, 2c-2k, 2m-2e-1]$

which is knotted.

$\{++-+-\}$ Assume $2i+1-2a \geqq 1$, $2j+1-2b \geqq 1$, $2k-2c \leqq -2$, $2l+1-2d \geqq 1$, and $2m-2e \leqq -2$. From equivalence 14 we get the knot $-[2i+1-2a, 2j-2b, 1, 2c-2k-2, 1, 2l-2d-1, 1, 2e-2m-1]$. The only possible non-positive terms are if $2j-2b=0, 2c-2k-2=0$, or $2l-2d-1=-1$. If $2l-2d-1>-1$ then we will still have a non trivial knot even when $2j-2b=0$ and $2c-2k-2=0$. If $2l-2d-1=-1$ then we get the knot $-[2i+1-2a, 2j-2b, 1, 2c-2k-3, 1, 2e-2m-2]$ which is knotted when $2j-2b>0$ adn $2c-2k>0$. If both $2j-2b=0$ and $2c-2k-2=0$ then we get the knot $-[2e-2m-2i+2a-3]$. We have already made at least $l+j+k+m+2$ changes. So we know that $a+e <= m+i-3$. This implies that $2e-2m-2i+2a-3>1$ and so we still have a non trivial knot.

$\{+++--\}$ From equivalence 15 we get the knot $-[2i+1-2a, 2j+1-2b, 2k-2c-1, 1, 2d-2l-2, 2e-2m]$. Only one term can be non-positive, namely $2d-2l-2=0$. When $2d-2l-2=0$ we still have the knot $-[2i+1-2a, 2j+1-2b, 2k-2c-1, 2e-2m+1]$ which is still knotted.

Part III. *Three positions are negative.*

$\{-,-,-,+,+\}$ From equivalence 16 we get the knot $[2a-2i-1, 2b-2j-1, 2c-2k-1, 1, 2l-2d, 2m-2e]$. The only possible non-positive term is when $2l-2d=0$ which will result in the non trivial knot $[2a-2i-1, 2b-2j-1, 2c-2k-1, 2m-2e+1]$.

$\{--+-+\}$ Assume $2i+1-2a \leqq -1$, $2j+1-2b \leqq -1$, $2k-2c \geqq 2$, $2l+1-2d \leqq -1$, and $2m-2e \geqq 2$. From equivalence 17 we get the knot $[2a-2i-1, 2b-2j-2, 1, 2k-2c-2, 1, 2d-2l-3, 1, 2m-2e-1]$ The possible non positive terms are if $2b-2j-2=0$, $2k-2c-2=0$, or $2d-2l-3=-1$. If $2d-2l-3>-1$ and the other two positions are zero it is easily seen that we still have a knot by our zero reduction. So, if $2d-2l-3=-1$ then we get the knot $[2a-2i-1, 2b-2j-2, 1, 2k-2c-3, 1, 2m-2e-2]$ which is knotted if $2b-2j-2>0$ and $2k-2c-2>0$. But, if $2b-2j-2=0$ and $2k-2c-2>0$ we get the knot $[2a-2i, 2k-2c-3, 1, 2m-2e-2]$ which is still knotted. If $2k-2c-2>0$ then we will get the knot $[2a-2i-1, 2b-2j-3, 1, 2m-2e-3]$. Notice that we have made $i+j+l+3$ changes, so we will still be knotted $since 2m-2e-3>1$. Then if both 2b-2j-2 = 0 and 2k-2c-2 = 0 we get the knot $[2m-2e-1-2a+2i]$. Notice that $b+c+d \geqq j+k+l+2$ which implies that $a+e \leqq i+m-3$. Hence, $2m-2e+2i-2a-1 \geqq 5$ and we have a non trivial knot.

$\{--++-\}$ To make these three positions negative we need at least $i+j+m+3$ changes. We are only concerned with those cases where less that

$i+k+m-1$ crossing changes are made. Since $j \geqq k$, we can ignore this case.

$\{-+--+\}$ Assume $2i+1-2a \leqq -1$, $2j+1-2b \geqq 1$, $2k-2c \leqq -2$, $2l+1-2d \leqq -1$, and $2m-2e \geqq 2$. From equivalence 19 we get the knot $[2a-2i-2, 1, 2j-2b-1, 1, 2c-2k-1, 2d-2l-2, 1, 2m-2e-1]$. The possible non-positive terms are when $2a-2i-2=0$, $2j-2b-1=-1$, or $2d-2l-2=0$. If $2j-2b-1>-1$, then when either or both of the other positions are zero we will still have a non trivial knot. If $2j-2b-1=-1, 2a-2i-2>0$, and $2d-2l-2>0$ then the resulting non trivial knot is $[2a-2i-3, 1, 2c-2k-2, 2d-2l-2, 1, 2m-2e-1]$. We will now consider the differing combinations of when $2a-2i-2=0$ and when $2d-2l-2=0$. If $2a-2i-2=0$ then the resulting knot is $[2d-2l-2, 1, 2m-2e-1]$ which will be knotted. If $2d-2l-2=0$ the we get the knot $[2a-2i-3, 1, 2c-2k-1, 2m-2e-1]$ which will also be knotted. Finally, if both $2a-2i-2=0$ and $2d-2l-2=0$ then we get the knot $[2m-2e-1]$. Notice we have made at least $i+j+l+1$ changes which implies that $2m-2e-1>1$ and therefore our knot is non trivial.

$\{-+-+-\}$ To make these three positions negative we need at least $i+k+m+3$ changes. We are only concerned with those cases where less that $i+k+m-1$ crossing changes are made.

$\{-++--\}$ Assume $2i+1-2a \leqq -1$, $2j+1-2b \geqq 1$, $2k-2c \geqq -2$, $2l+1-2d \leqq -1$, and $2m-2e \leqq 2$. From equivalence 21 we get the knot $-[2a-2i-2, 1, 2j-2b-2, 2k-2c-1, 1, 2d-2l-2, 2e-2m]$. The possible non-positive terms are if $2a-2i-2=0$, $2j-2b-2=0$, and $2d-2l-2=0$. Even with all of these equal to zero, we still get the knot $-[2e-2m+1]$ which is knotted since $2e-2m+1>1$.

$\{+---+\}$ Assume $2i+1-2a \geqq -1$, $2j+1-2b \leqq 1$, $2k-2c \leqq -2$, $2l+1-2d \leqq -1$, and $2m-2e \geqq 2$. From equivalence 22 we get the knot $[2i-2a, 1, 2b-2j-2, 2c-2k, 2d-2l-2, 1, 2m-2e-1]$. The possible non-positive terms are if $2i-2a=0$, $2b-2j-2=0$, or $2d-2l-2=0$. Notice that if any one or two of the positions are non-negative then we most certainly have a non trivial knot. If we make all three of these positions zero then we get the knot $[2m-2e-1]$. Notice that at least $i+j+l+1$ changes have been made, leaving at most $m-3$ changes. This implies that $2m-2e-1>1$ and hence we have a non trivial knot.

$\{+--+-\}$ Assume $2i+1-2a \geqq 1$, $2j+1-2b \leqq -1$, $2k-2c \leqq -2$, $2l+1-2d \geqq 1$, and $2m-2e \leqq -2$. From equivalence 23 we get the knot $-[2i-2a, 1, 2b-2j-2, 2c-2k-1, 1, 2l-2d-1, 1, 2e-2m-1]$. The non-positive terms are if $2b-2j-2=0$ or $2l-2d-1=-1$. Note that $2i-2a>0$ since we have alread made $j+k+m+3$

crossing changes which implies $a < i - 4$. If $2l - 2d - 1 \neq -1$ and $2b - 2j - 2 = 0$, we still we get a non trivial knot. If $2l - 2d - 1 = -1$ then we get the knot $-[2i-2a, 1, 2b-2j-2, 2c-2k-2, 1, 2e-2m-2]$ which is knotted if $2b - 2j - 2 \neq 0$. If $2b - 2j - 2 = 0$ then we get the knot $-[2i - 2a, 2c - 2k - 1, 1, 2e - 2m - 2]$ which will be knotted.

$\{+-+--\}$ Assume that $2i+1-2a \geqq 1$, $2j+1-2b \leqq -1$, $2k-2c \geqq 2$, $2l+1-2d \leqq -1$, and $2m - 2e \leqq -2$. From equivalence 24 we get the knot $-[2i-2a, 1, 2b-2j-3, 1, 2k-2c-2, 1, 2d-2l-2, 2e-2m]$. The non-positive terms are if $2b-2j-3 = -1$, $2k-2c-2 = 0$, or $2d-2l-2 = 0$. Note that we have already made at least $j+l+m+3$ changes which implies that $2i - 2a > 0$. If $2b - 2j - 3 \neq -1$ then we will still have a knot even when the other two position are zero. If $2b - 2j - 3 = -1$, then we get the knot $-[2i-2a-1, 1, 2k-2c-3, 1, 2d-2l-2, 2e-2m]$ which is knotted if $2k - 2c - 2 \neq 0$ and $2d - 2l - 2 \neq 0$. Now we need to consider combinations of $2k-2c-2 = 0$ and $2d-2l-2 = 0$. If only $2k - 2c = 0$ then we get the knot $-[2i - 2a - 2, 1, 2d - 2l - 3, 2e - 2m]$ which is non trivial. If $2d - 2l - 2 = 0$ then our equivalent knot is $-[2i - 2a - 1, 1, 2k - 2c - 3, 2e - 2m + 1]$ which is knotted. If both $2k-2c = 0$ and $2d-2l = 0$ then we get the knot $[2m-2e+2i-2a+1]$. Notice that $b+c+d \geqq j+k+l+1$ will imply that $a+e \leqq i+m-4$. And therefore $2m - 2e + 2i - 2a + 1 > 1$ and non trivial.

$\{++---\}$ From equivalence 25 we get the knot $-[2i+1-2a, 2j-2b, 1, 2c-2k-1, 2d-2l-1, 2e-2m]$. The only non-positive term is if $2j - 2b = 0$ which gives us the knot $-[2i+2-2a, 2c-2k-1, 2d-2l-1, 2e-2m]$ which is knotted.

$\{----+\}$ From equivalence 26 we get the knot $[2a - 2i - 1, 2b - 2j - 1, 2c - 2k, 2d - 2l - 2, 1, 2m - 2e - 1]$. The only possible non-positive term is when $2d - 2l - 2 = 0$ which gives us the knot $[2a - 2i - 1, 2b - 2j - 1, 2c - 2k + 1, 2m - 2e - 1]$ which is knotted.

$\{-+---\}$ To make these four positions negative we need at least $i+k+l+m+4$ changes. But, we are only concerned with those cases where less then $i + k + m - 1$ crossing changes are made.

$\{--+--\}$ To make these four positions negative we need at least $i+j+l+m+4$ changes. But, we are only concerned with those cases where less then $i + k + m - 1$ crossing changes are made.

$\{---+-\}$ To make these four positions negative we need at least $i+j+k+m+4$ changes. But, we are only concerned with those cases where less then $i + k + m - 1$ crossing changes are made.

$\{+----\}$ From equivalence 30 we have the knot $-[2i - 2a, 1, 2b - 2j - 2, 2c - 2k, 2d-2l-1, 2e-2m]$. The two non-positive terms are if $2i-2a = 0$ and if $2b-2j-2 = 0$ which gives us the knot $-[2d-2l-1, 2e-2m]$

which is knotted.

We have just show that the knot $[2i+1, 2j+1, 2k, 2l+1, 2m]$ will not be unknotted in fewer than $i+k+m$ changes. Hence, $[2i+1, 2j+1, 2k, 2l+1, 2m]$ with $j \geqq k, l \geqq 1$ has $U_{min}(k) = i+k+m$.

Prime Knot with Arbitrary Gap

From the previous section we have that the unknotting number of the Conway knot $K = [2i+1, 2j+1, 2k, 2l+1, 2m]$ with the restrictions $j, l \geqq 1$ and $j \geqq k$ is $i+k+m$. We also that all rational knots are prime [L]. Now all we need is a projection of K which has a smaller unknotting number. This can be achieved by looking at the even-continued fraction of this knot. The even continued fraction expansion of K is:

$$[2i+1, 2j+1, 2k, 2l+1, 2m] \equiv [2i+1, 2j+2, -2, \overbrace{2, -2}^{k-1 \text{ times}}, 2l+2, 2m]$$

Notice that the even continued fraction begins with an odd term. This is true for all knots (as opposed to links or knot-primes) since no sequence of evens in the form of a continued fraction can generate a rational of the form *odd/odd* and continued fraction of all knots are of this form. Thus the even continued fraction of a knot must contain at least one odd term[1]. As an example of the above equivalence consider the following equivalent knots:

$$[7, 11, 14, 3, 14] \equiv [7, 12, -2, 2, -2, 2, -2, 2, -2, 2, -2, 2, -2, 2, -2, 2, -2, 4, 14]$$

Now if we insist that $k = m$ we see that the non-minimal projection can be unknotted by making the changes:

$$\{i, 0, 0, \overbrace{1, 0}^{k-1 \text{ times}}, l+1, 0\}$$

Then we get the following knot and its reductions:

$$\begin{aligned}
&\equiv [1, 12, -2, \overbrace{0, -2}^{k-1 \text{ times}}, 0, 2k] \\
&\equiv [1, 12, -2k, 0, 2k] \\
&\equiv [1, 12, 2k - 2k] \\
&\equiv [1, 12, 0] \\
&\equiv [1]
\end{aligned}$$

[1] It is interesting to note that the infinite continued fraction $[2, -2, 2 \ldots]$ is exactly equal to one. Thus, knots can be written as an infinite even continued fraction since the last term in the expansion, which is exactly odd, differs from an even by one.

And the knot is unknotted in $i+k-1+l+1=i+k+l$ changes. At this point we can choose the parameters k and l to create a gap of arbitrary size. For example, if we pick $l \leqq k$ and remember that $k=m$ then we get a gap:

$$\mathrm{Gap}(K) \geqq i+2k-i-k-l=k-l \quad \text{and} \quad 1 \leqq l \leqq k$$

Thus we have a prime knot with the property that

$$\mathrm{Gap}(K) \geqq k-l$$

This is significant because there are no restrictions between k and l, thus the Gap is completely arbitrary. It was known previously that a knot with arbitrary gap could be constructed by forming the connected sum of an arbitrary number of knots with Gap$=1$. However, this type of construction does not yield a prime knot. The above family of knots is prime [L].

For our example knot, $K=[7,11,14,3,14]$ we have Gap $=k-l=7-1=6$.

Knots of the form $[2j+1,2k+1,2l,2m,2n]$ with $j \geqq l \geqq k \geqq m \geqq n$.

Using the method outlined in the previous section we will prove that $[2j+1,2k+1,2l,2m,2n]$ cannot be unknotted in fewer then $j+k+n+1$ crossing changes in its minimal projection. For this knot we need to assume that $j \geqq l \geqq k \geqq m \geqq n$. We will make the change $\{p,q,r,s,t\}$ where any of the p,q,r,s,t can be greater then or equal to zero. The resulting knot is $[2j+1-2p,2k+1-2q,2l-2r,2m-2s,2n-2t]$.

Observe that if any of the last three positions are changed to zero then our knot reduces to a three tangle Conway knot. If we make n changes to the 5th position then the knot reduces to $[2j+1,2k+1,2l]$ which will take $j+k+1$ changes to unknot [W]. If we make m changes to the 4th position then the resulting knot is $[2j+1,2k+1,2l+2n]$ which will take $j+k+1$ change to unknot [W]. And finally if we make l changes on the 3rd position then the knot reduces to $[2j+1,2k+1+2m,2n]$ which will take $j+n$ changes to unknot [W]. So we can assume throughout our proof that $2l-2r, 2m-2s$, and $2n-2t$ are all non zero.

Part I. *One Position is Negative.*

$\{-++++\}$ Assume that $2j+1-2p<0$ and the rest of our terms are positive. Then our resulting knot is $[2p-2j-2,1,2k-2q,2l-2r,2m-2s,2n-2t]$ by equivalence 1 and is non trivial even if $2p-2j-2=0$ and $2k-2q=0$.

$\{+-+++\}$ Assume that $2k+1-2q<0$ and the other four terms are positive. For simplicity we will consider when $2k+1-2q=-1$ and when $2k+1-2q<-1$. If $2k+1-2q=-1$ then our knot becomes $[2j-2p-1,1,2l-2r-2,2m-2s,2n-2t]$. If $2j-2p-1=-1$ then we get the knot $[2m-2s,2n-2t]$ which is non trivial. If $2j-2p-1>0$ and $2l-2r-2=0$ we get $[2j-2p-1,2m-2s+1,2n-2t]$ which is also non trivial. If $2k+1-2q<-1$ then our knot becomes

$[2j-2p, 1, 2q-2k-3, 1, 2l-2r-1, 2m-2s, 2n-2t]$. The only possible non positive term is if $2j-2p=0$. But even when $2j-2p=0$ we still get a non trivial knot.

$\{++-++\}$ Assume that $2l-2r<0$, then our knot becomes $[2j+1-2p, 2k-2q, 1, 2r-2l-2, 1, 2m-2s-1, 2n-2t]$. The only possible non positive terms are if $2k-2q=0$ or $2r-2l-2=0$. But if either or both are true it is easily seen that we still have a non trivial knot.

$\{+++-+\}$ Assume that $2m-2s<0$, then our knot becomes $[2j+1-2p, 2k-2q+1, 2l-2r-1, 1, 2s-2m-2, 1, 2n-2t]$. The only possible non positive term is if $2s-2m-2=0$, but then we have the knot $[2j+1-2p, 2k-2q+1, 2l-2r-1, 2, 2n-2t]$ which is also non trivial.

$\{++++-\}$ Assume that $2n-2t<0$, then our knot becomes $-[2j+1-2p, 2k-2q+1, 2l-2r-1, 2m-2s-1, 1, 2t-2n-1]$ which is non positive and hence non trivial.

Part II. *Two Positions Become Negative.*

$\{--+++\}$ Assume that $2j+1-2p<0$ and that $2k+1-2q$ is less than zero. Also, we know that $2l-2r, 2m-2s, 2n-2t$ are all greater than zero. Then our resulting knot from continued fraction identity 6 is $[2p-2j-1, 2q-2k-2, 1, 2l-2r-1, 2m-2s, 2n-2t]$. All of the terms will be greater than zero with the exception of $2q-2k-2$ which may equal zero. But, even if $2q-2k-2=0$ we still have $[2p-2j, 2l-2r-2, 2m-2s, 2n-2t]$ which is also non trivial.

$\{-+-++\}$ Assume that $2j+1-2p<0$ and that $2l-2r<0$. We also know that $2k+1-2q, 2m-2s$ and $2n-2t$ are all greater than zero. The resulting knot from identity 7 is $[2p-2j-2, 1, 2k-2q-1, 1, 2r-2l-2, 1, 2m-2s-1, 2n-2t]$. The possible non positive positions are when $2p-2j-2=0$, $2r-2l-2=0$ or $2k-2q-1=-1$.

Notice that since we have already made $j+l+2$ changes that we have $k+n-l-2$ changes left. Thus $q \leqq k-2$. Since $l \geqq n$ this implies that $2k-2q-1>-1$.

So if both of our positions are non negative then the knot becomes $[2k-2q-1, 2, 2m-2s-1, 2n-2t]$ which will still be non trivial.

$\{-++-+\}$ Assume that $2j+1-2p<0$ and that $2m-2s<0$. Then we also know that $2k+1-2q, 2l-2r$, and $2n-2t$ are all positive. Our resulting knot is $[2p-2j-2, 1, 2k-2q, 2l-2r-1, 1, 2s-2m-2, 1, 2n-2t-1]$. Again the only possible non-positive terms are when $2p-2j-2=0, 2k-2q=0$, or $2s-2m-2=0$. Even if all of these are zero we still get the knot $[2, 2n-2t-1]$, and since $2n-2t-1>0$ we know that it is non trivial.

$\{-+++-\}$ Assume $2j+1-2p<0, 2k+1-2q>0, 2l-2r>0, 2m-2s>0$, and $2n-2t<0$. Using equivalence 9, we get the knot $-[2p-2j-2, 1, 2k-2q, 2l-2r, 2m-2s-1, 1, 2t-2n-1]$. The only positions that may be non positive are if $2p-2j-2=0$ or $2k-2q=0$. But if both are positive we get the knot $[2m-2s-1, 1, 2t-2n-1]$ which is still non trivial.

$\{+--++\}$ Assume that $2j+1-2p>0, 2k+1-2q<0, 2l-2r<0, 2m-2s>0$, and $2n-2t>0$. Using equivalence 10 we get $[2j-2p, 1, 2q-2k-2, 2r-2l-1, 1, 2m-2s-1, 2n-2t]$. The only non positive terms are if $2j-2p=0$ or $2q-2k-2=0$. But even if they are both zero we still have the knot $[1, 2m-2s-1, 2n-2t]$ which is still knotted.

$\{+-+-+\}$ Assume that $2j+1-2p>0, 2k+1-2q<0, 2l-2r>0, 2m-2s<0$, and $2n-2t>0$. Using equivalence 11 we get the knot $[2j-2p, 1, 2q-2k-3, 1, 2l-2r-2, 1, 2s-2m-2, 1, 2n-2t-1]$. The possible non positive terms are if $2q-2k-3=-1, 2l-2r-2=0$, or $2s-2m-2=0$. Notice that we have already made at least $k+m+2$ changes. We can make at most $j+n-m-2$ changes. Since $m \geqq n$ we know that $2j-2p \geqq 5$. We need to try the different combinations of when $2q-2k-3=-1, 2l-2r-2=0$, or $2s-2m-2=0$. If all three of them are non positive then the knot becomes $[2j-2p, 1, -1, 3, 2n-2t-1]$. Which is equivalent to $[2-2j+2p, 2n-2t-1]$. Now we have made at least $k+l+m+1$ crossing changes. This implies that $p \leqq j-l-m-1$. So we know that $2-2j+2p<0$. Hence our equivalent knot is $[2j-2p-3, 1, 2n-2t-2]$ which is non trivial even if $2n-2t-2=0$ since $2j-2p-3>1$. If $2q-2k-3=-1$ and $2l-2r-2=0$ then we get the knot $[2j-2p-2, 1, 2s-2m-3, 1, 2n-2t-1]$ which is non negative and then hence non trivial. If $2q-2k-3=-1$ and $2s-2m-2=0$ then we get the knot $[2j-2p-1, 1, 2l-2r-3, 2, 2n-2t-1]$ which is also non trivial. If either $2s-2m-2=0$ or $2l-2r-2=0$ or both then one easily sees that we have a non trivial knot using the zero reduction property. If $2q-2k-3=-1$ then our knot is equivalent to $[2j-2p-1, 1, 2l-2r-3, 1, 2s-2m-2, 1, 2n-2t-1]$ which is non negative and hence non trivial.

$\{+-++-\}$ Assume $2j+1-2p>0, 2k+1-2q<0, 2l-2r>0, 2m-2s>0$, and $2n-2t<0$. Using equivalence 12 we get $-[2j-2p, 1, 2q-2k-3, 1, 2l-2r-1, 2m-2s-1, 1, 2t-2n-1]$. The only terms we need to be careful of are $2q-2k-3=-1$ and $2j-2p=0$. Notice that we have made a least $k+n+2$ crossing changes, so at most we have $j-2$ left. So we know that $2j-2p>0$. If $2q-2k-3=-1$ then our knot will become $-[2j-2p-1, 1, 2l-2r-2, 2m-2s-1, 1, 2t-2n-1]$. Now even if $2l-2r-2=0$, we still will have a non trivial knot.

$\{++--+\}$ Assume $2j+1-2p>0, 2k+1-2q>0, 2l-2r<0, 2m-2s<0$, and $2n-2t>0$. Using equivalence 13 we get $[2j+1-2p, 2k-$

$2q, 1, 2r-2l-1, 2s-2m-1, 1, 2n-2t-1]$. The only possible non positive term is if $2k-2q=0$. But, even if this is true we will still have a non trivial knot.

$\{++-+-\}$ Assume $2j+1-2p>0, 2k+1-2q>0, 2l-2r<0, 2m-2s>0$, and $2n-2t<0$. Using equivalence 14 we get $-[2j+1-2p, 2k-2q, 1, 2r-2l-2, 1, 2m-2s-2, 1, 2t-2n-1]$. The only possible non positive terms are if $2k-2q=0, 2r-2l-2=0$, and $2m-2s-2=0$. But, even if all are zero we still get $-[2j+4-2p, 2t-2n-1]$ which is non trivial.

$\{+++--\}$ Assume $2j+1-2p>0, 2k+1-2q>0, 2l-2r>0, 2m-2s<0$, and $2n-2t<0$. Using equivalence 15 we get the knot $-[2j+1-2p, 2k+1-2q, 2l-2r-1, 1, 2s-2m-1, 2t-2n]$ which is all negative, and therefore non trivial.

Part III. *Three positions are negative.*

$\{---++\}$ To make the first three positions negative we need to make at least $j+k+l+3$ changes. But we are only allowed $j+k+n$ changes and since$l \geqq n$ we are unable to make then $j+k+l+3$ changes.

$\{--+-+\}$ To make the 1st, 2nd and 4th positions negative we need to make $j+k+m+3$ changes. But again we are only allowed $j+k+n$. Since $m \geqq n$ we are unable to make these positions negative.

$\{--++-\}$ To make these positions negative we need to make $j+k+n+3$ changes, but we are allowed to make $j+k+n$ changes. So, we cannot make these positions negative.

$\{-+--+\}$ To make these positions negative we need to make $j+l+m+3$ changes, but we are allowed only $j+k+n$ changes. Since $m \geqq n$ we are unable to make these positions negative.

$\{-+-+-\}$ To make these positions negative we need to make $j+l+n+3$ changes, but we are only allowed $j+k+n$ changes. Since $l \geqq k$, we cannot make these three positions negative.

$\{-++--\}$ Assume that $2j+1-2p<0, 2k+1-2q>0, 2l-2r>0, 2m-2s<0$, and $2n-2t<0$. Then our resulting knot by 21 is $-[2p-2j-2, 1, 2k-2q, 2l-2r-1, 1, 2s-2m-1, 2t-2n]$. The only possible non negative terms are if $2p-2j-2=0$ or if $2k-2q=0$. But, even if they are both zero we still have $-[1, 2s-2m-1, 2t-2n]$ which is non-trivial.

$\{+---+\}$ Assume that $2j+1-2p>0, 2k+1-2q<0, 2l-2r<0, 2m-2s<0$, and $2n-2t>0$. Then our resulting knot by 22 is $[2j-2p, 1, 2q-2k-2, 2r-2l, 2s-2m-1, 1, 2n-2t]$. The only possible non positive terms are if $2j-2p=0$ and if $2q-2k-2=0$. But, even if both of these equal zero we still have the knot $[2s-2m-1, 1, 2n-2t]$ which will still be non trivial.

$\{+--+-\}$ Assume that $2j+1-2p>0, 2k+1-2q<0, 2l-2r<0, 2m-2s>0$, and $2n-2t<0$. Then our resulting knot will be $-[2j-2p, 1, 2q-2k-2, 2r-2l-2, 1, 2m-2s-2, 1, 2t-2n-1]$. Notice that we have already made at least $k+l+n+3$ crossing changes. We have at most $j-l-3$ left to make. The only possible non-positive terms are $2q-2k-2=0$,$2j-2p=0$, $2r-2l-2=0$, and $2m-2s-2=0$. If all of them are zero then we get the knot $-[2, 2t-2n-1]$ which is still knotted since $2t-2n \geqq 2$.

$\{+-+--\}$ Assume that $2j+1-2p>0, 2k+1-2q<0, 2l-2r>0, 2m-2s<0$, and $2n-2t<0$. Then our resulting knot is $-[2j-2p, 1, 2q-2k-3, 1, 2l-2r-2, 1, 2s-2m-1, 2t-2n]$. Since we have made at least $k+m+n+3$ changes we have at most $j-m-3$ changes left to make. So we know that $2j-2p>7$. The terms that may be non positive are if $2q-2k-3=-1$ and if $2l-2r-2=0$. If $2q-2k-3=-1$ and $2l-2r-2=0$ then our resulting knot is $-[2j-2p-2, 1, 2s-2m-2, 2t-2n]$. From above we know that $2j-2p-2>0$, so even if $2s-2m-2=0$ we still have a non trivial knot.

$\{++---\}$ Assume that $2j+1-2p>0, 2k+1-2q>0, 2l-2r<0, 2m-2s<0$,and $2n-2t<0$. Then our resulting knot is $-[2j+1-2p, 2k-2q, 1, 2r-2l-1, 2s-2m, 2t-2n]$. The only term which can be non negative is when $2k-2q=0$. But, even if $2k-2q=0$ we still get the knot $-[2j+2-2p, 2r-2l-1, 2s-2m, 2t-2n]$ which is non-trivial.

Part IV. *Four positions are changed to negative*

If $2j+1-2p<0$ then $p \geqq j+1$. Then we need $q+r+s+t \leqq k+n-1$. Even if we choose the next largest position to recieve zero changes $(r=0)$, we still have $q+s+t \leqq k+n-1$ where we need $q+s+t=k+m+n+3$. Hence, we have reached a contradiction. So, we cannot make the first position negative and still have four negative positions. So the only case we need consider is when we make the last four cases negative. For simplicity we will assume we have the knot $[a, -b, -c, -d, -e]$ where a, b, c, d and e can be any value. The equivalent knot will be $-[a-1, 1, b-1, c, d, e]$. If a and b are greater then 1 then we are done. If $a=1$ and $b>1$ then we get $-[b-1, c, d, e]$ which is still knotted. If $a>1$ and $b=1$ then we get $-[a-1, c+1, d, e]$ which is also knotted. If $a=1$ and $b=1$ then we will get $-[d, e]$ which is also knotted.

It has just been show that the knot $[2j+1, 2k+1, 2l, 2m, 2n]$ with $j \geqq l \geqq k \geqq m \geqq n$ cannot be unknotted in fewer than $j+k+n+1$ changes. Since its minimal projection can be unknotted in $j+k+n+1$ changes we know that $U_{min}(k)=j+k+n+1$.

Prime Knot with Gap One

Previously we have proved that the knot $[2j+1, 2k+1, 2l, 2m, 2n]$ with $j \geqq l \geqq k \geqq l \geqq m \geqq n$ has $U_{min}(k)=j+k+n+1$. We now describe a projection of this knot where $U(k)=j+k+n$. This implies that the knot $[2j+1, 2k+1, 2l, 2m, 2n]$ will have $Gap(k) \geqq 1$.

The projection of k that we looked at is its even continued fraction expansion:

$$[2j+1, 2k+1, 2l, 2m, 2n] \equiv [\ \overbrace{2, -2}^{k-1 \text{ times}}\ , 2k+2, 2l, 2m, 2n]$$

We know that these knots are equivalent by looking at their continued fractions. To see that $[\ \overbrace{2, -2}^{k-1 \text{ times}}\ , 2k+2, 2l, 2m, 2n]$ can be unknotted $j+k+n$ changes consider making j-1 changes on the set of negative and positive $2's$, $k+1$ changes on $2k+2$, and n changes on the last position. The resulting knot is $[\ \overbrace{0, -2}^{j-l-1 \text{ times}}\ , 2, -2, \overbrace{0, -2}^{l \text{ times}}, 0, 2l, 2m, 0]$. After using our zero reduction property the trivial knot will result. Therefore, we are able to unknot the $[2j+1, 2k+1, 2l, 2m, 2n]$ knot in non minimal projection in less then $j+k+n+1$ crossing changes producing a Gap $\geqq 1$.

Conclusion

We have shown that for several families of knots the unknotting number of the actual knot will be lower then the unknotting number for its minimal projection. Although we did not directly find the exact unknotting number for these knots, we were able to give a upper bound on the unknotting number. Current results and previous ones by Murasugi try to find a lower bound on the unknotting number [M]. They seem to indicate that one may be able to compute the lower bound exactly and therefore get the exact Gap for the knot. Further research into these areas should provide the exact gaps for all Conway knots. The one missing piece is to develop an easy way to unknot the minimal projection of any Conway knot. We conjecture that to unknot a minimal projection of a Conway knot with arbitrarily large tangles, the only crossing changes that need to be checked are if a position is changed to -1, 1, or 0. We also conjecture that the Murasugi lower bound specifies the positions that must be unwound to unknot the minimal projection.

Continued Fraction Identities for 5-Component Conway Knots

In the following tables, the variables a, b, c, d, e are considered to be positive integers > 0. These identities can all be proved using the continued fraction relationship of Conway Knots.

1.$[-a, b, c, d, e] \equiv [a-1, 1, b-1, c, d, e]$
2.$[a, -b, c, d, e] \equiv [a-1, 1, b-2, 1, c-1, d, e]$
3.$[a, b, -c, d, e] \equiv [a, b-1, 1, c-2, 1, d-1, e]$
4.$[a, b, c, -d, e] \equiv [a, b, c-1, 1, d-2, 1, e-1]$
5.$[a, b, c, d, -e] \equiv -[a, b, c, d-1, 1, e-1]$
6.$[-a, -b, c, d, e] \equiv [a, b-1, 1, c-1, d, e]$
7.$[-a, b, -c, d, e] \equiv [a-1, 1, b-2, 1, c-2, 1, d-1, e]$
8.$[-a, b, c, -d, e] \equiv [a-1, 1, b-1, c-1, 1, d-2, 1, e-1]$
9.$[-a, b, c, d, -e] \equiv -[a-1, 1, b-1, c, d-1, 1, e-1]$
10.$[a, -b, -c, d, e] \equiv [a-1, 1, b-1, c-1, 1, d-1, e]$
11.$[a, -b, c, -d, e] \equiv [a-1, 1, b-2, 1, c-2, 1, d-2, 1, e-1]$
12.$[a, -b, c, d, -e] \equiv -[a-1, 1, b-2, 1, c-1, d-1, 1, e-1]$
13.$[a, b, -c, -d, e] \equiv [a, b-1, 1, c-1, d-1, 1, e-1]$
14.$[a, b, -c, d, -e] \equiv -[a, b-1, 1, c-2, 1, d-2, 1, e-1]$
15.$[a, b, c, -d, -e] \equiv -[a, b, c-1, 1, d-1, e]$
16.$[-a, -b, -c, d, e] \equiv [a, b, c-1, 1, d-1, e]$
17.$[-a, -b, c, -d, e] \equiv [a, b-1, 1, c-2, 1, d-2, 1, e-1]$
18.$[-a, -b, c, d, -e] \equiv -[a, b-1, 1, c-1, d-1, 1, e-1]$
19.$[-a, b, -c, -d, e] \equiv [a-1, 1, b-2, 1, c-1, d-1, 1, e-1]$
20.$[-a, b, -c, d, -e] \equiv -[a-1, 1, b-2, 1, c-2, 1, d-2, 1, e-1]$
21.$[-a, b, c, -d, -e] \equiv -[a-1, 1, b-1, c-1, 1, d-1, e]$
22.$[a, -b, -c, -d, e] \equiv [a-1, 1, b-1, c, d-1, 1, e-1]$
23.$[a, -b, -c, d, -e] \equiv -[a-1, 1, b-1, c-1, 1, d-2, 1, e-1]$
24.$[a, -b, c, -d, -e] \equiv -[a-1, 1, b-2, 1, c-2, 1, d-1, e]$
25.$[a, b, -c, -d, -e] \equiv -[a, b-1, 1, c-1, d, e]$
26.$[-a, -b, -c, -d, e] \equiv [a, b, c, d-1, 1, e-1]$
27.$[-a, -b, -c, d, -e] \equiv -[a, b, c-1, 1, d-2, 1, e-1]$
28.$[-a, -b, c, -d, -e] \equiv -[a, b-1, 1, c-2, 1, d-1, e]$
29.$[-a, b, -c, -d, -e] \equiv -[a-1, 1, b-2, 1, c-1, d, e]$
30.$[a, -b, -c, -d, -e] \equiv -[a-1, 1, b-1, c, d, e]$

31.$[-1, b, c, d, e] \equiv [b-1, c, d, e]$

32.$[a, -1, c, d, e] \equiv [a-2, 1, c-2, d, e]$

33.$[a, b, -1, d, e] \equiv [a, b-2, 1, d-2, e]$

34.$[a, b, c, -1, e] \equiv [a, b, c-2, 1, e-2]$

35.$[a, b, c, d, -1] \equiv [a, b, c, d-1]$

36.$[-1, 1, c, d, e] \equiv [d, e]$

37.$[-1, -1, c, d, e] \equiv [2, c-1, d, e]$

38.$[a, 1, -1, d, e] \equiv [d-a-1, e]$

39.$[a, -1, 1, d, e] \equiv [d+1-a, e]$

40.$[a, -1, -1, d, e] \equiv [a-1, 1, 1, d-1, e]$

41.$[a, b, -1, 1, e] \equiv [a, e+1-b]$

42.$[a, b, 1, -1, e] \equiv [a-1, 1, e-b-2]$

43.$[a_1, \ldots, a_n, 1, -1, 1, a_{n+1}, \ldots, a_m] \equiv [a_1, \ldots, a_n - 1, 1, a_{n+1} - 1, \ldots, a_m]$

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