SUFFICIENT CONDITIONS FOR GLOBAL STABILITY IN POPULATION MODELS

NANCY HEINSCHEL

August 12, 1994

ABSTRACT. Seven common population models for which local stability is equivalent to global stability are examined. Using methods suggested by Cull and Singer, we look for a single set of sufficient conditions for global stability satisfied by these seven models. Additionally, we hope these conditions will be relatively easy to test. Four models are proven to be globally stable by a Schwarzian derivative test. Methods for restricting other models to fit the criteria of the test are discussed. One model is found to have a region of positive Schwarzian derivative, modifying a claim by Singer.

Introduction

Population models have been formulated to model the growth and decay of a typical biological population. Characteristics of these models reflect the growth of a population until it reaches some environmental maximum capacity, and the subsequent decline in populations larger than that capacity. In "nicely" behaved population models, the population may oscillate between growth and decay, but will eventually reach a stable size at which birth and death rates are equal, regardless of the initial size of the population. This convergence to a stable size, called global stability, is a characteristic of the population models we will be examining. However, it is not a property inherent in our definition of population model. In fact, functions which qualify as population models can behave in numerous ways, from cycling infinitely through a number of sizes to becoming chaotic, which do not result in a stable population size. It is also possible to have a model where convergence to a stable size occurs for all initial population sizes in a small region about the stable size, but not for other initial sizes. This is known as local stability. It is important for those applying population models to know whether or not their models are globally stable. Models having this property are predictable, while those that do not can exhibit unexpected behavior.

Using the definition of global stability or equivalent conditions to determine whether a model is globally stable is quite difficult, if not impossible. Sets of sufficient conditions for global stability which are relatively easy to check exist, but each only applies to a limited number of the seven commonly studied models which we are interested in examining. Intuitively, there should be a simple relation among these and other globally stable models, which causes them to have this property. Our goal is to find a single set of easy to test

REU Summer Research Program, Oregon State University; Advisor Paul Cull .

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

sufficient conditions for global stability which are satisfied by the seven population models under consideration and could also be applied to other models outside our area of inquiry.

Previous Work

Authors such as Fisher *et al.* (1979) and Goh (1979) construct Liapunov functions in order to prove that models are globally stable. The required calculations are laborious, so we do not pursue this method any further in this work.

The models that we will study are proven in Cull (1988) to have the property of equivalent local and global stability. Cull requires two distinct theorems to prove global stability for all seven models; each of Cull's theorems is satisfied by only five of the seven models. The theorems are relatively easy to use, but we seek to unite all seven models under a single set of sufficient conditions. The first of the theorems uses the first three derivatives of the population model, checking them for size and positivity or negativity in various regions. The second examines the derivatives of functions formed using the population model, again looking for positivity or negativity in certain regions. Additionally, Cull's results include a theorem and corollary which give conditions equivalent to the definition of global stability. The theorem, while quite intuitive, is difficult to apply to a given model to determine stability. The corollary helps us to check some models for stability very easily (those with no critical point smaller than the equilibrium point), but for others is not easily applied.

Cull normalizes all of his models to have the same equilibrium point. For four of our models we choose to use the variations on Cull's models mentioned by Singer (1978). Singer's work considers functions more general than our population models, but includes a theorem which gives sufficient conditions for a property equivalent to global stability. This requires that the Schwarzian derivative of our function, a formula involving its first three derivatives, be negative everywhere. Singer claims that this condition is satisfied by his four models. In order to meet the other criteria of Singer's theorem we must restrict our models to a closed interval. Hence we look for our models to have negative Schwarzian everywhere so that this restriction can be made arbitrarily.

We will be testing models presented by Cull and Singer to see if they fulfill the requirements of Singer's theorem. If the theorem does not apply directly, we will consider different ways of viewing the models, using information found in Cull's work, in order to allow use of the theorem.

Definitions

A *population model* is a function of the form

$$x_{t+1} = f(x_t)$$

where f is a continuous function from the nonnegative reals to the nonnegative reals and there is a positive number $\bar{x}$, the equilibrium point, such that

$$\begin{aligned} &f(0) = 0 \\ &f(x) > x \qquad \text{for } 0 < x < \bar{x} \\ &f(x) = x \qquad \text{for } x = \bar{x} \\ &f(x) < x \qquad \text{for } x > \bar{x} \end{aligned}$$

and if $f'(x_M) = 0$ and $x_M \leq \bar{x}$ then

$$\begin{aligned} f'(x) &> 0 \qquad \text{for } 0 \leq x < x_M \\ f'(x) &< 0 \qquad \text{for } x > x_M \text{ such that } f(x) > 0. \end{aligned}$$

We will allow $f(x) = 0$ for all $x > \hat{x}$ yielding the possibility that f is not strictly differentiable at $\hat{x}$. Otherwise we assume that the derivative is continuous whenever it exists.

A population model is *globally stable* if and only if for all x_0 such that $f(x_0) > 0$ we have

$$\lim_{t\to\infty} x_t = \bar{x}$$

where $\bar{x}$ is the unique equilibrium point of $x_{t+1} = f(x_t)$.

A population model is *locally stable* if and only if there is some small neighborhood of $\bar{x}$ such that for all x_0 in this neighborhood, x_t is in this neighborhood for all t, and $\lim_{t\to\infty} x_t = \bar{x}$.

The *Schwarzian derivative* (Schwarzian) of f at a point x is given by:

$$S(f,x) = \frac{f^{(3)}(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

for any real valued function f with at least three continuous derivatives.

Let $\{x_n\}$ be a sequence of points defined recursively by a function f as $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \ldots$, and such that $f(x_q) = x_p$ for some $1 \leq p < q$. Then the sequence $x = \{x_p, x_{p+1}, \ldots, x_q\}$ is an *orbit* of period $q - p + 1$. Given another recursively defined sequence $\{y_0, y_1, \ldots\}$, we define the distance from any element y_i of the sequence to x as $d(y_i, x) = \min_{x_j \in x} |y_i - x_j|$. If $\lim_{n\to\infty} d(y_n, x) = 0$, regardless of the choice of y_0, we call the sequence $\{x_p, x_{p+1}, \ldots, x_q\}$, a *stable orbit.*

Finally, an *endomorphism* is a function $f : [0,1] \to [0,1]$. For our purposes we will also assume that endomorphisms possess at least three continuous derivatives.

Models

Model Number	*Function*	*Author Cited*	*Theorems Satisfied*
1	$f_1(x) = xe^{r\left(1-\frac{x}{k}\right)}$	Singer	$\mathcal{A},\mathcal{B},\mathcal{S}$
2	$f_2(x) = x\left(1 + r\left(1 - \frac{x}{k}\right)\right)$	Singer	$\mathcal{A},\mathcal{B},\mathcal{S}$
3	$f_3(x) = x(1 - r\ln x)$	Cull	$\mathcal{A}$
4	$f_4(x) = x\left(\frac{1}{b+cx} - d\right)$	Cull	$\mathcal{A},\mathcal{S}$
5	$f_5(x) = \frac{rx}{1+e^{-A\left(1-\frac{x}{B}\right)}}$	Singer	$\mathcal{A},\mathcal{B},\mathcal{S}$
6	$f_6(x) = \frac{rx}{\left(1+\frac{x}{B}\right)^b}, b > 1$	Singer	$\mathcal{B}$
7	$f_7(x) = \frac{rx}{1+(r-1)x^c}$	Cull	$\mathcal{B}$

Local Stability Implies Global Stability

The following theorems, which we denote $\mathcal{A}$ and $\mathcal{B}$ for convenience, were proven by Cull in order to justify that local stability implies global stability in our seven models.

Theorem $\mathcal{A}$. *If a population model has a maximum x_M in $(0, x_M)$ and satisfies:*

(1) $f''(x) < 0$ *for x in* $[x_M, \bar{x})$;
(2) $f^{(3)}(x) \geq 0$ *for all x such that* $f''(x) < 0$ *and f'' has at most one sign change; and*
(3) $|f'(\bar{x})| \leq 1$,

then the model is globally stable.

Theorem $\mathcal{A}$ is used by Cull to prove that local stability implies global stability for Models 1, 2, 3, 4, and 5. For Model 5, calculation of the third derivative becomes somewhat tedious. Otherwise, this theorem is an easy way to check that the models are globally stable whenever they are locally stable (condition (3) of the theorem is a necessary condition for local stability). However, to prove global stability for the remaining two models we must turn to Cull's Theorem $\mathcal{B}$.

Theorem $\mathcal{B}$. *For a population model f, let* $k = k(x) = \frac{x}{f(x)}$. *Let the function g be defined by* $\frac{k}{k'} = g(x) + Bx$ *where B is a constant chosen to make $g(x)$ nonnegative. If the population model satisfies:*

(1) $f'(\bar{x}) = -1$;
(2) $k' \leq 2$ *on* $[x_M, \bar{x})$;
(3) $g(x) \geq 0$ *on* $[x_M, f(x_M)]$;
(4) $g'(x) \leq 0$ *on* $[x_M, f(x_M)]$; *and*
(5) $g''(x) \geq 0$ *on* $[x_M, f(x_M)]$,

then the model is globally stable.

Theorem $\mathcal{B}$ is used by Cull to prove that local stability implies global stability for Models 1, 2, 5, 6, and 7. Calculation of the derivatives becomes complex for Models 5, 6, and 7. Consequently, this theorem is more difficult to use than Theorem $\mathcal{A}$. However, it is still a relatively easy theorem to use, and is necessary to prove global stability for the models that are not covered by Theorem $\mathcal{A}$.

Another of Cull's theorems will be necessary for the implentation of Singer's theorem to our models. We will call this Theorem $\mathcal{E}$ as it results in an equivalent definition of global stability.

Theorem $\mathcal{E}$. *A population model is globally stable if and only if it has no orbits of period two.*

Theorem $\mathcal{E}$ is intuitive: clearly if an orbit of period two exists in a population model, a single limit can not be achieved by any recursively defined sequence which includes the orbit. The other direction of implication is not so easily seen, but is proven in Cull. Unfortunately, this theorem is difficult to apply to a given model in order to determine whether or not it is globally stable. Also of interest is Cull's Corollary to Theorem $\mathcal{E}$.

Corollary. *A population model is globally stable if and only if either:*

(1) *there is no maximum of $f(x)$ in* $(0, \bar{x})$; *or*
(2) *there is a maximum of $f(x)$ at x_M in* $(0, \bar{x})$ *and* $f(f(x)) > x$ *for all x in* $[x_M, \bar{x})$.

The first claim of the theorem allows Cull to eliminate a number of cases before applying Theorems $\mathcal{A}$ and $\mathcal{B}$. Additionally, it indicates that these cases are in some sense trivial.

Accordingly, we can consider ignoring cases with no maximum in $(0, \bar{x})$ when trying to adjust our models to better meet the requirements of Singer's theorem.

The Schwarzian Derivative

Hoping to find a single set of sufficient conditions for global stability, we consider the work of David Singer. We call Singer's result Theorem $\mathcal{S}$ (for $\mathcal{S}$inger or $\mathcal{S}$chwarzian).

Theorem $\mathcal{S}$. *Let $\mathcal{G}$ be the set of all endomorphisms which satisfy:*

(1) $f(0) = f(1) = 0$;
(2) f *has a unique critical point in* $[0, 1]$; *and*
(3) $S(f, x) < 0$ *everywhere.*

Then for any f in $\mathcal{G}$ there is at most one stable orbit in $(0, 1)$.

From the Corollary to Theorem $\mathcal{E}$ we know that all of the models of particular interest to us have a unique critical point (x_M) in the closed interval $[0, \bar{x}]$. Clearly, a corresponding theorem with any other closed interval replacing $[0, 1]$ (in all parts of the theorem as well as in the definition of endomorhism) would yield the same results. Our models are of the form $f : [0, \infty) \to [0, \infty)$. Ideally we would like our models to have negative Schwarzian for all values of x. Given this result, we could choose a closed interval $[a, b]$ such that for all $x_0 \in [a, b]$, $x_t \in [a, b]$ for all t and also so that $x_M \in [a, b]$ and $\bar{x} \in [a, b]$ allowing our model to meet the requirements of the theorem. Suppose a model given by $f(x)$ satisfies the conditions of Theorem $\mathcal{E}$ in a closed interval containing $\bar{x}$. The requirement that $f(\bar{x}) = \bar{x}$ in a population model creates a stable orbit of period one, namely $\{\bar{x}, \bar{x}, \bar{x}, \dots\}$, within the interval. This must be the single stable orbit guaranteed by Theorem $\mathcal{S}$. Hence the model can not have a stable orbit of period two in the closed interval. It follows from Theorem $\mathcal{E}$ that the limited model is globally stable.

Consequently, we must begin by finding the Schwarzian derivative of each of our models. Singer claims that condition (3) of his theorem holds for Models I, II, V, and VI, which we attempt to verify.

Calculation of the Schwarzian

(Model 1)

$$f_1(x) = xe^{r\left(1-\frac{x}{k}\right)}$$

$$f_1'(x) = \left(1 - \frac{rx}{k}\right) e^{r\left(1-\frac{x}{k}\right)}$$

$$f_1''(x) = -\frac{r}{k}\left(2 - \frac{rx}{k}\right) e^{r\left(1-\frac{x}{k}\right)}$$

$$f_1^{(3)}(x) = \frac{r^2}{k^2}\left(3 - \frac{rx}{k}\right) e^{r\left(1-\frac{x}{k}\right)}$$

$$S(f_1, x) = \frac{-r^2}{2k^2\left(1 - \frac{rx}{k}\right)^2}\left(\frac{r^2x^2}{k^2} - 4\frac{rx}{k} + 6\right)$$

(Model 2)

$$f_2(x) = x\left(1 + r\left(1 - \frac{x}{k}\right)\right)$$
$$f_2'(x) = 1 + r - 2\frac{rx}{k}$$
$$f_2''(x) = -2\frac{r}{k}$$
$$f_2^{(3)}(x) = 0$$
$$S(f_2, x) = \frac{-6r^2}{k^2\left(1 - r - 2\frac{rx}{k}\right)^2}$$

(Model 3)

$$f_3(x) = x(1 - rlnx)$$
$$f_3'(x) = 1 - r - rlnx$$
$$f_3''(x) = -\frac{r}{x}$$
$$f_3^{(3)}(x) = \frac{r}{x^2}$$
$$S(f_3, x) = r\left[\frac{2 - 5r - 2rlnx}{2x^2(1 - r - rlnx)}\right]$$

(Model 4)

$$f_4(x) = x\left(\frac{1}{b + cx} - d\right)$$
$$f_4'(x) = \frac{b}{(b + cx)^2} - d$$
$$f_4''(x) = \frac{-2bc}{(b + cx)^3}$$
$$f_4^{(3)}(x) = \frac{6bc^2}{(b + cx)^4}$$
$$S(f_4, x) = \frac{-6bc^2d}{[b - d(b + cx)^2]^2}$$

(Model 5)

$$f_5(x) = \frac{rx}{1+e^{-A\left(1-\frac{x}{B}\right)}}$$

$$f_5'(x) = r\left[\frac{1+(1-\frac{xA}{B})e^{-A\left(1-\frac{x}{B}\right)}}{\left(1+e^{-A\left(1-\frac{x}{B}\right)}\right)^2}\right]$$

$$f_5''(x) = \frac{-rA}{B}e^{-A\left(1-\frac{x}{B}\right)}\left[\frac{\frac{xA}{B}\left(1-e^{-A\left(1-\frac{x}{B}\right)}\right)+2\left(1+e^{-A\left(1-\frac{x}{B}\right)}\right)}{\left(1+e^{-A\left(1-\frac{x}{B}\right)}\right)^3}\right]$$

$$\begin{aligned} f_5^{(3)}(x) = \left(\frac{-rA^2}{B^2}\right) & \\ & \left\{\frac{\left[\frac{xA}{B}\left(1-e^{-A\left(1-\frac{x}{B}\right)}\right)+2\left(1+e^{-A\left(1-\frac{x}{B}\right)}\right)+(1-\frac{xA}{B})\,e^{-A\left(1-\frac{x}{B}\right)}+1\right]}{e^{A\left(1-\frac{x}{B}\right)}\left(1+e^{-A\left(1-\frac{x}{B}\right)}\right)^5}\right\} \\ & + \left(\frac{rA^2}{B^2}\right)\left\{\frac{3e^{-A\left(1-\frac{x}{B}\right)}\left[\frac{xA}{B}\left(1-e^{-A\left(1-\frac{x}{B}\right)}\right)+2\left(1+e^{-A\left(1-\frac{x}{B}\right)}\right)\right]}{e^{A\left(1-\frac{x}{B}\right)}\left(1+e^{-A\left(1-\frac{x}{B}\right)}\right)^4}\right\} \end{aligned}$$

$$\begin{aligned} S(f_5,x) = -\left(\frac{A^2}{2B^2e^{A\left(1-\frac{x}{B}\right)}}\right) & \\ & \left\{\frac{6+2\frac{xA}{B}+\frac{18+\frac{x^2A^2}{B^2}}{e^{A\left(1-\frac{x}{B}\right)}}+\frac{18-6\frac{xA}{B}+2\frac{x^2A^2}{B^2}}{e^{2A\left(1-\frac{x}{B}\right)}}+\frac{6-4\frac{xA}{B}+\frac{x^2A^2}{B^2}}{e^{3A\left(1-\frac{x}{B}\right)}}}{\left(1+e^{-A\left(1-\frac{x}{B}\right)}\right)^2\left[1+(1-\frac{xA}{B})\,e^{-A\left(1-\frac{x}{B}\right)}\right]^2}\right\} \end{aligned}$$

(Model 6)

$$f_6(x) = \frac{rx}{\left(1+\frac{x}{B}\right)^b} \qquad b>1$$

$$f_6'(x) = r\left[\frac{1+\frac{x}{B}(1-b)}{\left(1+\frac{x}{B}\right)^{b+1}}\right]$$

$$f_6''(x) = \frac{-rb}{B}\left[\frac{2+\frac{x}{B}(1-b)}{\left(1+\frac{x}{B}\right)^{b+2}}\right]$$

$$f_6^{(3)}(x) = \frac{rb(b+1)}{B^2}\left[\frac{3+\frac{x}{B}(1-b)}{\left(1+\frac{x}{b}\right)^{b+3}}\right]$$

$$S(f_6,x) = \frac{b\left\{2(b+1)\left[3+\frac{x}{B}(1-b)\right]\left[1+\frac{x}{B}(1-b)\right] - 3b\left[2+\frac{x}{B}(1-b)\right]^2\right\}}{2B^2\left(1+\frac{x}{B}\right)^2\left[1+\frac{x}{B}(1-b)\right]^2}$$

(Model 7)

$$f_7(x) = \frac{rx}{1+(r-1)x^c}$$

$$f_7'(x) = \frac{r\left[1-(c-1)(r-1)x^c\right]}{\left[1+(r-1)x^c\right]^2}$$

$$f_7'' = \frac{-r(r-1)c\left\{x^{c-1}\left[c+1-(c-1)(r-1)x^c\right]\right\}}{\left[1+(r-1)x^c\right]^3}$$

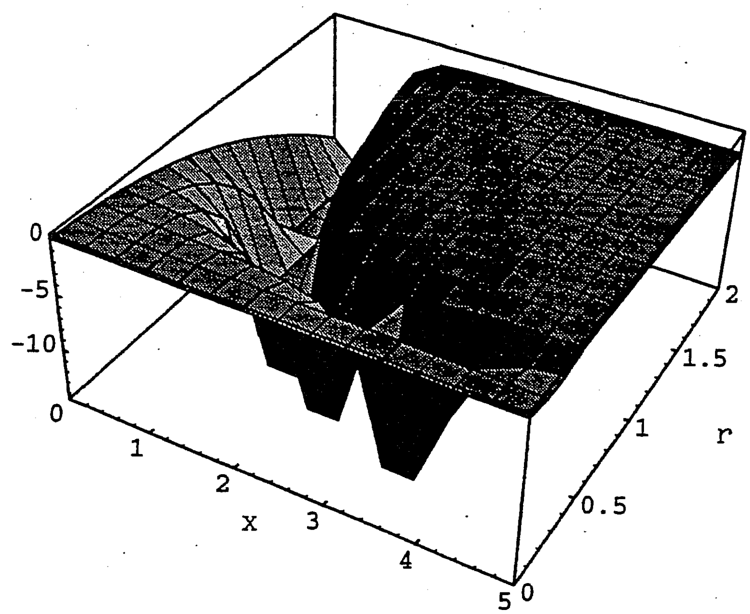
$$f_7^{(3)}(x) = r(r-1)cx^{c-2}\left\{\frac{(c-1)(r-1)x^c\left[c-2-(r-1)(c+1)x^c\right]-(c+1)\left[c-1-3c(r-1)x^c\right]}{\left[1+(r-1)x^c\right]^4}\right\}$$

$$S(f_7,x) = 2(r-1)cx^{c-2}\left[1-(c-1)(r-1)x^c\right]\left\{\frac{(c-1)(r-1)x^c\left[c-2-(r-1)(c+1)x^c\right]-(c+1)\left[(c-1)-3c(r-1)x^c\right]}{2\left[1+(r-1)x^c\right]^2\left[1-(c-1)(r-1)x^c\right]^2}\right\} - \frac{3(r-1)^2c^2x^{2c-2}\left[c+1-(c-1)(r-1)x^c\right]^2}{2\left[1+(r-1)x^c\right]^2\left[1-(c-1)(r-1)x^c\right]^2}$$

Interpretation of the Schwarzian

In all of the figures, except for the Schwarzian of Model 7, we fix all but one parameter and x equal to 1 so that we can graph the Schwarzians in three dimensions. For Model 7, we fix $r = 2$. Note the similar troughlike structures of each of the Schwarzian graphs, particularly in regions where the Schwarzian is negative.

FIGURE A. Model 1 Schwarzian



First, we note that the quadratic $z^2 - 4z + 6$ has no real roots. For $z = 1$ the quadratic is positive (it equals 3). Consequently, Model 1 has negative Schwarzian for all real values of $\frac{rx}{k}$ (see Figure A).

Since the third derivative of Model 2 is zero, the Schwarzian of Model 2 is trivially negative for all x (see Figure B).

For all nonnegative b and d the Schwarzian of Model 4 is clearly negative for all x (see Figure C).

Model 5 is also negative for all values of x and the parameters, but the proof of this is somewhat more complex. Note that a negative term has been factored out of the main fraction in braces. The only negative terms that remain inside the braces are $-6\frac{xA}{B}$ and $-4\frac{xA}{B}$. However,

$$\frac{xA}{B} \geq 3 \quad \Rightarrow \quad 2\frac{x^2A^2}{B^2} \geq 6\frac{xA}{B}$$

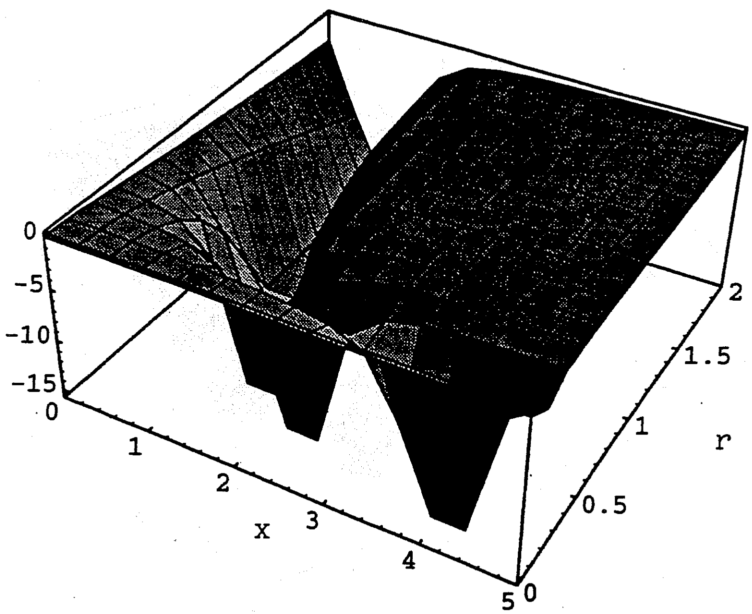
$$\Rightarrow \quad \frac{-6\frac{xA}{B} + 2\frac{x^2A^2}{B^2}}{e^{2A(1-\frac{x}{B})}} \geq 0$$

and

$$\frac{xA}{B} < 3 \quad \Rightarrow \quad 6\frac{xA}{B} < 18$$

$$\Rightarrow \quad \frac{18 - 6\frac{xA}{B}}{e^{2A(1-\frac{x}{B})}} > 0$$

FIGURE B. Model 2 Schwarzian



Thus, in both cases,

$$\frac{18 - 6\frac{xA}{B} + 2\frac{x^2A^2}{B^2}}{e^{2A\left(1-\frac{x}{B}\right)}} \geq 0$$

for all nonnegative values of x,A, and B. Similarly,

$$\frac{xA}{B} \geq 4 \qquad \Rightarrow \qquad \frac{x^2A^2}{B^2} \geq 4\frac{xA}{B}$$

$$\Rightarrow \qquad \frac{-4\frac{xA}{B} + \frac{x^2A^2}{B^2}}{e^{3A\left(1-\frac{x}{B}\right)}} \geq 0$$

and

$$\frac{xA}{B} \leq \frac{3}{2} \qquad \Rightarrow \qquad 4\frac{xA}{B} \leq 6$$

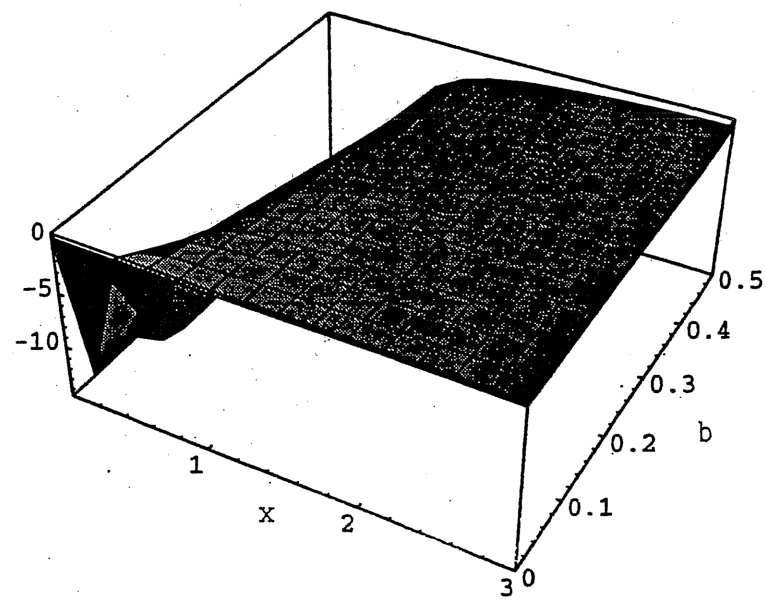
$$\Rightarrow \qquad \frac{6 - 4\frac{xA}{B}}{e^{3A\left(1-\frac{x}{B}\right)}} \geq 0$$

and

$$\frac{3}{2} < \frac{xA}{B} \leq \frac{33}{16}$$

$$\Rightarrow \qquad \frac{9}{4} < \frac{x^2A^2}{B^2} \text{and} \Rightarrow \qquad 4\frac{xA}{B} < \frac{33}{4} = 6 + \frac{9}{4}$$

FIGURE C. Model 4 Schwarzian



$$\Rightarrow \frac{6+\frac{x^2A^2}{B^2}}{e^{3A\left(1-\frac{x}{B}\right)}} > \frac{33}{4}\frac{1}{e^{3A\left(1-\frac{x}{B}\right)}} \geq \frac{4\frac{xA}{B}}{e^{3A\left(1-\frac{x}{B}\right)}}$$

The same method applied to

$$\frac{33}{16} < \frac{xA}{B} \leq \frac{2625}{1024},$$

$$\frac{2625}{1024} < \frac{xA}{B} \leq \frac{13182081}{4194304},$$

$$\frac{13182081}{4194304} < \frac{xA}{B} \leq \frac{2.793203758 \times 10^{14}}{7.036874416 \times 10^{13}},$$

and

$$\frac{2.793203758 \times 10^{14}}{7.036874416 \times 10^{13}} < \frac{xA}{B} \leq \frac{1.077304333 \times 10^{29}}{1.980704062 \times 10^{28}}$$

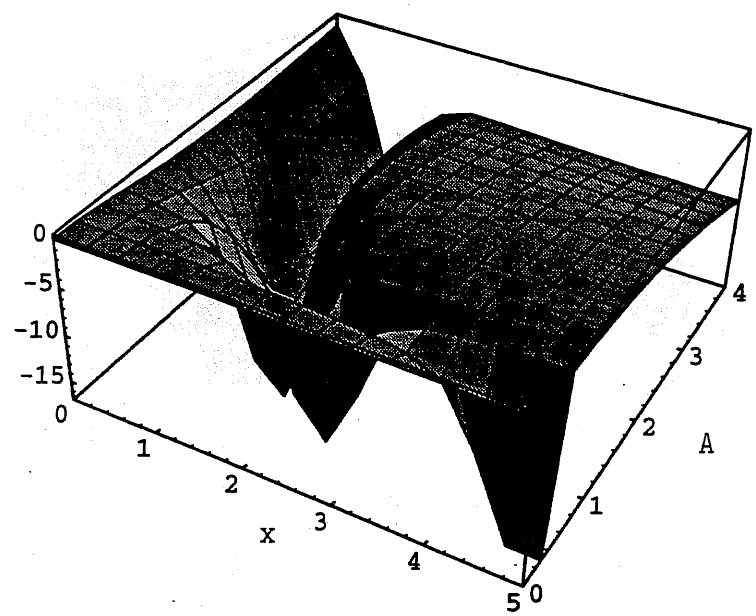
yields that

$$\frac{6+\frac{x^2A^2}{B^2}}{e^{3A\left(1-\frac{x}{B}\right)}} \geq \frac{4\frac{xA}{B}}{e^{3A\left(1-\frac{x}{B}\right)}}$$

for all x such that $\frac{3}{2} < x < 4$. Consequently,

$$\frac{6-4\frac{xA}{B}+\frac{x^2A^2}{B^2}}{e^{3A\left(1-\frac{x}{B}\right)}} \geq 0$$

FIGURE D. Model 5 Schwarzian



for all nonnegative x. We conclude that the Schwarzian of Model 5 is negative for all nonnegative x and A, and all positive B (see Figure D).

Since lnx is negative for $x < 1$, the Schwarzian of Model 3 is negative if and only if

$$x > e^{\left(\frac{1}{r}-\frac{5}{2}\right)}$$

(see Figure E). In the next section, we consider limiting our concentration to intervals in which the Schwarzian of Model 3 remains negative.

Singer claims that Model 6 satisfies condition (3) of Theorem $\mathcal{S}$, that is, $S(f_6, x)$ is negative everywhere. However, letting

$$x = 10.5, \qquad B = 1, \qquad \text{and} \qquad b = 1.5,$$

we discover that

$$S(f_6, x) = 0.000088303 > 0$$

contradicting Singer's claim. We conjecture that Singer may have intended that we consider only limited ranges for x when calculating the Schwarzian. Indeed, away from the region in which we chose the values for our parameters, the Schwarzian for Model 6 appears to be negative, except when $b \leq 1$ (see Figures F and G), which Singer does not allow.

Letting

$$r = 2, \qquad \text{and} \qquad c = 1,$$

in Model 7, yields $S(f_7, x) = 0$ for all positive x. However, we notice that for $r = 2$ and $c > 2$, the Schwarzian appears to be everywhere negative (see Figure H).

FIGURE E. Model 3 Schwarzian

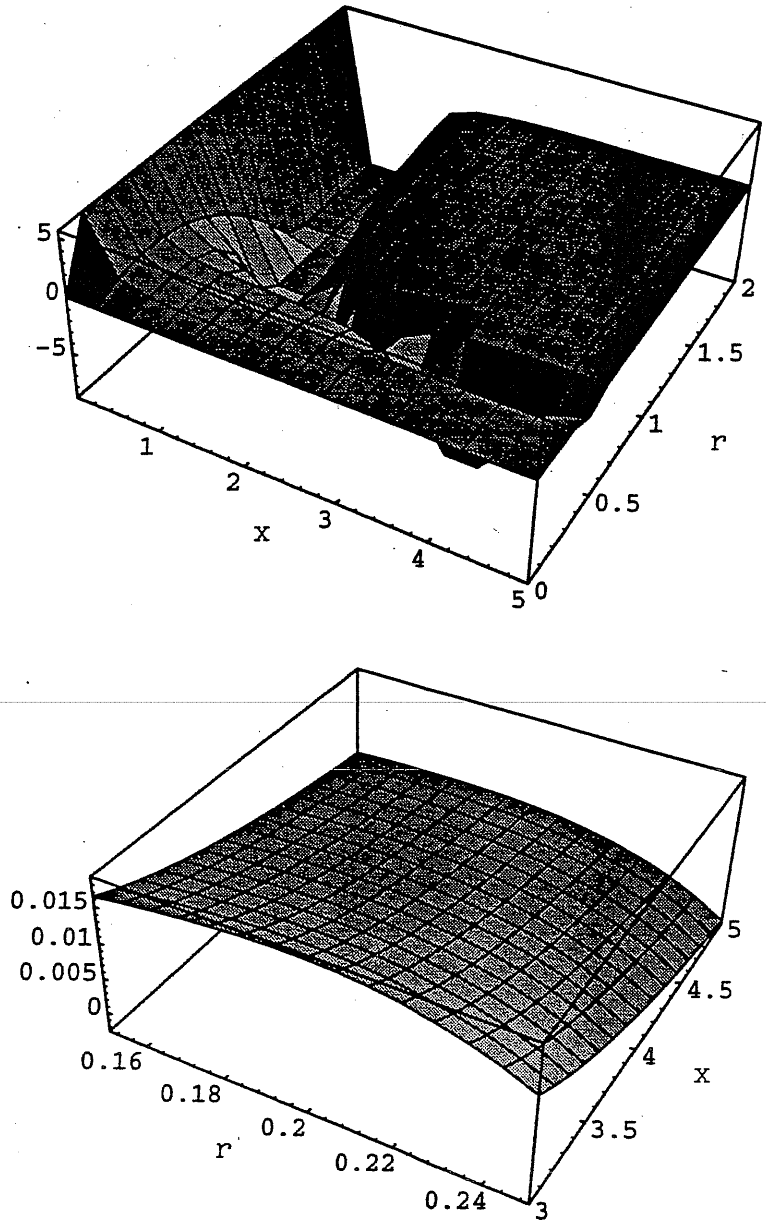
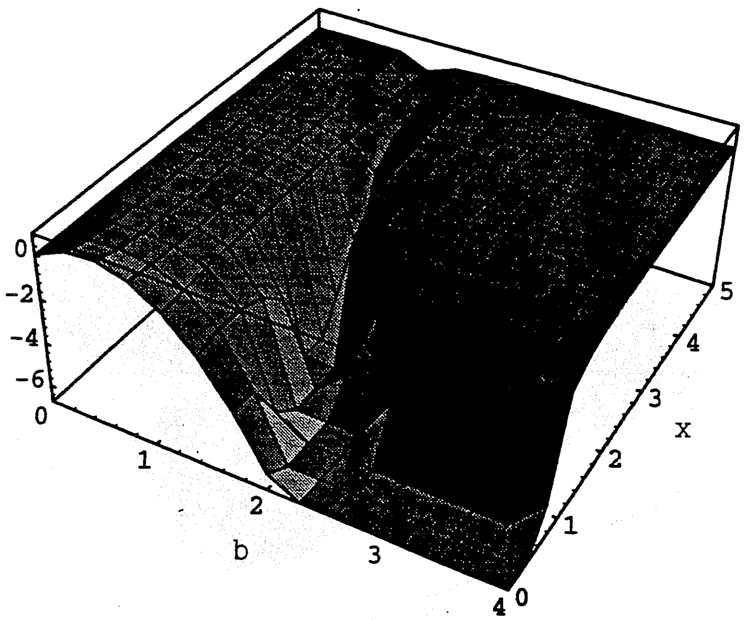


FIGURE F. Model 6 Schwarzian



Further Application of Theorem $\mathcal{S}$

We consider only Model 3. It is likely that Models 6 and 7 can be approached similarly, but time constraints have limited us to concentrate on only one model.

Suppose $r \geq 1$, this being our primary region of interest, since there is no maximum in the interval $(0, \bar{x})$ for $0 < r < 1$. Recall that the Schwarzian is nonnegative only for $x \leq e^{\left(\frac{1}{r}-\frac{5}{2}\right)}$, so we need to avoid this region.

Since we want to concentrate on some closed interval which contains x_M and $\bar{x}$, and $f_3(x_M)$ is maximal, we consider using $f_3^{-1}(\bar{x})$ and $f_3(x_M)$ as our left and right endpoints, respectively. Note that $x_M = e^{\left(\frac{1}{r}-1\right)}$, so

$$f_3(x_M) = re^{\left(\frac{1}{r}-1\right)}$$

and

$$f_3(f_3(x_M)) = r^2 e^{\left(\frac{1}{r}-1\right)},$$

so $x_M > e^{\left(\frac{1}{r}-\frac{5}{2}\right)}$ for all $r \geq 1$. Now, since there is only one maximum of the function, and it occurs at a value less than $\bar{x}$, $f_3^{-1}(\bar{x}) = \{\bar{x}, b\}$ where b is a value less than x_M. We want to use b as our left endpoint. Clearly, we can only do so for r such that $b > e^{\left(\frac{1}{r}-\frac{5}{2}\right)}$, so we restrict our attention to those cases. For $x \in [b, \bar{x})$ we have

$$b \leq x < f_3(x) \leq f_3(x_M)$$

FIGURE G. Model 6 Nonnegative Schwarzian

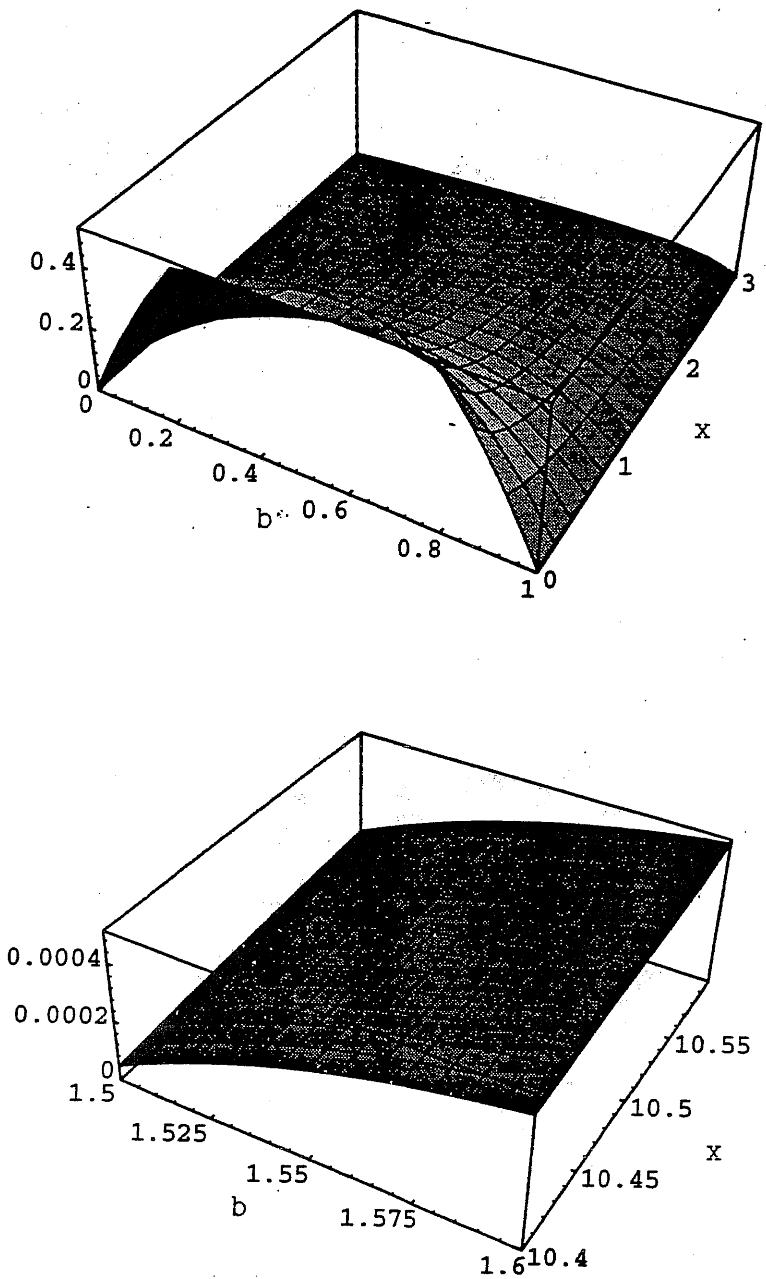
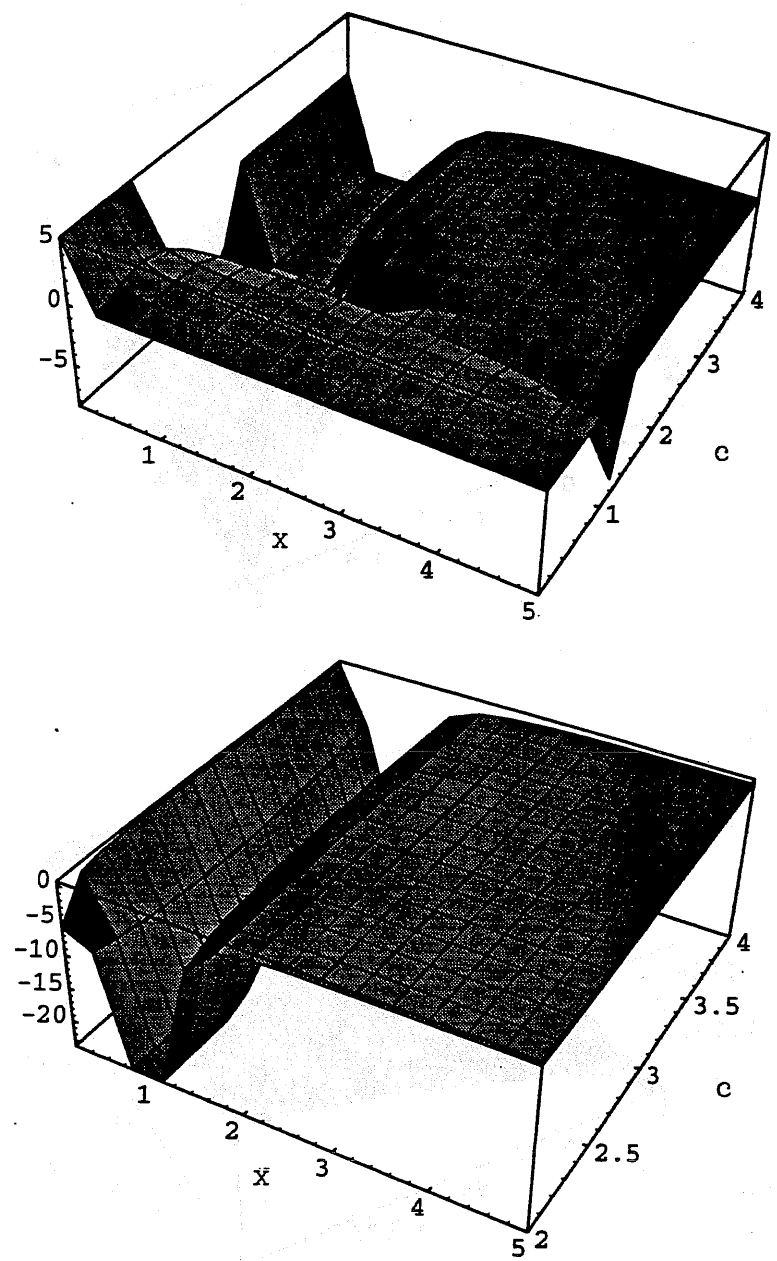


FIGURE H. Model 7 Schwarzian



by definition of population model and x_M. For $x \in [\bar{x}, f_3(x_M)]$ we have

$$b < x_M < f_3(f_3(x_M)) < f_3(x) < x < f_3(x_M)$$

where the first inequality comes from our choice of b; the second inequality is a result of the Corollary to Theorem $\mathcal{E}$ (since we know f_3 is globally stable by Theorem $\mathcal{A}$); the third and fourth occur because f_3 is a decreasing function for $x > \bar{x}$ by the definition of population model; and the final inequality is a result of our choice of x. Consequently, if $r \geq 1$ and r is chosen such that $b > e^{\left(\frac{1}{r} - \frac{5}{2}\right)}$, we have $x_t \in [b, x_M]$ for all t whenever $x_0 \in [b, x_M]$. Since all areas of negative Schwarzian occur for $x < b$, we have found a closed interval in which the Schwarzian of f_3 is everywhere negative and such that any recursively defined sequence of points initially within the interval remain within the interval. Hence, we can now apply Theorem $\mathcal{S}$ to this limited region of Model 3.

Conclusion

Theorem $\mathcal{S}$ is too restrictive to justify global stability for all seven models under consideration. However, it is a relatively easy test to apply, and four of the models satisfy its conditions. We can limit the range of our concentration for the other models in order to allow them to meet the criteria of the theorem. We are left with several questions related to our study. We still would like to find a single, easy to test set of sufficient conditions for global stability satisfied absolutely by all seven of our models. Models 3, 6, and 7 could be studied further to see if Theorem $\mathcal{S}$ can be better applied to them. Perhaps a clarification of any limits on the range of x intended by Singer (as indicated by the region of positive Schwarzian found in Model 6) would aid us in application of his theorem to our other models. Finally, we are left wondering what relation makes our seven models simple enough to be globally stable when they are not linked by any of the primary theorems we have considered.

References

1. H. Bai-lin, *Elementary Symbolic Dynamics and Chaos in Dissipative Systems*, World Scientific Publishing Co., 1989, pp. 38–40.
2. P. Cull, *Stability of discrete one-dimensional population models*, Bull. math. Biol. **50** (1988), 67–75.
3. M. E. Fisher, B. S. Goh, and T. L. Vincent, *Some stability conditions for discrete-time single species models*, Bull. math. Biol. **41** (1979), 861–875.
4. James Gleick, *Chaos: Making a New Science*, Penguin Books, 1987.
5. B. S. Goh, *Management and Analysis of Biological Populations*, Elsevier, 1979.
6. R. M. May and G. F. Oster, *Bifurcations and dynamic complexity in simple ecological models*, Amer. Natur. **110** (1976), 573–599.
7. D. Singer, *Stable orbits and bifurcation of maps of the interval*, SIAM J. Appl. Math. **35** (1978), 260–267.

University of California, Davis;

E-mail: neheinschel@ucdavis.edu