ON ISOLATED VALUES IN THE MARKOFF SPECTRUM

AMY POOL AND SUZANNE ZAWISTOWSKI

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ABSTRACT. In his doctoral thesis, David Crisp looks at isolated values of the Markoff spectrum. This is a further investigation of this topic, and includes an examination of the use of backwards continued fractions, as well as the application of Crisp's technique for showing that sequences have isolated values in the Markoff spectrum.

INTRODUCTION

The purpose of this paper is to examine isolated Markoff values. Our work was based on Chapter 6 of David Crisp's thesis, *The Markoff spectrum and geodesics on the punctured torus*, University of Adelaide. We define what it means for a sequence to have an isolated Markoff value, and note those which are known to have isolated Markoff values. We then outline the method Crisp uses to prove that certain families of sequences have isolated Markoff values. We divide his method into three steps, and state the lemmas necessary to accomplish each step.

We worked to convert his lemmas from using ordinary continued fraction to using backwards continued fractions. Our hope was that backwards continued fractions might simplify the process of looking for isolated Markoff values, however problems occured with the conversion of Crisp's lemmas towards this purpose.

As an alternative, we began examining specific sequences using Crisp's method. We were able to show that several infinite families of sequences had locally isolated Markoff values. We also worked through specific sequences, showing that Crisp's method could be adapted for use in a systematic manner. It seems likely that this technique could be applied for automated use. In addition, we extended Crisp's method using specific computations of Markoff values. It appears that this may also be adaptable for use by computer.

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DEFINITIONS

Definition. The *ordinary continued fraction* representation of a real number α is

$$\alpha = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

where a_i is a positive integer for $i \geq 1$, and a_0 is a non-negative integer. If α has a finite continued fraction representation, then it can be expressed as the ratio

$$\alpha = [a_0; a_1, a_2, \ldots, a_n] = \frac{K(a_0, a_1, a_2, \ldots, a_n)}{K(a_1, a_2, \ldots, a_n)},$$

where the function K is defined as

$$\begin{aligned} &K(a_0) = a_0; \\ &K(a_0, a_1) = a_0 a_1 + 1; \\ &K(a_0, a_1, \ldots, a_n) = a_n K(a_0, a_1, \ldots, a_{n-1}) + K(a_0, a_1, \ldots, a_{n-2}). \end{aligned}$$

We also know that

$$K(a_0, a_1, \ldots, a_n, 1, 1) = K(a_0, a_1, \ldots, a_n, 2),$$
$$K(a_0, a_1, \ldots, a_n) = K(a_n, a_{n-1}, \ldots, a_0).$$

Furthermore, if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$\begin{aligned} a &= K(a_0, a_1, \ldots, a_n); \\ b &= K(a_0, a_1, \ldots, a_{n-1}); \\ c &= K(a_1, a_2, \ldots, a_n); \\ d &= K(a_1, a_2, \ldots, a_{n-1}); \end{aligned}$$

and

(1) $$f(x) = \frac{ax + b}{cx + d} = [a_0; a_1, a_2, \ldots, a_n, x].$$

The determinant of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (-1)^{n+1}$, so

(2) $$f(x) - f(y) = \frac{(ad - bc)(x - y)}{(cx + d)(cy + d)} = \frac{(-1)^{n+1}(x - y)}{(cx + d)(cy + d)}.$$

Definition. Let $\mathcal{A} = \dots, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, \dots$ be a doubly infinite sequence of positive integers. The *Markoff value* of $\mathcal{A}$ is

$$M(\mathcal{A}) = \sup_n \lambda_n(\mathcal{A}),$$

where

$$\lambda_n = [a_n; a_{n+1}, a_{n+2}, \dots] + [0; a_{n-1}, a_{n-2}, \dots].$$

Note that if $\mathcal{A}$ is periodic then it can be re-indexed so that $M(\mathcal{A}) = \lambda_0(\mathcal{A})$. When this condition holds, we will indicate the a_0 position in the sequence with an asterisk.

For the sake of simplification, Crisp concentrated on sequences composed solely of 1's and 2's, and we shall do likewise.

Definition. The *Markoff spectrum* is defined as

$$MS = \{M(\mathcal{A}) : \mathcal{A} \text{ is a doubly infinite sequence of positive integers}\}.$$

Definition. Let the distance between two doubly infinite sequences, $\mathcal{A}$ and $\mathcal{B}$, be

$$d(\mathcal{A}, \mathcal{B}) = \begin{cases} 0, & \text{if } \mathcal{A} = \mathcal{B} \\ 1/(k+1), & \text{if } \mathcal{A} \neq \mathcal{B}, \end{cases}$$

where k is the largest integer such that $a_i = b_i$ for $-k < i < k$.

Definition. Let $\mathcal{A} = \{a_i\}_{i=-\infty}^{+\infty}$ be a doubly infinite sequence of positive integers. $M(\mathcal{A})$ is *locally isolated* if and only if there exist $\epsilon, \delta > 0$ such that for every doubly infinite sequence $\mathcal{B}$, if $d(\mathcal{A}, \mathcal{B}) < \delta$ then either

$$M(\mathcal{A}) = M(\mathcal{B}) \qquad \text{or} \qquad |M(\mathcal{A}) - M(\mathcal{B})| > \epsilon.$$

Definition. The doubly infinite sequence $\mathcal{A}$ has an *isolated Markoff value* if and only if there exists $\epsilon > 0$ such that for every doubly infinite sequence $\mathcal{B} \neq \mathcal{A}$,

$$\text{if } |M(\mathcal{A}) - M(\mathcal{B})| < \epsilon \qquad \text{then } M(\mathcal{A}) = M(\mathcal{B}).$$

Known Isolated Families

Before searching for new infinite families of sequences that have isolated Markoff values, we note the known isolated families. These may suggest new sequences that have isolated points in the Markoff spectrum. From previous work we know that the following infinite families of sequences have isolated Markoff values:

$$\mathcal{A}_1 = \{\overline{2, \underbrace{1, \dots, 1}_{2n}}\} \qquad n \geq 1$$

$$\mathcal{A}_2 = \{\overline{1, \underbrace{2, \dots, 2}_{2n}}\} \qquad n \geq 1$$

$$\mathcal{A}_3 = \{\overline{1, \underbrace{2, \dots, 2}_{2n}, 1, \underbrace{2, \dots, 2}_{2n+2}}\} \qquad n \geq 1$$

Gbur demonstrated in [G] that $\mathcal{A}_1$ have isolated Markoff values. $\mathcal{A}_2$ and $\mathcal{A}_3$ are shown to have isolated Markoff values by Crisp [C] .

An Outline of Crisp's Method

In his thesis, Crisp describes two new families of sequences that have isolated Markoff values. We broke down his method for doing this into the following steps:

Step A. Show that $\lambda_0(\mathcal{A}) = M(\mathcal{A})$.
Step B. Prove that $M(\mathcal{A})$ is locally isolated.
Step C. Assume $M(\mathcal{A})$ is <u>not</u> isolated. Then there is a sequence of integer sequences whose Markoff values converge to $M(\mathcal{A})$. Since the metric space consisting of integer sequences is compact, there must be a subsequence of these sequences that converges to a sequence $\mathcal{B}$ such that $\mathcal{B} \neq \mathcal{A}$ and $M(\mathcal{B}) = M(\mathcal{A})$. Prove that no such sequence $\mathcal{B}$ exists.

Because of the periodicity of the sequences Crisp examines, Step A can be accomplished by showing that $\lambda_0(\mathcal{A}) \geq \lambda_i(\mathcal{A})$ for $-n \leq i \leq 0$, where n is the period of the sequence. The following lemmas are useful for showing this.

Lemma 1. *Let $\mathcal{A} = \{a_i\}_{i=-\infty}^{+\infty}$ be a doubly infinite sequence with $a_i \in \{1,2\}$ for all i. Then $\lambda_n < \lambda_m$ if $a_n < a_m$.*

Proof. Let $a_n < a_m$. Then it must be the case that $a_n = 1$ and $a_m = 2$. Furthermore, it can be verified that

$$\frac{-1+\sqrt{3}}{2} \leq [0; a_i, a_{i+1}, \ldots] \leq -1+\sqrt{3}, \qquad \text{for all } i,$$

and thus,

$$-1+\sqrt{3} \leq [0; a_{i+1}, a_{i+2}, \ldots] + [0; a_{i-1}, a_{i-2}, \ldots] \leq -2+2\sqrt{3}.$$

Since $a_n = 1$, it can be seen that

$$a_n + [0; a_{n+1}, a_{n+2}, \ldots] + [0; a_{n-1}, a_{n-2}, \ldots] \leq -1+2\sqrt{3},$$

and likewise, since $a_m = 2$, it must be the case that

$$1+\sqrt{3} \leq a_m + [0; a_{m+1}, a_{m+2}, \ldots] + [0; a_{m-1}, a_{m-2}, \ldots].$$

Therefore,

$$\lambda_n < \lambda_m. \quad \square$$

Lemma 2 (Crisp, Lemma 6.5). *Let $\mathcal{A} = \{a_i\}_{i=-\infty}^{+\infty}$ be a sequence of positive integers and suppose $n \geq 1$ and $a_1, a_2, \ldots, a_{n-1}$ is symmetric. Then $\lambda_0(\mathcal{A}) \geq \lambda_n(\mathcal{A})$ if and only if*

$$[a_0; a_{-1}, a_{-2}, \ldots] \geq [a_n; a_{n+1}, a_{n+2}, \ldots].$$

Step B is fairly easy to do using the following. Crisp adapted this lemma from the work of Davis and Kinney [D].

Lemma 3 (Crisp, Remark 6.2). *Let $\mathcal{A}$ be a periodic sequence of 1's and 2's with $M(\mathcal{A}) = \lambda_0(\mathcal{A})$. If there is some odd integer k such that $a_{k-i} = a_i$ for all integers i, then M takes a locally isolated value at $\mathcal{A}$.*

The following lemmas are used to accomplish Step C. This is the most involved part of Crisp's method. Lemmas 6 and 7 as stated here in weaker forms than in Crisp's thesis, but are adequate and more clear for our purposes.

Lemma 4 (Crisp, Remark 6.4). *Let $\mathcal{A} = \{a_i\}_{i=-\infty}^{+\infty}$ and $\mathcal{B} = \{b_i\}_{i=-\infty}^{+\infty}$ be sequences of 1's and 2's with $M(\mathcal{A}) = \lambda_0(\mathcal{A})$ and $M(\mathcal{B}) = \lambda_0(\mathcal{B})$. If $a_{-1}, a_0, a_1 = 2, 2, 1$ then $M(\mathcal{B})$ is bound away from $M(\mathcal{A})$ unless $b_{-1}, b_0, b_1 = 2, 2, 1$ or $b_{-1}, b_0, b_1 = 1, 2, 2$. Likewise, if $a_{-1}, a_0, a_1, a_2 = 2, 2, 1, 1$ then $M(\mathcal{B})$ is bound away from $M(\mathcal{A})$ unless $b_{-1}, b_0, b_1, b_2 = 2, 2, 1, 1$ or $b_{-2}, b_{-1}, b_0, b_1 = 1, 1, 2, 2$. This result is due to the work of Bumby [B].*

Lemma 5 (Crisp, Lemma 6.6). *Let $\mathcal{A} = \{\overline{a_{-(m+1)}, \ldots, a_{-1}, a_0, a_1, \ldots, a_n}\}$ be a doubly infinite sequence of 1's and 2's satisfying $M(\mathcal{A}) = \lambda_0(\mathcal{A})$ and suppose both the sequences*

$$a_{-m}, \ldots, a_{-2}, a_{-1} \qquad \text{and} \qquad a_1, a_2, \ldots, a_n$$

are symmetric and $n, m \geq 0$ are both even and $a_{-(m+1)} = a_0$. Then there is a constant $\delta > 0$ such that if $\mathcal{B} = \{b_i\}_{i=-\infty}^{+\infty}$ is any sequence of 1's and 2's other than $\mathcal{A}$ satisfying $M(\mathcal{B}) = \lambda_0(\mathcal{B})$ and $b_i = a_i$ for $-m \leq i \leq n$ then $M(\mathcal{B}) - M(\mathcal{A}) \geq \delta$.

Lemma 6 (Crisp, Lemma 6.2). *Let $\mathcal{A} = \{a_i\}_{i=-\infty}^{+\infty}$ and $\mathcal{B} = \{b_i\}_{i=-\infty}^{+\infty}$ be sequences of 1's and 2's and suppose there are integers $m, n \geq 0$ such that $b_i = a_i$ for $-m \leq i \leq -n$. If further, $(-1)^n(a_{n+1} - b_{n+1}) > 0$ and $(-1)^m(a_{-(m+1)} - b_{-(m+1)}) > 0$ then*

$$\lambda_0(\mathcal{B}) \neq \lambda_0(\mathcal{A})$$

Lemma 7 (Crisp, Lemma 6.3). *Let $\mathcal{A} = \{a_i\}_{i=-\infty}^{+\infty}$ and $\mathcal{B} = \{b_i\}_{i=-\infty}^{+\infty}$ be sequences of 1's and 2's and suppose there are integers $m, n \geq 0$ such that $b_i = a_i$ for $-m \leq i \leq -n$ and suppose also that $(-1)^n(a_{n+1} - b_{n+1}) > 0$. Then $\lambda_0(\mathcal{B}) \neq \lambda_0(\mathcal{A}) > 0$ if*

$$K(a_1, a_2, \ldots, a_n, 2) \leq K(a_{-1}, a_{-2}, \ldots, a_{-m}, 1) \neq 2.$$

Also note that if $(-1)^m(a_{-(m+1)} - b_{-(m+1)}) > 0$, and

$$K(a_{-1}, a_{-2}, \ldots, a_{-m}, 2) \leq K(a_1, a_2, \ldots, a_n, 1) \neq 2,$$

then $\lambda_0(\mathcal{B}) \neq \lambda_0(\mathcal{A})$.

For both of Crisp's families, Lemma 4 limits the possibilities for $\mathcal{B}$ to sequences with $b_{-1}, b_0, b_1 = 2, 2, 1$. (Note that if $\bar{\mathcal{B}}$ is the reverse sequence of $\mathcal{B}$, with $\bar{b}_0 = b_0$, then $M(\bar{\mathcal{B}}) = M(\mathcal{B})$, thus it is unnecessary to also check $\mathcal{B}$'s of the form $b_{-1}, b_0, b_1 = 1, 2, 2$.) Similarly, Lemma 5 rules out $\mathcal{B}$'s that have $b_i = a_i$ near the center, but have differing values away from the center. For instance, with $\mathcal{A}_2 = \{\overline{1, \underbrace{2, \ldots, 2}_{2n}}\}$, Lemma 5 rules out all $\mathcal{B}$ of the form

$$\mathcal{B} = \ldots, \underbrace{2, \ldots, 2}_{2n-2}, 2^*, 1, \underbrace{2, \ldots, 2}_{2n}, 1, \ldots.$$

This then leaves a relatively small group of possible $\mathcal{B}$'s which need to be tested. Crisp divides this group into three cases. The above example is divided into

case 1. $\mathcal{B} = \ldots, \underbrace{2, \ldots, 2}_{j+1}, 2^*, 1, \underbrace{2, \ldots, 2}_{j}, \ldots \qquad 0 \leq j \leq 2n-2;$

case 2. $\mathcal{B} = \ldots, 1, \underbrace{2, \ldots, 2}_{j}, 2^*, 1, \underbrace{2, \ldots, 2}_{j}, \ldots \qquad 1 \leq j \leq 2n-2;$

case 3. $\mathcal{B} = \ldots, \underbrace{2, \ldots, 2}_{2n-1}, 2^*, 1, \underbrace{2, \ldots, 2}_{2n-1}, \ldots.$

These cases can be handled using Lemmas 2, 6 and 7, with Lemma 7 used most prominently.

Backwards Continued Fractions

With an understanding of Crisp's method for showing that sequences have isolated Markoff values, we undertook the task of converting his lemmas for use with backwards continued fractions. First let us define some terms.

Definition. The *backwards continued fraction* representation of a real number β is

$$\beta = [|b_0; b_1, b_2, b_3, \ldots|] = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\ddots}}}}$$

If the function L is defined recursively by

$$L(a_0,) = a_0;$$
$$L(a_0, a_1) = a_0 a_1 - 1;$$
$$L(a_0, a_1, \ldots, a_n) = a_n L(a_0, a_1, \ldots, a_{n-1}) - L(a_0, a_1, \ldots, a_{n-2});$$

then

$$\frac{L(a_0, a_1, a_2, \ldots, a_n)}{L(a_1, a_2, \ldots, a_n)} = [|a_0; a_1, a_2, \ldots, a_n|].$$

If

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & -1 \\ 1 & 0 \end{pmatrix},$$

then

$$\begin{aligned} a &= L(a_0, a_1, \dots, a_n); \\ b &= -L(a_0, a_1, \dots, a_{n-1}); \\ c &= L(a_1, a_2, \dots, a_n); \\ d &= -L(a_1, a_2, \dots, a_{n-1}); \end{aligned}$$

and

$$g(x) = \frac{ax+b}{cx+d} = [|a_0;, a_1, a_2, \dots, a_n, x|]. \tag{3}$$

Since $ad - bc = 1^{n+1}$,

$$g(x) - g(y) = \frac{x-y}{(cx+d)(cy+d)}. \tag{4}$$

When compared with equation (2) for ordinary continued fractions, this equation is noticably simpler, lacking the $(-1)^{n+1}$ term. Our original hope was that similarly, other equations used by Crisp would simplify when applied to backwards continued fractions.

Conversion Algorithm. Given the ordinary continued fraction

$$\alpha = [a_0; a_1, a_2, a_3, \dots]$$

we can convert it to a backwards continued fraction representation so that

$$\alpha = [|a_0+1; \underbrace{2,\dots,2}_{a_1-1}, a_2+2, \underbrace{2,\dots,2}_{a_3-1}, a_4+2, \dots|].$$

Thus, given the doubly infinite sequence $\mathcal{A}$,

$$\begin{aligned} \lambda_0(\mathcal{A}) &= [|a_0+1; \underbrace{2,\dots,2}_{a_1-1}, a_2+2, \underbrace{2,\dots,2}_{a_3-1}, a_4+2, \dots|] \\ &\quad + [|0+1; \underbrace{2,\dots,2}_{a_{-1}-1}, a_{-2}+2, \underbrace{2,\dots,2}_{a_{-3}-1}, a_{-4}+2, \dots|] \\ &= [|a_0+1; \underbrace{2,\dots,2}_{a_1-1}, a_2+2, \underbrace{2,\dots,2}_{a_3-1}, a_4+2, \dots|] + 1 \\ &\quad + [|0+1; \underbrace{2,\dots,2}_{a_{-1}-1}, a_{-2}+2, \underbrace{2,\dots,2}_{a_{-3}-1}, a_{-4}+2, \dots|] - 1 \\ &= [|a_0+2; \underbrace{2,\dots,2}_{a_1-1}, a_2+2, \underbrace{2,\dots,2}_{a_3-1}, a_4+2, \dots|] \\ &\quad + [|0; \underbrace{2,\dots,2}_{a_{-1}-1}, a_{-2}+2, \underbrace{2,\dots,2}_{a_{-3}-1}, a_{-4}+2, \dots|]. \end{aligned}$$

We were able to verify this conversion algorithm in the work of Zagier [Z].

Definition. Let $\tilde{\mathcal{A}} = \{a_i\}_{i=-\infty}^{+\infty}$ be a doubly infinite sequence of integers with $2 \leq a_i$. Then we define

$$\tilde{\lambda}_n(\tilde{\mathcal{A}}) = [|a_n; a_{n+1}, a_{n+2}, \dots |] + [|0; a_{n-1}, a_{n-2}, \dots |].$$

Likewise, we define

$$\tilde{M}(\tilde{\mathcal{A}}) = \sup_n \tilde{\lambda}_n.$$

If $\tilde{M}(\tilde{\mathcal{A}}) = M(\mathcal{A})$, then we shall say that $\tilde{\mathcal{A}}$ is the *converted sequence* of $\mathcal{A}$.

Examination of Known Families as Converted Sequences. Having armed ourselves with a conversion algorithm, we thought it might be of interest to look at the converted sequences of the families which we already knew to have isolated Markoff values. The converted sequences of the first two known families are

$$\mathcal{A}_1 = \{\overline{2, \underbrace{1, \dots, 1}_{2n}}\} \qquad \tilde{\mathcal{A}}_1 = \{\overline{2, \underbrace{3, \dots, 3}_{n}, 4, \underbrace{3, \dots, 3}_{n}, 2}\} \qquad a_0 = 4$$

$$\mathcal{A}_2 = \{\overline{1, \underbrace{2, \dots, 2}_{2n}}\} \qquad \tilde{\mathcal{A}}_2 = \{\overline{\underbrace{4, 2, \dots, 4, 2}_{n\text{pairs}}, 3, \underbrace{2, 4, \dots, 2, 4}_{n\text{pairs}}}\} \qquad a_{-1}, a_0, a_1 = 4, 4, 2$$

No pattern was immediately apparent, but further examination might suggest new sequences to investigate.

Results and Difficulties with Conversion of Lemmas. We were interested in converting Crisp's lemmas into analogous forms for use with backwards continued fractions. Our hope was that this would simplify the steps necessary to prove that an infinite family of sequences had isolated Markoff values, or alternatively, that these new lemmas would suggest new families for testing. We found that the analog to Lemma 2 holds for backwards continued fractions.

Lemma 8. *Let $\tilde{\mathcal{A}} = \{a_i\}_{i=-\infty}^{+\infty}$ be a sequence of integers with $2 \leq a_i$ for all i, and let a_1, a_2, ..., a_{n-1} be symmetric for some $n \geq 1$. Then*

$$\tilde{\lambda}_0(\tilde{\mathcal{A}}) \geq \tilde{\lambda}_n(\tilde{\mathcal{A}}) \text{ if and only if } [|a_0; a_{-1}, a_{-2}, \dots |] \geq [|a_n; a_{n+1}, a_{n+2}, \dots |].$$

Proof. Define $\alpha = [|a_0; a_{-1}, a_{-2}, \dots |]$, and $\beta = [|a_n; a_{n+1}, a_{n+2}, \dots |]$, and let f be the function defined as

$$f(x) = \frac{ax+b}{cx+d} = [|0; a_1, a_2, \dots, a_n, x|]$$

. Then

$$
\begin{aligned}
&\tilde{\lambda}_0(\mathcal{A}) \geq \tilde{\lambda}_n(\mathcal{A}) \\
\Leftrightarrow&[|a_0; a_1, a_2, \dots |] + [|0; a_{-1}, a_{-2}, \dots |] \\
&\qquad \geq [|a_n; a_{n+1}, a_{n+2}, \dots |] + [|0; a_{n-1}, a_{n-2}, \dots |] \\
\Leftrightarrow&[|0; a_1, a_2, \dots |] + \alpha \geq \beta + [|0; a_{n-1}, a_{n-2}, \dots, a_1, a_0, \dots |] \\
\Leftrightarrow&[|0; a_1, a_2, \dots, a_{n-1}, a_n, \dots |] + \alpha \geq \beta + [|0; a_1, a_2, \dots, a_{n-1}, a_0, a_{-1}, a_{-2}, \dots |] \\
\Leftrightarrow&[|0; a_1, a_2, \dots, a_{n-1}, \beta|] + \alpha \geq \beta + [|0; a_1, a_2, \dots, a_{n-1}, \alpha|] \\
\Leftrightarrow&f(\beta) + \alpha \geq \beta + f(\alpha) \\
\Leftrightarrow&\alpha - \beta \geq f(\alpha) - f(\beta) \\
\Leftrightarrow&\alpha - \beta \geq \frac{\alpha - \beta}{(c\alpha + d)(c\beta + d)} \\
\Leftrightarrow&\alpha - \beta \geq 0 \\
\Leftrightarrow&\alpha \geq \beta \\
\Leftrightarrow&[|a_0; a_{-1}, a{-2}, \dots |] \geq [|a_n; a_{n+1}, a_{n+2}, \dots |]. \quad \square
\end{aligned}
$$

Unfortunately, Lemma 5 presented difficulties when we attempted to convert it. Crisp's proof of this lemma begins as follows:

$$
\begin{aligned}
\text{Set}\quad & \alpha_1 = [a_{n+1}; a_{n+2}, a_{n+3}, \dots] \\
& \alpha_2 = [a_{-(m+1)}; a_{-(m+2)}, a_{-(m+3)}, \dots] \\
& \beta_1 = [b_{n+1}; b_{n+2}, b_{n+3}, \dots] \\
& \beta_2 = [b_{-(m+1)}; b_{-(m+2)}, b_{-(m+3)}, \dots] \\
\text{Then}\quad & M(\mathcal{A}) = [a_0; a_1, a_2, \dots, a_n, \alpha_1] + [0; a_{-1}, a_{-2}, \dots, a_{-m}, \alpha_2] \\
& M(\mathcal{B}) = [a_0; a_1, a_2, \dots, a_n, \beta_1] + [0; a_{-1}, a_{-2}, \dots, a_{-m}, \beta_2].
\end{aligned}
$$

These equations are of the same form as equation (1). Keeping in mind that the hypothesis of this lemma requires that n and m are even, Crisp uses the fact that $(-1)^{n+1} = -1$ and thus $f(\alpha_1) - f(\beta_1) = -\frac{\alpha_1 - \beta_1}{(c\alpha_1 + d)(c\beta_1 + d)}$, and similarly with m. This is a condition that can never be achieved with the simpler form found in equation (4). Directly converting this lemma for use with backwards continued fractions is not possible, although it may be possible to develop a lemma or lemmas that would provide a similar tool.

Putting Crisp's Method to Work

Having encountered a stumbling block in our work with backwards continued fractions, we turned our attention to using Crisp's method on select sequences of 1's and 2's. Our sequences included both infinite families similar to those examined by Crisp, and specific sequences.

Some Infinite Families.

We were able to show that several similar infinite families were locally isolated, including

$$\mathcal{A}_a = \{\overline{1,1,\underbrace{2,\ldots,2}_{2n}}\}$$
$$\mathcal{A}_b = \{\overline{1,\underbrace{2,\ldots,2}_{2n},1,\underbrace{2,\ldots,2}_{2m}}\}, \qquad n < m-2$$
$$\mathcal{A}_c = \{\overline{2,2,2,\underbrace{1,\ldots,1}_{2n}}\}$$
$$\mathcal{A}_d = \{\overline{\underbrace{1,\ldots,1}_{2n}\underbrace{2,\ldots,2}_{2n+2}}\}.$$

We were also able to apply Lemmas 4 and 5 to these families, thus limiting the remaining possible cases for $\mathcal{B}$. However, when testing the analog of Crisp's case 1, problems occured. Crisp's primary tool for proving each case is Lemma 7. With our sequences, we were unable to fill the hypothesis of $K(a_1, a_2, \ldots, a_n, 2) \leq K(a_{-1}, a{-}2, \ldots, a{-}m, 1)$ necessary to show that $\lambda_0(\mathcal{B}) \neq \lambda_0(\mathcal{A})$. With $\mathcal{A}_a$, $\mathcal{A}_c$, and $\mathcal{A}_d$, the additional 1's to the right of a_0 force $K(a_1, a_2, \ldots, a_n, 2)$ to always be greater than $K(a_{-1}, a_{-2}, \ldots, a_{-m}, 1)$. In the case of $\mathcal{A}_b$, a similar problem occured. This hypothesis of Lemma 7 created an obstacle to selecting sequences for which this method would work.

Some Specific Sequences

$$\mathcal{A}_e = \{\overline{1,2,2}\}$$
$$\mathcal{A}_f = \{\overline{1,1,2,2}\}$$
$$\mathcal{A}_g = \{\overline{1,1,2,2,2}\}$$

The first two of these sequences were both already known to have isolated Markoff values. $\mathcal{A}_e$ is actually the case of n=1 in Crisp's first family. $\mathcal{A}_f$ represents one of the Markoff numbers, which are already known to be isolated. Our interest in these sequences was in how Crisp's tools could be used to determine if a sequence had an isolated Markoff value. We were unsure if $\mathcal{A}_g$ had an isolated Markoff value, but suspected that it did since its Markoff value fell in the area of the spectrum where there are no intervals.

All three sequences had locally isolated Markoff values and we were able to apply Lemmas 4 and 5 to greatly limit the number of possible $\mathcal{B}$ sequences which might have the same Markoff values. With $\mathcal{A}_e$ and $\mathcal{A}_f$ it was then possible to use Lemmas 2, 6 and 7 to show that none of the remaining possible cases for $\mathcal{B}$ (or subcases of $\mathcal{B}$) could have the same Markoff value as the $\mathcal{A}$ sequence, thus proving that our sequences were isolated.

Lemma 9. *The doubly infinite sequence $\mathcal{A}_e = \{\overline{1,2,2}\}$ has an isolated Markoff value.*

Proof. Let $\mathcal{A}_e = \{\overline{1,2,2}\}$ with $a_{-1}, a_0, a_1 = 2, 2, 1$. Note that

$$\lambda_{-1}(\mathcal{A}_e) = [2; 2, 1, \ldots] + [0; 1, 2, 2, \ldots] = [0; 2, 1, \ldots] + [2; 1, 2, \ldots] = \lambda_0(\mathcal{A}_e).$$

Also, it follows from Lemma 1 that $\lambda_1(\mathcal{A}_e) < \lambda_0(\mathcal{A}_e)$. Because of the periodicity of $\mathcal{A}_e$, this is sufficient to show that $M(\mathcal{A}_e) = \lambda_0(\mathcal{A}_e)$. Since $a_{-1-i} = a_i$ for all i, Lemma 3 implies that $\mathcal{A}_e$ has a locally isolated Markoff value.

Assume $\mathcal{A}_e$ is not isolated. Then there must exist a doubly infinite sequence $\mathcal{B}$ such that $\mathcal{B} \neq \mathcal{A}_e$ and $M(\mathcal{B}) = \lambda_0(B) = M(\mathcal{A}_e)$. From Lemma 4, we know that $\mathcal{B}$ must be of the form $b_{-1}, b_0, b_1 = 2, 2, 1$ or $b_{-1}, b_0, b_1 = 1, 2, 2$, but because the Markoff values of $\mathcal{B}$ and its reverse are the same, we can assume that $\mathcal{B}$ has the first form. Further, we can rewrite $\mathcal{A}_e$ as

$$\mathcal{A}_e = \{\overline{2,2,1,2,2,1}\} = \ldots, 2, 2^*, 1, 2, 2, 1, \ldots$$

thus satisfying the hypothesis for Lemma 5. This implies that if $M(\mathcal{B}) = M(\mathcal{A}_e)$, it must be the case that $\mathcal{B}$ is not of the form

$$\mathcal{B} = \ldots, 2^*, 1, 2, 2, 1 \ldots.$$

This leaves only three possible cases for $\mathcal{B}$. They are

case 1. $\mathcal{B} = \ldots, 2, 2^*, 1, 1, \ldots$

case 2. $\mathcal{B} = \ldots, 2, 2^*, 1, 2, 1, \ldots$

case 3. $\mathcal{B} = \ldots, 2, 2^*, 1, 2, 2, 2, \ldots.$

We shall handle these three cases individually.

case 1. First note that $a_i = b_i$ for $-1 \leq i \leq 1$, and $(-1)^n(b_{n+1} - a_{n+1}) = (-1)^1(1-2) > 0$. Also,

$$K(1,2) \leq K(2,1) \neq 2,$$

which implies $\lambda_0(\mathcal{A}) \neq \lambda_0(\mathcal{B})$ by Lemma 7.

case 2. By our assumption, $\lambda_0(\mathcal{B}) \geq \lambda_2(\mathcal{B})$. It follows from Lemma 2 that

$$[2^*; 2, b_{-2}, \ldots] \geq [2; 1, b_4, \ldots].$$

Define the function f by

$$f(x) = [2; x]$$

and let $\alpha = [2; b_{-2}, b_{-3}, \ldots], \beta = [1; b_4, b_5, \ldots]$. Thus,

$$f(\alpha) \geq f(\beta)$$

$$f(\alpha) - f(\beta) \geq 0$$

$$\frac{(-1)^1(\alpha - \beta)}{(c\alpha + d)(c\beta + d)} \geq 0.$$

But this contradicts the fact that $\alpha > \beta$. Therefore, $\lambda_0(\mathcal{B}) \neq M(\mathcal{B})$.

case 3. We need to consider two subcases for this case. First, let $b_{-2} = 1$. By an argument similar to case 2, it can be shown that $\lambda_0(\mathcal{B}) \neq M(\mathcal{B})$. Now let $b_{-2} = 2$. Note that $a_i = b_i$ for $-1 \leq i \leq 3$ and

$$(-1)^3(a_4 - b_4) > 0 \qquad (-1)^1(a_{-2} - b_{-2}) > 0.$$

Thus Lemma 6 implies that $\lambda_0(\mathcal{B}) \neq \lambda_0(\mathcal{A})$.

We have shown that there is no $\mathcal{B}$ such that $M(\mathcal{B}) = \lambda_0(\mathcal{B}) = M(\mathcal{A})$. Therefore $\mathcal{A}$ must have an isolated Markoff value. □

Lemma 10. *The doubly infinite sequence $\mathcal{A}_f = \{\overline{1,1,2,2}\}$ has an isolated Markoff value.*

Proof. Let $\mathcal{A}_f = \{\overline{1,1,2,2}\}$ with $a_{-1}, a_0, a_1 = 2, 2, 1$. Using an argument similar to that in Lemma 9, it can be shown that $\lambda_0(\mathcal{A}_f) = M(\mathcal{A}_f)$. Note that $a_{-1-i} = a_i$ for all i, so Lemma 3 implies that $\mathcal{A}_f$ has a locally isolated Markoff value. Assume $\mathcal{A}_f$ is not isolated. Then there must exist a doubly infinite sequence $\mathcal{B}$ such that $\mathcal{B} \neq \mathcal{A}_f$ and $M(\mathcal{B}) = \lambda_0(B) = M(\mathcal{A}_f)$. Similar to the proof of Lemma 9, we can use Lemma 4 to see that $\mathcal{B}$ must be of the form $b_{-1}, b_0, b_1, b_2 = 2, 2, 1, 1$. Further, we can rewrite $\mathcal{A}_f$ as

$$\mathcal{A}_f = \{\overline{2,2,1,1,2,2,1,1}\} = \ldots, 2, 2^*, 1, 1, 2, 2, 1, 1 \ldots$$

thus satisfying the hypothesis for Lemma 5. This implies that if $M(\mathcal{B}) = M(\mathcal{A}_f)$, it must be the case that $\mathcal{B}$ is not of the form

$$\mathcal{B} = \ldots, 2^*, 1, 1, 2, 2, 1, 1, \ldots.$$

The remaining possible cases for $\mathcal{B}$ are

case 1.	$\mathcal{B} = \ldots, 2, 2^*, 1, 1, 1, \ldots$
case 2.	$\mathcal{B} = \ldots, 2, 2^*, 1, 1, 2, 1, \ldots$
case 3.	$\mathcal{B} = \ldots, 2, 2^*, 1, 1, 2, 2, 2, \ldots$
case 4.	$\mathcal{B} = \ldots, 2, 2^*, 1, 1, 2, 2, 1, 2, \ldots.$

case 1. We need two subcases for this case. Let $b_{-2} = 1$. Then $b_i = a_i$ for $-2 \leq i \leq 2$, and $(-1)^2(a_3 - b_3) > 0$. Also,

$$K(1,1,2) \leq K(2,1,1) \neq 2$$

so Lemma 7 indicates that $\lambda_0(\mathcal{B}) \neq \lambda_0(\mathcal{A}_f)$. Now consider $b_{-2} = 2$. In this case notice that $b_i = a_i$ for$-1 \leq i \leq 2$. Further,

$$(-1)^2(a_3 - b_3) > 0 \qquad (-1)^1(a_{-2} - b_{-2}) > 0.$$

Thus Lemma 6 implies that $\lambda_0(\mathcal{B}) \neq \lambda_0(\mathcal{A}_f)$.

case 2. As in case 2 of the proof of Lemma 9, we can show that $\lambda_0(\mathcal{B}) \neq M(\mathcal{B})$ using Lemma 2.

case 3. Similar to case 1, we need to consider two subcases. The first, when $b_{-2} = 1$, can be handled in the same method as the previous case. Consider when $b_{-2} = 2$. Notice that $a_i = b_i$ for $-1 \leq i \leq 4$, and $(-1)^1(a_{-2} - b_{-2}) > 0$. We can see that

$$K(2,2) \leq K(1,1,2,2,1),$$

and thus by Lemma 7, $\lambda_0(\mathcal{B}) \neq \lambda_0(\mathcal{A}_f)$.

case 4. This case requires several subcases. The details of the proofs of each subcase are similar to the cases already discussed, so we shall merely indicate which lemmas were used to handle each subcase.

subcase a:	$b_{-2} = 2$	Lemma 6
subcase b:	$b_{-2} = 1, b_{-3} = 2$	Lemma 7
subcase c:	$b_{-2} = 1, b_{-3} = 1, b_{-4} = 1$	Lemma 7
subcase d:	$b_{-2} = 1, b_{-3} = 1, b_{-4} = 2, b_{-5} = 1$	Lemma 6
subcase e:	$b_{-2} = 1, b_{-3} = 1, b_{-4} = 2, b_{-5} = 2, b_{-6} = 1$	Lemma 7
subcase f:	$b_{-2} = 1, b_{-3} = 1, b_{-4} = 2, b_{-5} = 2, b_{-6} = 2$	Lemma 6

Thus there is no $\mathcal{B}$ such that $M(\mathcal{B}) = \lambda_0(\mathcal{B}) = M(\mathcal{A}_f)$. Therefore $\mathcal{A}_f$ must have an isolated Markoff value. □

Our third sequence, $\mathcal{A}_g$, was not so straightforward. Although the majority of cases and subcases were easily proved using Lemmas 2, 6 and 7, there were two subcases for which these tools were not enough. In both spots, the subcases had been taken to a level where $b_i = a_i$ for $-m \leq i \leq n$ and both $b_{-(m+1)} \neq a_{-(m+1)}$ and $b_{n+1} \neq a_{n+1}$. The lemmas remained inconclusive to this point, and looking at any further subcases would have no effect on the results of Lemma 6 and 7. We continued examining these subcases without the lemmas by looking at the possible values which $\lambda_0(\mathcal{B})$ could take on. If a point could be reached in a subcase where the minimum possible $\lambda_0(\mathcal{B})$ was greater than $\lambda_0(\mathcal{A})$, or the maximum possible $\lambda_0(\mathcal{B})$ was less than $\lambda_0(\mathcal{A})$, then it would be proved that $M(\mathcal{A}) \neq M(\mathcal{B})$ for that subcase. Using this technique for several levels of subcases did not produce the results we were looking for, but a continued examination of these subcases could eventually produce results. The following remark demonstrates the accomplished progress for showing that $\mathcal{A}_g$ has an isolated Markoff value.

Remark 1. *Progress made towards showing $\mathcal{A}_g$ has an isolated Markoff value.*

Calculations. It can be shown with the techniques above that $\lambda_0(\mathcal{A}_g) = M(\mathcal{A}_g)$, and that $M(\mathcal{A}_g)$ is locally isolated. Therefore, if $\mathcal{A}$ does not have an isolated Markoff value, there must be a doubly infinite sequence $\mathcal{B}$ such that $\mathcal{B} \neq \mathcal{A}_g$ and $M(\mathcal{B}) = \lambda_0(\mathcal{B}) = M(\mathcal{A}_g)$. We work to show that no such $\mathcal{B}$ exists. The basic cases we need to consider, as determined by Lemmas 4 and 5, are

case 1.	$\mathcal{B} =$	$\ldots, 1, 2, 2^*, 1, 1, \ldots$
case 2.	$\mathcal{B} =$	$\ldots, 2, 2, 2, 2^*, 1, 1, \ldots$
case 3.	$\mathcal{B} =$	$\ldots, 2, 1, 2, 2, 2^*, 1, 1, \ldots$
case 4.	$\mathcal{B} =$	$\ldots, 1, 1, 1, 2, 2, 2^*, 1, 1, \ldots$
case 5.	$\mathcal{B} =$	$\ldots, 1, 2, 1, 1, 2, 2, 2^*, 1, 1, \ldots.$

In order to work with cases similar to those in Lemmas 9 and 10, we shall examine the reverse sequences of $\mathcal{A}_g$ and $\mathcal{B}$. Thus we shall think of $\mathcal{A}_g$ as having the form

$$\mathcal{A}_g = \ldots, 1, 1, 2^*, 2, 2, 1, 1, 2, 2, 2, \ldots,$$

and likewise with each case of $\mathcal{B}$. In a similar manner to case 4 of the previous proof, we shall omit the details and list the lemmas which we applied to each subcase.

case 1.	
subcase a. $b_{-3} = 1$	see below
subcase b. $b_{-3} = 2$	Lemma 7
case 2.	
subcase a. $b_{-3} = 1$	Lemma 7
subcase b. $b_{-3} = 2, b_{-4} = 1$	Lemma 6
subcase c. $b_{-3} = 2, b_{-4} = 2$	Lemma 7
case 3.	Lemma 2
case 4.	
subcase a. $b_{-3} = 1$	Lemma 6
subcase b. $b_{-3} = 2$	Lemma 2
case 5.	
subcase a. $b_{-3} = 1$	Lemma 7
subcase b. $b_{-3} = 2, b_{-4} = 1$	Lemma 6
subcase c. $b_{-3} = 2, b_{-4} = 2, b_{-5} = 1$	Lemma 7
subcase d. $b_{-3} = 2, b_{-4} = 2, b_{-5} = 2, b_{-6} = 2$	see below
subcase e. $b_{-3} = 2, b_{-4} = 2, b_{-5} = 2, b_{-6} = 1, b_{-7} = 1$	Lemma 7
subcase f. $b_{-3} = 2, b_{-4} = 2, b_{-5} = 2, b_{-6} = 1, b_{-7} = 2$	Lemma 6

Most of the cases for the $\mathcal{B}$ sequence were easily proven using the lemmas. However, cases 1a and 5d could not be handled using these methods. As an alternative we compared the possible values of $\lambda_0(\mathcal{B})$ with $M(\mathcal{A}_g)$. We can determine that $M(\mathcal{A}_g) = \frac{650+190\sqrt{13}}{228+60\sqrt{13}} \approx 3.0046$. By computing the minimum and maximum possible values for $\lambda_0(\mathcal{B})$ we hoped to be able to show that $\lambda_0(\mathcal{B}) \neq \lambda_0(\mathcal{A}_g)$. We also continued to use Lemma 2 when appropriate.

case 1a. We begin by noting that

$$\lambda_0(\mathcal{B}) = [2;2,1,\alpha] + [0;1,1,1,\beta] = 2 + \frac{1+\alpha}{2+3\alpha} + \frac{1+2\beta}{2+3\beta},$$

where α and β represent any continued fraction composed of 1's and 2's. We know that

$$\frac{1+\sqrt{3}}{2} \leq \alpha, \beta \leq 1+\sqrt{3},$$

so it must be the case that

$$2.9780 \approx 2 + \frac{1+(1+\sqrt{3})}{2+3(1+\sqrt{3})} + \frac{1+2(\frac{1+\sqrt{3}}{2})}{2+3(\frac{1+\sqrt{3}}{2})} \leq \lambda_0(\mathcal{B}) \leq 2 + \frac{1+(\frac{1+\sqrt{3}}{2})}{2+3(\frac{1+\sqrt{3}}{2})} + \frac{1+2(1+\sqrt{3})}{2+3(1+\sqrt{3})} \approx 3.0220.$$

This test is inconclusive, so we examine subcases of it.

subcase a. $b\text{–}4 = 1$	$2.9780 \lessapprox \lambda_0(\mathcal{B}) \lessapprox 3.0083$
subcase b. $b\text{–}4 = 1, b\text{–}5 = 1$	$3.0074 \lessapprox \lambda_0(\mathcal{B}) \lessapprox 3.0391$
subcase c. $b\text{–}4 = 1, b\text{–}5 = 2$	$2.9780 \lessapprox \lambda_0(\mathcal{B}) \lessapprox 3.0015$
subcase d. $b\text{–}4 = 2$	$2.9961 \lessapprox \lambda_0(\mathcal{B}) \lessapprox 3.0220$
subcase e. $b\text{–}4 = 2, b\text{–}5 = 1$	$2.9984 \lessapprox \lambda_0(\mathcal{B}) \lessapprox 3.0220$
subcase f. $b\text{–}4 = 2, b\text{–}5 = 2$	$2.9961 \lessapprox \lambda_0(\mathcal{B}) \lessapprox 3.0187$
subcase g. $b\text{–}4 = 2, b\text{–}5 = 2, b\text{–}6 = 1$	$2.9961 \lessapprox \lambda_0(\mathcal{B}) \lessapprox 3.0183$
subcase h. $b\text{–}4 = 2, b\text{–}5 = 2, b\text{–}6 = 2$	$2.9966 \lessapprox \lambda_0(\mathcal{B}) \lessapprox 3.0187$

Subcase a also is inconclusive, but note that subcases b and c both show that $\lambda_0(\mathcal{B}) \neq \lambda_0(\mathcal{A}_g)$. These cover all possible subcases in which $b_{-4} = 1$. The rest of the subcases are inconclusive, but Lemma 2 can be used with both subcases e and h to show that $\lambda_0(\mathcal{B}) \neq M(\mathcal{B})$, thus eliminating the need to examine more subcases in those directions. Those subcases which remain unresolved (where $b\text{–}4 = 2, b\text{–}5 = 2$, and $b_{-6} = 1$) can be examined further using these computational techniques. Case 5d can be investigated similarly.

Computational Possibilities

The systematic techniques used in Lemmas 9 and 10 and Remark 1 suggest that it may be worthwhile to develop a computer program to assist in testing for isolated Markoff values. The simplest of such programs could compute the range of possible values of $\lambda_0(\mathcal{B})$ for a given partially defined sequence $\mathcal{B}$. This would eliminate the most tedious part of working through a proof such as case 1a of Remark 1. In addition, if the program could be expanded to recognize and test subcases in which applying Lemma 2 might be of use, entire cases such as 1a could be handled recursively by the program. It should also be possible to design an algorithm for a fully automated system which would take a given sequence $\mathcal{A}$ and a possible basic case for $\mathcal{B}$ and systematically test $\mathcal{B}$ and subcases of $\mathcal{B}$ using Lemmas 2, 6 and 7, as well as using the computational technique above. This could be a valuable tool for finding new sequences with isolated Markoff values.

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