GEODESICS WITH THREE INTERSECTIONS ON THE PUNCTURED TORUS

Mande Butler, Jeanne Carton, Emil Kraft

Summer 1995

1 Background

We classify the free homotopy classes of loops on a once-punctured torus whose self-intersection number is three. In particular, these include the classes which, for any Riemannian metric, admit geodesics of three self-intersections. Our classification is up to homeomorphism type; that is, we identify two free homotopy classes if there is a self homeomorphism of the punctured torus taking one class to the other. In the setting of a punctured torus, this is equivalent to classification of the free homotopy classes up to the action of automorphisms of the fundamental group. Our work is an extension of David Crisp's Ph.D thesis which involved classes of loops with a single self-intersection [C], and a paper by Dziadosz, Insel, and Wiles concerning classes of loops with two self-intersections [DIW]. The remaining background information in this section is as stated in [CDGISW].

Let T be a punctured torus with a Riemannian metric. The fundamental group of T, $\pi_1(T)$, is isomorphic to the free group on two letters, F(a,b). We fix such an isomorphism. There is a natural bijection between free homotopy classes of closed curves on T and conjugacy classes of elements of F(a,b). A free homotopy class is said to be primitive if it is not a non-trivial power of some other class.

A closed curve on T is freely homotopic to a geodesic if and only if the curve lies in a primitive free homotopy class which contains no simple curve bounding either a disc or a punctured disc. A geodesic has the minimal number of self-intersections amongst all curves in its free homotopy class.

J. Nielsen [N] showed that every automorphism of $\pi_1(T)$ can be realized, up to inner automorphism, by some homeomorphism. He further showed that every automorphism of $\pi_1(T) \cong F(\text{a,b})$ takes $aba^{-1}b^{-1}$ to $x(aba^{-1}b^{-1})^{\pm 1}x^{-1}$ for some x an element of $F(\text{a,b})$.

Theorem 1.1 *Let T_1 and T_2 be two Riemannian punctured tori. Let (A_1,B_1) and (A_2,B_2) be fixed generating pairs of their respective fundamental groups. Let n be a positive integer. Let $W(a,b)$ be a word in the letters a and b. If a geodesic on T_1 in the free homotopy class of $W(A_1,B_1)$ has n intersections, then any geodesic on T_2 in the free homotopy class of $W(A_2, B_2)$ also has n intersections.*

For proof see [CDGISW].

Corollary 1.1 *A classification for a particular Riemannian punctured torus of the automorphism classes of those free homotopy classes which contain closed geodesics which are n-times self-intersecting is in fact valid for all such punctured tori.*

We have thus shown that the results of [CM], as proven in [C], are in fact valid for all Riemannian punctured tori. Indeed, the result is a topological one. In what follows, we specialize to a fixed metric only when we wish to show a geodesic in a particular free homotopy class has exactly the claimed number of self-intersections. In fact the surface we use is the quotient of the Poincaré upper half-plane by the commutator subgroup of the modular surface, exactly that studied by these other authors. We thus discuss this specialization.

We consider the particular once-punctured torus T, the quotient of the Poincaré upper half-plane, $\mathcal{H}$, by the commutator subgroup of the modular group, Γ'. This torus has constant curvature minus one - thus, there is at most one geodesic in each class - and admits H as its universal Riemannian cover. The action of Γ on $\mathcal{H}$ is given by linear fractional transformations. We use a standard fundamental domain $\mathcal{D}$ for this action - a quadrilateral with hyperbolic line segment boundaries of vertices -1,0,1,∞.

We take A=$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and B=$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ as generators of $\Gamma' = F(A, B)$. Thus, given a word W in A and B, by matrix multiplication we find a corresponding matrix in Γ'. A hyperbolic element of Γ' is one of trace greater than 2 in absolute value. Each such fixes an axis - a geodesic of $\mathcal{H}$, thus a semicircle with center on $\mathrm{R}\cup\infty$. Free homotopy classes correspond to conjugacy classes in Γ'. A free homotopy class has a closed geodesic in it if and only if the corresponding conjugacy class in hyperbolic. The geodesic in such a homotopy class is then the projection of the axes fixed by the elements of the conjugacy class. Indeed, the geodesic is the 1-1 projection of those segments of the fixed axes which lie in $\mathcal{D}$. Furthermore, for a reduced word in A and B representing an element of the conjugacy class, the cyclic permutations of this word determine all axes which will have geodesic segments within $\mathcal{D}$. We refer to [C] for a more detailed presentation of these standard facts.

Thus, to find the number of self-intersections of the unique geodesic in the (hyperbolic) class [W], for a given word W in A and B, we find the number of intersections within $\mathcal{D}$ of the axes of those matrices which arise from the cyclic permutations of the word of W. That is, for each cyclic permutation of W we find the corresponding element $M \in \Gamma'$ as a matrix. There is a straightforward formula for the endpoints $p_1, p_2 \in \mathbf{R}$ of the axis of M. Thus, we find each axis. We then simply count the number of intersections which lie in $\mathcal{D}$ of these finitely many axes. Note that this determination of the number of self-intersections of the geodesic of W has been reduced to arithmetic.

2 Introduction

Our goal is to classify geodesics on T with three transverse self intersections. To do so, we consider four simple loops, as justified by the following lemma taken from Crisp, Dziadosz, Garity, Insel, Schmidt, Wiles [CDGISW].

Lemma 2.1 *Up to free homotopy any l on T with k transverse self intersection points can be formed as the compostition of $k+1$ simple loops which intersect at a single point.*

In answer to the question of what types of simple loops are found on the punctured torus, we state a result that follows directly form the classification of surfaces. See Birman and Series [BS] for a discussion of this and other results about simple loops on surfaces.

Theorem 2.1 *The conjugacy class of a simple loop l of T is either,*

(i) the identity and l bound a disc,

(ii) one of $[aba^{-1}b^{-1}]$ or $[bab^{-1}a^{-1}]$ and l bounds a punctured disc, or

(iii) $[w]$ where w is a generator of $\pi_1(T)$ and l does not separate T.

In the next section we classify the types of generators found on T up to reflection and orientation. The method by which we schematically represent T is from [C]. T is cut along some loop l whose image in $\pi_1(T)$ is a generator to obtain a disc bounded by l containing the puncture and a hole also bounded by l. This operation is illustrated by Figure 1.

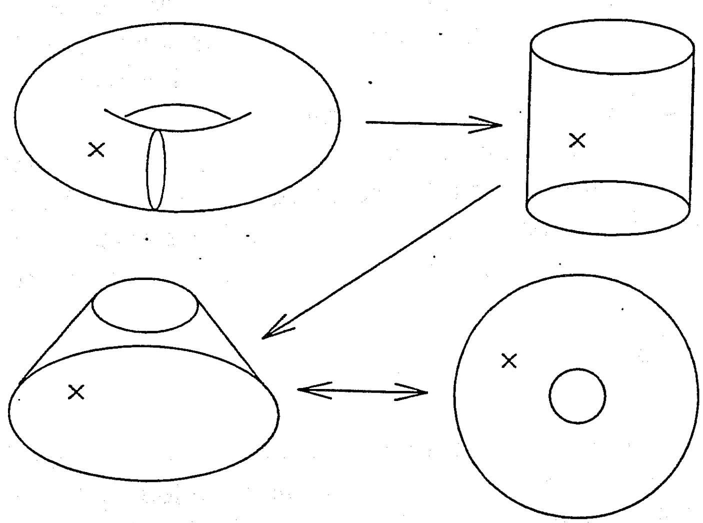


Figure 1

3 Discussion of Loops

Theorem 3.1 *Given a generating pair* $\{a,b\}$ *for* $\pi_1(T)$*, any other generator representable by a simple loop is equivalent to one of the following six, up to orientation and reflection,*

(i) a

(ii) b

(iii) bab^{-1}

(iv) aba^{-1}

(v) ab

proof: Cut T along a. Now a and b can be schematically expressed as in Figure 2.

A loop, l, can either start and end on the same side or different sides of a.

Assume it starts and ends on the same side of a. If l bounds nothing it is an identity loop and hence not a generator (by Theorem above). If l bounds just the puncture it is not a generator. If l bounds the puncture and the hole, it is homotopic to a. If it bounds the hole but not the puncture it is a seperating curve and thus a generator.

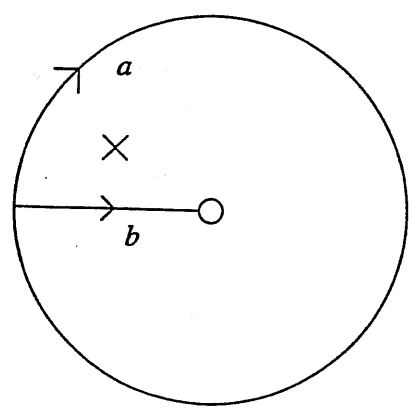


Figure 2

If the loop, b, is present in the bouquet, all loops around the hole but not the puncture intersect it more than once except for bab^{-1}, $ba^{-1}b^{-1}$, $b^{-1}ab$, and $b^{-1}a^{-1}b$ which are the same up to orientation and reflection.

If the loop, b, is not present, all such loops can be deformed to the above four. Refer to this class of generators as bab^{-1} (iii).

Assume a loop, l, starts and ends at different sides of a. Then l starts and ends at either the same side or different sides of b. Suppose l starts and ends at the same side of b. If the puncture is not contained in the region strictly bounded by b and l, the loop l is homotopic to b. If the puncture is contained in the region strictly bounded by b and l, the loop is aba^{-1}, $ab^{-1}a^{-1}$, $a^{-1}ba$, or $a^{-1}b^{-1}$, which are all the same up to orientation and reflection. Refer to this class as aba^{-1} (iv).

Suppose it starts and ends at different sides of b. All such loops intersect a or b more than once except ab, $a^{-1}b$, ab^{-1}, $a^{-1}b^{-1}$, ba, $b^{-1}a$, ba^{-1}, or $b^{-1}a^{-1}$, which are all the same up to orientation and reflection. Refer to this class as ab (v). ♡

We now present a lemma before introducing a theorem classifying the possible generators on T.

Lemma 3.1 *Given three non-homotopic generators of $\pi_1(T)$, representable by simple loops, that intersect only at the basepoint, each one of them must form a basis with at least one of the other two.*

proof: Suppose l_1, l_2, and l_3 are such non-homotopic generators. WLOG cut along l_1 - call it a. Assume neither l_2 nor l_3 forms a basis with l_1. l_2 and l_3 are not homotopic but bab^{-1} is the only generator which doesn't form a basis with a so we have a contradiction. ♡.

Theorem 3.2 *Let $\{l_1, l_2, ..., l_n\}$ be a collection of non-homotopic generators of $\pi_1(T)$ representable by simple loops which intersect only at the basepoint. Then $n \leq 4$ and generators for $\pi_1(T)$ can be chosen so that $\{l_1, l_2, ..., l_n\}$ is one of:*

(i) a;

(ii) a, b;

(iii) a, bab^{-1};

(iv) a, b, bab^{-1};

(v) a, b, ab;

(vi) a, b, ab, aba^{-1}.

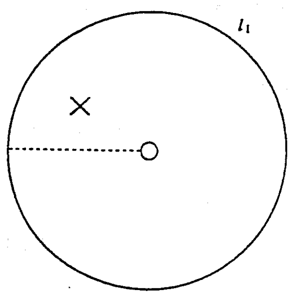


Figure A

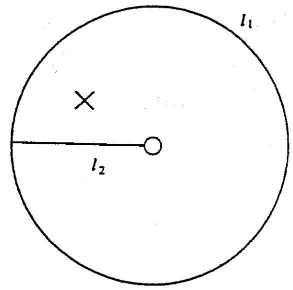


Figure B

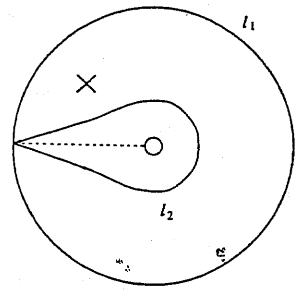


Figure C

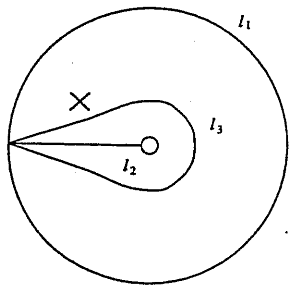


Figure D

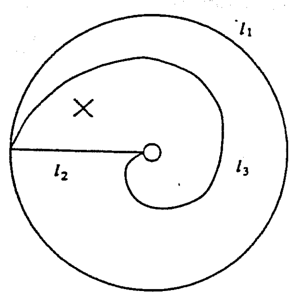


Figure E

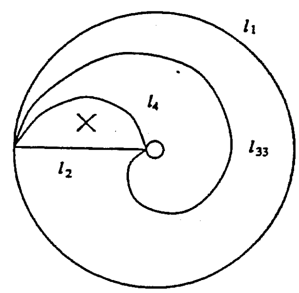


Figure F

proof: Assume n=1. Cut along l_1 and call it a (i).

Assume n=2, thus there are two generators l_1 and l_2. Cut along l_1 — call it a. Either the l_2 doesn't form a basis with l_1 or it does. Suppose it doesn't. Then l_2 is bab^{-1}, the only generator that doesn't form a basis with a (iii). Suppose l_2 forms a basis with l_1. Then take l_2 to be b (ii).

Assume n=3, thus there are three generators, l_1, l_2, and l_3. By the above lemma, each generator must form a basis with at least one of the other two. So of the pairs (l_1, l_2), (l_2, l_3), and (l_3, l_1), two or three of them form a basis.

WLOG assume (l_1, l_2) and (l_2, l_3) are both bases pairs and (l_3, l_1) isn't. Cut along l_1 — call it a. Call l_2 b. Then l_3 must be bab^{-1} since bab^{-1} is the only generator that forms a basis with b and not with a (iv).

Now assume all three pairs form bases. Cut along l_1 — call it a. Call l_2 b. ab is the only generator that forms a basis with a and with b so call l_3 ab (v).

Assume n=4, thus there are four generators, l_1, l_2, l_3, and l_4. Assume each possible pair forms a basis. WLOG cut along l_1 — call it a. Call l_2 b. The only generator that forms a basis with a and with b is ab. So there do not exist four such generators. So at least one pair doesn't form a basis. WLOG say (l_2, l_3) doesn't form a basis. By multiple applications of Lemma 3.1, l_2 must form a basis with l_1 and with l_4. Similarly, l_3 must form a basis with l_1 and l_4. Cut along l_1 — call it a. Call l_2 b. l_3 must be aba^{-1} since it is not a basis with l_2 (b) and is with l_1 (a). Finally, l_4 forms a basis with l_2 (b) and with l_3 (aba^{-1}) so l_4 must be ab (vi). Note that l_4 and l_1 are forced to be a basis pair.

Assume n=5, thus there are five generators. This would be a combination of every generator from Theorem 3.1. aba^{-1} and bab^{-1} can't both exist in the same bouquet because they intersect each other more than once. So there is no five non-homotopic generator case. Clearly, given n≥6, there will be homotopic repeats. ♡

Theorem 3.3 *Any configuration of four non-trivial simple loops on T, which intersect only at the basepoint, is automorphic to one of the following nineteen configurations.*

proof: Assume there are zero generators and four loops around the puncture. Cutting along any generator will yield:

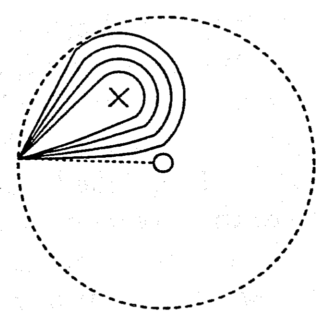


Figure 3.1

Assume there is one generator and three loops around the punture. Cut along the generator to yield:

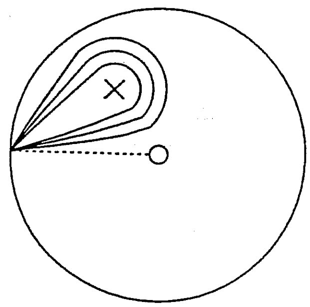


Figure 3.2

Assume there are two generators and two loops around the puncture. Add a homotopic loop to l_1 of Figure A (this is denoted in the following text by adding one loop to (l_1,A). Cut along l_1 to yield:

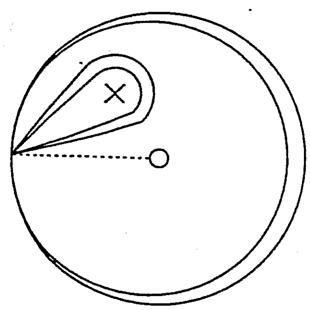


Figure 3.3

Observe Figure B to yield:

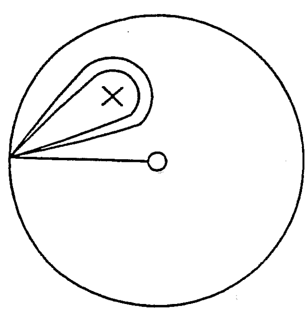


Figure 3.4

Obersve Figure C to yield:

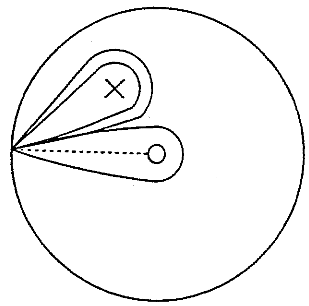


Figure 3.5

Assume there are three generators and one loop around the puncture. Add two homotopic loops to (l_1,A) and cut along l_1 to yield:

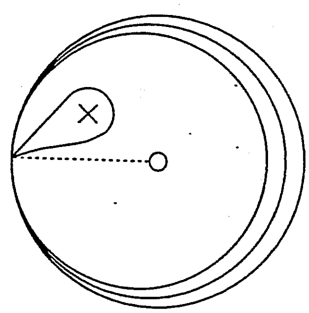


Figure 3.6

Add one homotopic loop to (l_1,B) and cut along l_1, or add one homotopic loop to (l_2,B) and cut along l_2 to yield:

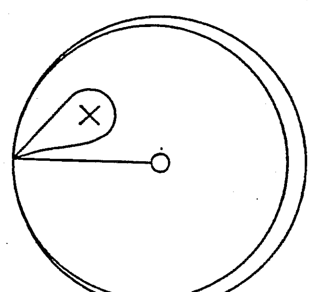


Figure 3.7

Add one homotopic loop to (l_1,C) and cut along l_1, or add one homotopic loop to (l_2,C) and cut along l_2 to yield:

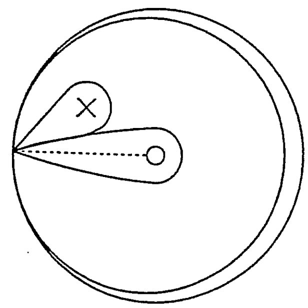


Figure 3.8

Observe D to yield:

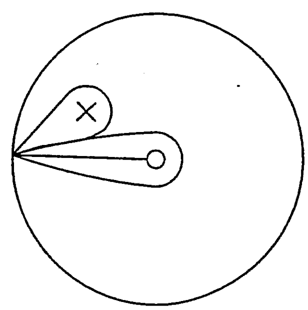


Figure 3.9

Observe E to yield:

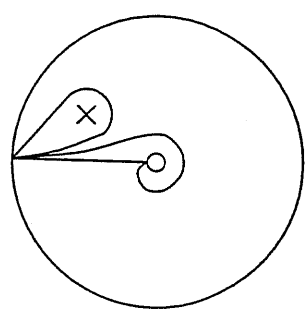


Figure 3.10

Assume there are four generators and zero loops around the puncture. Add three homotopic loops to (l_1,A) and cut along l_1 to yield:

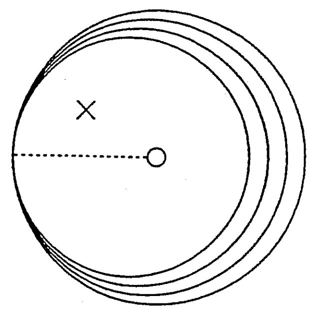


Figure 3.11

Add two homotopic loops to (l_1,B) and cut along l_1, or add two homotopic loops to (l_2,B) and cut along l_2 to yield:

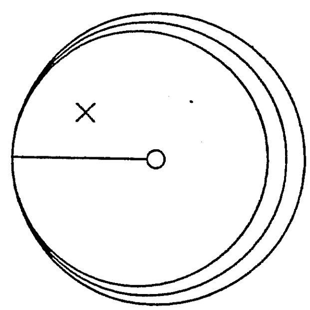


Figure 3.12

Add one homotopic loop to (l_1,B) and one homotopic loop to (l_2,B) and cut along l_1 to yield:

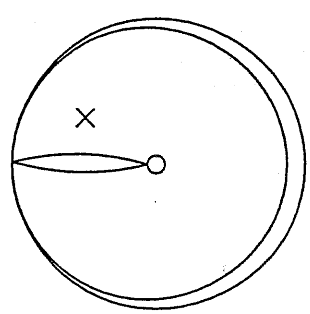


Figure 3.13

Add two homotopic loops to (l_1,C) and cut along l_1, or add two homotopic loops to (l_2,C) and cut along l_2 to yield:

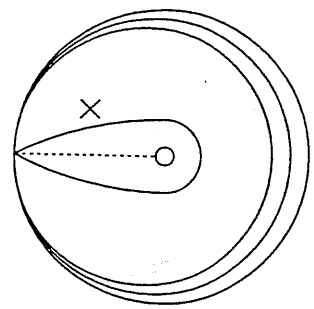


Figure 3.14

Add one homotopic loop to (l_1,C) and one homotopic loop to (l_2,C) and cut along l_1 to yield:

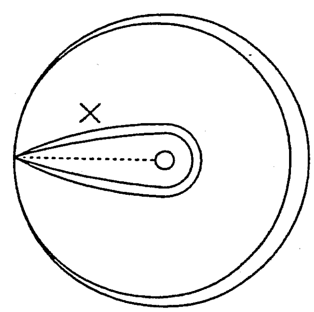


Figure 3.15

Add one homotopic loop to (l_1,D) and cut along l_1, or add one homotopic loop to (l_3,D) and cut along l_3 to yield:

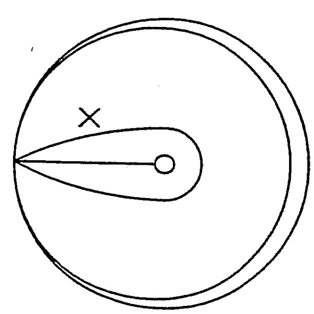


Figure 3.16

Add one homotopic loop to (l_2,D) and cut along l_2 to yield:

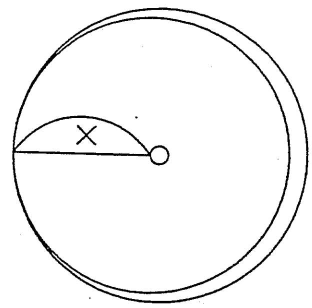


Figure 3.17

Add one homotopic loop to (l_1,E) and cut along l_1, or add one homotopic loop to (l_2,E) and cut along l_2, or add one homotopic loop to (l_3,E) and cut along l_3 to yield:

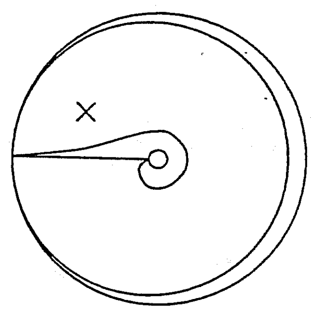


Figure 3.18

Observe F to yield:

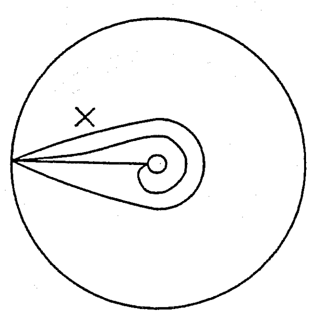


Figure 3.19

♡

4 Cross Method Analysis

Upon classifying the unique pictures of various combinations of four simple loops on a punctured torus, it is possible to use a technique derived from the basepoint analysis in [CDGISW]. When dealing with three rather than two intersections, the basepoint graph of each picture consists of eight segments joined at a single point (as seen in Figure 4). Yet, with three intersections, when the eight point star is pulled apart to reveal the individual intersections, two possible configurations arise (Figures 4.1b and 4.2b).

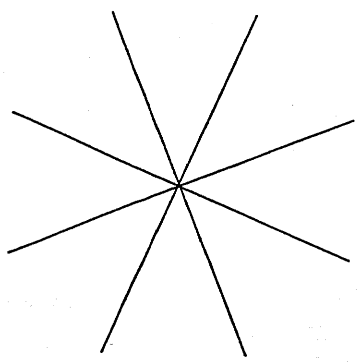


Figure 4

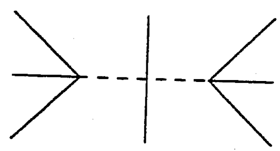


Figure 4.1a

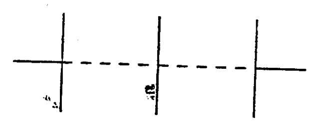


Figure 4.1b

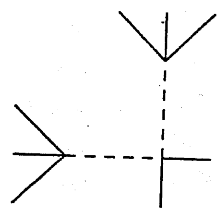


Figure 4.2a

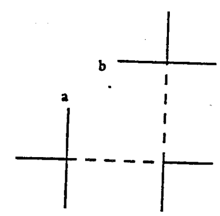


Figure 4.2b

In order to simplify analysis we would like to show that the basepoints of all possible loops with three intersection can be expressed in one of these forms. A counter-example shows that not all loops with three intersections can be drawn as Figure 5.

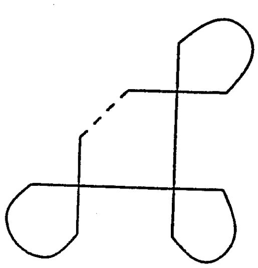


Figure 5

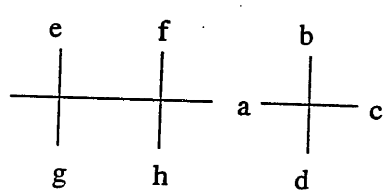


Figure 6.1

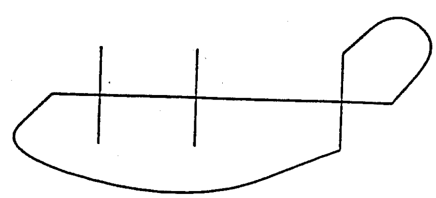


Figure 6.2

Lemma 4.1 *Given a loop on T with three transverse intersections that intersect at only one point, the basepoint can be pulled apart along two segments as shown in Figure 4.2b.*

proof: Assume there exists a curve with three self intersections whose basepoint cannot be deformed to Figure 4.2b. Beginning with three single crosses, two must first be joined by a segment (Figure 6.1). Next, a third cross must be joined. When adding a third cross, in order to avoid any configuration that can be deformed into Figure 4.2b, none of the four segments a-d can be connected to segments e-h. Therefore, two segments of the new cross must connect with the unlabeled segments of the two adjoined crosses, leaving the other two to be joined in a closed loop (Figure 6.2). Yet, after such a connection is made, the entire three intersecion loop cannot be traced. Thus every curve with three transverse intersections can be expressed in the form shown in Figure 4.2b. ♡

Now using the fact that all basepoints can be expressed in this form, each of the unique pictures can be analyzed more efficiently. The uniqueness of Figure 4.2b is that the segments labeled a and b must be located on adjacent segments of the basepoint star. Thus the possibilities of various combinations of loops is drastically reduced. After full analysis, a classification of loops was developed.

Theorem 4.1 *The conjugacy class in $\pi_1(T)$ of a loop on T with three non-trivial self-intersections is one of*

(a) $[g(a^4)]$

(b) $[(aba^{-1}b^{-1})^4]$ *or* $[(bab^{-1}a^{-1})^4]$

(c) $[g(a(aba^{-1}b^{-1})^3)]$

(d) $[g(aa(aba^{-1}b^{-1})^2)]$

(e) $[g(aaba^{-1}bab^{-1}a^{-1}b)]$

(f) $[g(aaba^{-1}a^{-1}b^{-1}aba^{-1}b^{-1})]$

(g) $[g(abab^{-1}a^{-1}bab^{-1}a^{-1}bab^{-1})]$

(h) $[g(aaaaba^{-1}b^{-1})]$

(i) $[g(aabba^{-1}b^{-1})]$

(j) $[g(aabab^{-1}aba^{-1}b^{-1}]$

(k) $[g(aaba^{-1}b^{-1}abab^{-1})]$

(l) $[g(aabab^{-1}a^{-1}bab^{-1})]$

(m) $[g(aaaba^{-1}a^{-1}b^{-1})]$

(n) $[g(aaaabb)]$

(o) $[g(aaabab^{-1})]$

(p) $[g(aabaab^{-1})]$

(q) $[g(abab^{-1}a^{-1}ba^{-1}b^{-1})]$

for some $g \in Aut\ \pi_1(T)$.

proof: **Case 1:** First the possibility of identity loops must be addressed. If one of our four loops is the identity, by its nature, the loop must connect adjacent segments of the basepoint. In fact, if an identity exists, in order to maintain all three intersections, it must connect segments a and b. If it were to connect any of the other segments, one or more intersections would be lost when it was pulled away. By connecting segments a and b, Figure 6.1 appears. Since the center loop is the identity, the basepoint can be pulled back into a six star which denotes three simple loops. We can refer back to [CDGISW] where a full case analysis of the combinations of three simple loops is presented. Two three intersectors were discovered in their analysis, namely (q) and (i) in the previous Theorem.

Cases 2-20: Since Figures 3.1-3.19 exhaust all possible combinations of non-trivial loops, the cross method, performed on each, eliminating all simple, one and two intersectors, produces a complete list of loops with three intersections (see Table 1 — Note: Loops are labelled $a = \ell_1$, continuing clockwise in order of appearance on the basepoint graph.). A sample analysis is as follows. ♡

Analyzing Figure 7.1, we can create its basepoint star as in Figure 7.2. Exiting the star along a loop is denoted by the number of the loop and reentering the star is denoted by that same number prime. Since the figure is not symmetric about the horizontal, all possible adjacent pairs of segments on the star must be fit to a and b on Figure 4.2b and tested.

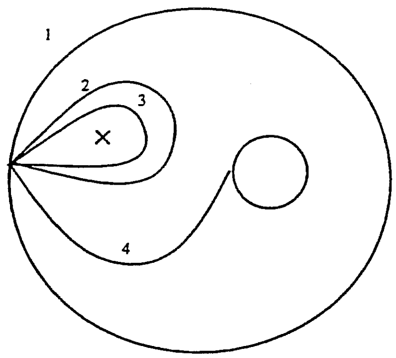


Figure 7.1

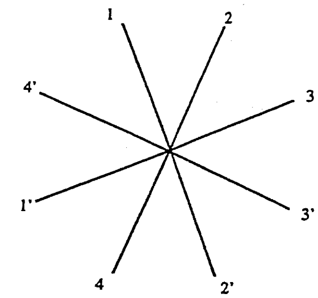
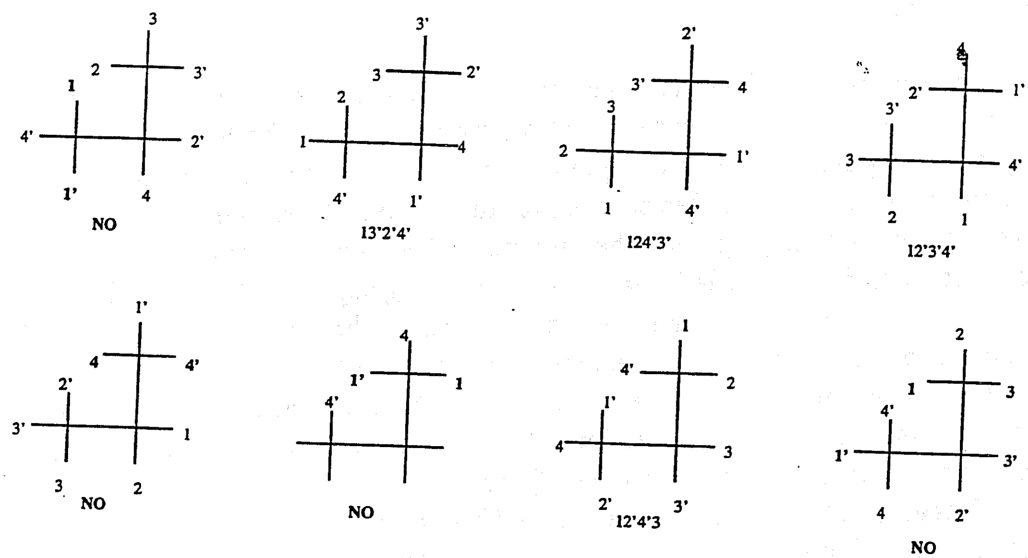


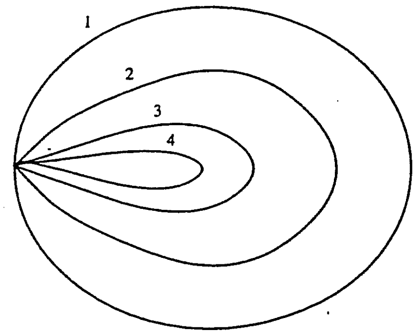
Figure 7.2

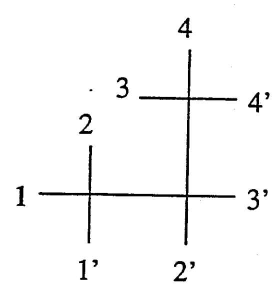


The eight labelled crosses above allow us to find which loop combinations can possibly yield three intersections. If a number and its prime are on opposite ends of a segment, the loop continually repeats itself and we cannot trace out three transverse intersections. Therefore, those cases can be eliminated. As in the last cross, if only two loops can be traced those cases can be elimated as well.

Therefore, the only configurations left to be analyzed are $\ell_1\ell_3^{-1}\ell_2^{-1}\ell_4^{-1}$, $\ell_1\ell_2\ell_4^{-1}\ell_3^{-1}$, $\ell_1\ell_2^{-1}\ell_3^{-1}\ell_4^{-1}$, and $\ell_1\ell_2^{-1}\ell_4^{-1}\ell_3$. Due to the lack of symmetry in this particular case, each of these loop combinations and their inverses must be analyzed. (If horizontal symmetry occurs within the picture, a combination and its inverse always produce the same word.) Now, assign each loop its corresponding word. $\ell_1 = a$, $\ell_2 = aba^{-1}b^{-1}$, $\ell_3 = aba^{-1}b^{-1}$, $\ell_4 = b$. Plugging in these words or their inverses, $\ell_1\ell_3^{-1}\ell_2^{-1}\ell_4^{-1}$ yields $abab^{-1}a^{-1}bab^{-1}a^{-1}b^{-1}$ which can be reduced to $aaba^{-1}bab^{-1}a^{-1}b$. $(\ell_1\ell_3^{-1}\ell_2^{-1}\ell_4^{-1})^{-1}$ yields the same word. $\ell_1\ell_2\ell_4^{-1}\ell_3^{-1}$ and its inverse both yield $aaba^{-1}b^{-1}b^{-1}aba^{-1}b^{-1}$ which through a series of steps reduces to $aaba^{-1}b^{-1}$ which is a one intersector case and can be eliminated. $\ell_1\ell_2^{-1}\ell_3^{-1}\ell_4^{-1}$ and its inverse yield the same as the first case since the words for ℓ_2^{-1} and ℓ_3^{-1} are equivalent. Finally, $\ell_1\ell_2^{-1}\ell_4^{-1}\ell_3$ and its inverse yield $abab^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}$ which also reduces to the same word as do the first and third combinations.

To simplify a bit, all pictures consisting of four concentric loops, (Figures 3.1, 3.2, 3.3, 3.6, 3.11, 3.14, and 3.15) can be analyzed together using an un-oriented picture. Figure 8 only produces three intersections with the pattern $\ell_1\ell_2\ell_4\ell_3$.





1243

Figure 8

Figure 3.1 yields $(aba^{-1}b^{-1})^4$. Figure 3.2 yields $aaba^{-1}b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1}$. Figure 3.3 yields $aaaba^{-1}b^{-1}aba^{-1}b^{-1}$. Figure 3.6 yields $aaaba^{-1}b^{-1}$. Figure 3.11 yields a^4. Figure 3.14 yields $aaabab^{-1}$, and Figure 3.15 yields $aabaab^{-1}$.

Continuing this process, each picture can be analyzed. After the possible three intersection loop configurations are discovered through the cross method, the words for each loop are joined. Some reduction is still necessary at this point. Each word must be reduced to shortest length and compared to see if it corresponds to any simple, one-intersection, or two-intersection loops. No cases of greater than three intersections occur due to the nature of the method. Table 1 shows the results of this method in their entirety. ♡

Theorem 4.2 *A closed geodesic on T has three self-intersections if and only if it is in one of the following free homotopy classes*

(a) $[g(a(aba^{-1}b^{-1})^3)]$

(b) $[g(aa(aba^{-1}b^{-1})^2)]$

(c) $[g(aaba^{-1}bab^{-1}a^{-1}b)]$

(d) $[g(aaba^{-1}a^{-1}b^{-1}aba-1b-1)]$

(e) $[g(abab^{-1}a-1bab^{-1}a-1bab^{-1})]$

(f) $[g(aaaaba^{-1}b^{-1})]$

(g) $[g(aabba^{-1}b^{-1})]$

(h) $[g(aabab^{-1}aba^{-1}b^{-1}]$

(i) $[g(aaba^{-1}b^{-1}abab^{-1})]$

(j) $[g(aabab^{-1}a^{-1}bab^{-1})]$

(k) $[g(aaaba^{-1}a^{-1}b^{-1})]$

(l) $[g(aaaabb)]$

(m) $[g(aaabab^{-1})]$

(n) $[g(aabaab^{-1})]$

(o) $[g(abab^{-1}a^{-1}ba^{-1}b^{-1})]$

for some $g \in Aut\ pi_1(T)$.

proof: To find all classes of geodesics with three self-intersection, apply Theorem 1.2 and eliminate all non primitive loops (namely, a^4, $(aba^{-1}b^{-1})^4$, and $(bab^{-1}a^{-1})^4$). (The terminology 'non primitive' is explained in detail in (C).) Thus the only candidates left are classes (a)-(o) above.

TABLE 1

Figure	Possible Loops	Reduced Word	Intersections	Classification
3.1	$\ell_1\ell_2\ell_4\ell_3$	$(aba^{-1}b^{-1})^4$	3	(b)
3.2	$\ell_1\ell_2\ell_4\ell_3$	$a(aba^{-1}b^{-1})^3$	3	(c)
3.3	$\ell_1\ell_2\ell_4\ell_3$	$aa(aba^{-1}b^{-1})^2$	3	(d)
3.4	$\ell_1\ell_3^{-1}\ell_2^{-1}\ell_4^{-1}$	$aaba^{-1}bab^{-1}a^{-1}b$	3	(e)
	$(\ell_1\ell_3^{-1}\ell_2^{-1}\ell_4^{-1})^{-1}$	$aaba^{-1}bab^{-1}a^{-1}b$	3	(e)
	$\ell_1\ell_2^{-1}\ell_4^{-1}\ell_3$	$aaba^{-1}bab^{-1}a^{-1}b$	3	(e)
	$(\ell_1\ell_2^{-1}\ell_4^{-1}\ell_3)^{-1}$	$aaba^{-1}bab^{-1}a^{-1}b$	3	(e)
	$\ell_1\ell_2^{-1}\ell_3^{-1}\ell_4^{-1}$	$aaba^{-1}bab^{-1}a^{-1}b$	3	(e)
	$(\ell_1\ell_2^{-1}\ell_3^{-1}\ell_4^{-1})^{-1}$	$aaba^{-1}bab^{-1}a^{-1}b$	3	(e)
	$\ell_1\ell_2\ell_4^{-1}\ell_3^{-1}$	$aaba^{-1}b^{-1}$	1	
	$(\ell_1\ell_2\ell_4^{-1}\ell_3^{-1})^{-1}$	$aaba^{-1}b^{-1}$	1	
3.5	$\ell_1\ell_2\ell_3\ell_4^{-1}$	$aaba^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}$	3	(f)
	$(\ell_1\ell_2\ell_3\ell_4^{-1})^{-1}$	$aaba^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}$	3	(f)
	$\ell_1\ell_2\ell_4^{-1}\ell_3$	$aaba^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}$	3	(f)
	$(\ell_1\ell_2\ell_4^{-1}\ell_3)^{-1}$	$aaba^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}$	3	(f)
	$\ell_1\ell_3\ell_2\ell_4^{-1}$	$aaba^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}$	3	(f)
	$(\ell_1\ell_3\ell_2\ell_4^{-1})$	$aaba^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}$	3	(f)
	$\ell_1\ell_2^{-1}\ell_3^{-1}\ell_4$	$abab^{-1}a^{-1}bab^{-1}a^{-1}bab^{-1}$	3	(g)
	$(\ell_1\ell_2^{-1}\ell_3^{-1}\ell_4)^{-1}$	$abab^{-1}a^{-1}bab^{-1}a^{-1}bab^{-1}$	3	(g)
	$\ell_1\ell_2^{-1}\ell_4^{-1}\ell_3^{-1}$	$aba^{-1}b^{-1}$	0	
	$(\ell_1\ell_2^{-1}\ell_4^{-1}\ell_3^{-1})^{-1}$	$aba^{-1}b^{-1}$	0	
3.6	$\ell_1\ell_2\ell_4\ell_3$	$aaaaba^{-1}b^{-1}$	3	(h)
3.7	$\ell_1\ell_4\ell_3\ell_2$	$aabba^{-1}b^{-1}$	3	(i)
	$(\ell_1\ell_4\ell_3\ell_2)^{-1}$	$aaba^{-1}b^{-1}$	1	
	$\ell_1\ell_2\ell_4^{-1}\ell_3$	$aaba^{-1}b^{-1}$	1	
	$(\ell_1\ell_2\ell_4^{-1}\ell_3)^{-1}$	$aaba^{-1}b^{-1}$	1	
	$\ell_1\ell_3^{-1}\ell_4\ell_2$	$aaba^{-1}b^{-1}$	1	
	$(\ell_1\ell_3^{-1}\ell_4\ell_2)^{-1}$	$aaba^{-1}b^{-1}$	1	
	$\ell_1\ell_4\ell_2\ell_3$	$aaba^{-1}b^{-1}$	1	
	$(\ell_1\ell_4\ell_2\ell_3)^{-1}$	$aaba^{-1}b^{-1}$	1	

Figure	Possible Loops	Reduced Word	Intersections	Classification
3.8	$\ell_1\ell_2\ell_4\ell_3$	$aabab^{-1}aba^{-1}b^{-1}$	3	(j)
	$(\ell_1\ell_2\ell_4\ell_3)^{-1}$	$aaba^{-1}b^{-1}abab^{-1}$	3	(k)
	$\ell_1\ell_4\ell_3\ell_2$	$aabab^{-1}aba^{-1}b^{-1}$	3	(j)
	$(\ell_1\ell_4\ell_3\ell_2)^{-1}$	$aaba^{-1}b^{-1}abab^{-1}$	3	(k)
	$\ell_1\ell_2\ell_3^{-1}\ell_4$	$aabab^{-1}a^{-1}bab^{-1}$	3	(l)
	$(\ell_1\ell_2\ell_3^{-1}\ell_4)^{-1}$	$aabab^{-1}a^{-1}bab^{-1}$	3	(l)
	$\ell_1\ell_3\ell_2\ell_4$	$aaba^{-1}b^{-1}abab^{-1}$	3	(k)
	$(\ell_1\ell_3\ell_2\ell_4)^{-1}$	$aabab^{-1}aba^{-1}b^{-1}$	3	(j)
	$\ell_1\ell_3\ell_4^{-1}\ell_2$	$aaaba^{-1}a^{-1}b^{-1}$	3	(m)
	$(\ell_1\ell_3\ell_4^{-1}\ell_2)^{-1}$	$aaaba^{-1}a^{-1}b^{-1}$	3	(m)
	$\ell_1\ell_4^{-1}\ell_2\ell_3^{-1}$	$aabab^{-1}$	2	
	$(\ell_1\ell_4^{-1}\ell_2\ell_3^{-1})^{-1}$	a	0	
3.9	$\ell_1\ell_4\ell_2^{-1}\ell_3$	$aaba^{-1}bab^{-1}a^{-1}b$	3	(e)
	$(\ell_1\ell_4\ell_2^{-1}\ell_3)^{-1}$	$aaba^{-1}bab^{-1}a^{-1}b$	3	(e)
	$\ell_1\ell_2\ell_4\ell_3^{-1}$	$aabba^{-1}b^{-1}$	3	(i)
	$(\ell_1\ell_2\ell_4\ell_3^{-1})^{-1}$	$aabba^{-1}b^{-1}$	3	(i)
	$\ell_1\ell_4\ell_2\ell_3^{-1}$	$aabba^{-1}b^{-1}$	3	(i)
	$(\ell_1\ell_4\ell_2\ell_3^{-1})^{-1}$	$aabba^{-1}b^{-1}$	3	(i)
	$\ell_1\ell_3\ell_4^{-1}\ell_2$	$aaba^{-1}bab^{-1}a^{-1}b$	3	(e)
	$(\ell_1\ell_3\ell_4^{-1}\ell_2)^{-1}$	$aaba^{-1}bab^{-1}a^{-1}b$	3	(e)
3.10	$\ell_1\ell_4^{-1}\ell_2^{-1}\ell_3^{-1}$	$aaaabb$	3	(n)
	$(\ell_1\ell_4^{-1}\ell_2^{-1}\ell_3^{-1})^{-1}$	$aaaabb$	3	(n)
	$\ell_1\ell_2^{-1}\ell_4^{-1}\ell_3^{-1}$	$aaaabb$	3	(n)
	$(\ell_1\ell_2^{-1}\ell_4^{-1}\ell_3^{-1})^{-1}$	$aaaabb$	3	(n)
3.11	$\ell_1\ell_2\ell_4\ell_3$	a^4	3	(a)
3.12	$\ell_1\ell_2\ell_4\ell_3^{-1}$	a^2	1	
	$\ell_1\ell_4\ell_2^{-1}\ell_3^{-1}$	a^2	1	
3.13	$\ell_1\ell_3\ell_2^{-1}\ell_4$	$abab^{-1}$	1	
	$\ell_1\ell_4\ell_2\ell_3^{-1}$	$abab^{-1}$	1	
3.14	$\ell_1\ell_2\ell_4\ell_3$	$aaabab^{-1}$	3	(o)
3.15	$\ell_1\ell_2\ell_4\ell_3$	$aabaab^{-1}$	3	(p)
3.16	$\ell_1\ell_2\ell_4\ell_3^{-1}$	$aabba^{-1}b^{-1}$	3	(i)
	$\ell_1\ell_4\ell_2^{-1}\ell_3$	$aba^{-1}bab^{-1}$	2	

Figure	Possible Loops	Reduced Word	Intersections	Classification
3.17	$\ell_1\ell_3\ell_2^{-1}\ell_4$	$aaaabb$	3	(n)
	$\ell_1\ell_4\ell_2\ell_3^{-1}$	a^2	1	
3.18	$\ell_1\ell_4\ell_2^{-1}\ell_3^{-1}$	a	0	
	$\ell_1\ell_3\ell_2^{-1}\ell_4^{-1}$	$aaba^{-1}b^{-1}$	1	
3.19	$\ell_1\ell_3^{-1}\ell_2^{-1}\ell_4$	$aabba^{-1}b^{-1}$	3	(i)
	$(\ell_1\ell_3^{-1}\ell_2^{-1}\ell_4)^{-1}$	a^3	2	
	$\ell_1\ell_3\ell_2^{-1}\ell_4^{-1}$	$aabba^{-1}b^{-1}$	3	(i)
	$(\ell_1\ell_3\ell_2^{-1}\ell_4^{-1})^{-1}$	a	0	

5 Distinctness

Note that two loops on T cannot be in automorphic conjugacy classes in $\pi_1(T)$ if they are not in automorphic conjugacy classes on the normal torus T'. When we remove the puncture from T, loops can be deformed as follows:

(i) $[aaba^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}]$, $[abab^{-1}a^{-1}ba^{-1}b^{-1}]$ are in $[Id]'$

(ii) $[a(aba^{-1}b^{-1})^3]$, $[aaba^{-1}bab^{-1}a^{-1}b]$, $[aabba^{-1}b^{-1}]$, and $[aaaba^{-1}a^{-1}b^{-1}]$ are in $[g(a)]'$

(iii) $[aa(aba^{-1}b^{-1})^2]$, $[abab^{-1}a^{-1}bab^{-1}a^{-1}bab^{-1}]$, and $[aaaabb]$ are in $[g(a^2)]'$

(iv) $[aaaaba^{-1}b^{-1}]$, $[aabab^{-1}aba^{-1}b^{-1}]$, $[aaba^{-1}b^{-1}abab^{-1}]$, and $[aabab^{-1}a^{-1}bab^{-1}]$ are in $[g(a^3)]'$

(v) $[aaabab^{-1}]$, and $[aabaab^{-1}]$ are in $[g(a^4)]'$

for some $g \in Aut\pi_1(T)$.

With this result, distinctness within each category would imply total distinctness. To discuss distinctness within the categories we must introduce the concepts of a minimal word and of direct reduction.

Let w be a word in $\pi_1(T)$. Let $[g(w)]$ denote the automorphism class of w. Let $L(w)$ denote the length of w. If there exists no word $v \in [g(w)]$ such that $L(v) < L(w)$ then w is a minimal word.

Let $\mathcal{A}$ be the set of all possible automorphisms from $\pi_1(T)$ to itself. Then $f : (a,b) \mapsto (b,a), g : (a,b) \mapsto (a^{-1},b)$, and $h : (a,b) \mapsto (ab,b)$ span $\mathcal{A}$, and are named fundamental automorphisms. This is known because f, g, and h can easily be shown to span a set of three other automorphisms proven to span $\mathcal{A}$ and presented in [C].

Consider an automorphism γ. Factor it into its fundamental automorphisms.

$$\gamma = fgfgfhgfhgf \cdots fghgfgf$$

Think of applying γ to a word by applying the fundamental automorphisms of its factorization one by one. Repeated applications of f or g in any order will never change the length of a word, only interchanging letters and inverses. Since any automorphism that factors into only f's and g's will never change the word's length, automorphisms of this type are called inconsequential. Automorphisms that take a word to cyclic permutations of itself, also never change length and are thus inconsequential.

The only time a reduction or enlargement can occur is through the application of an h. Now reassociate the string of fundamental automotphisms into fundimental components, or groupings each of which contain only one h.

$$\gamma = (fgfgfhg)(fhgf) \cdots (fghgfgf)$$

Let the number of h's in factorization of γ be k. Rename the ith fundamental component c_i for all i between 1 and k.

$$\gamma = c_k c_{k-1} \cdots c_2 c_1$$

Because a given c_i doesn't factor into f's and g's alone it may reduce or enlarge a word. Automorphisms of this type are then named consequential automorphisms. Furthermore in factoring γ into a composition of fundamental components we've factored γ into a maximal number of consequential automorphisms.

If there exists an automorphism γ from some $w \in \pi_1(T)$ to a minimal word such that,

$$L(w) > L(c_1(w)) > L(c_2c_1(w)) > \cdots > L(c_{k-1} \cdots c_2c_1(w)) > L(\gamma(w))$$

then w reduces directly.

Conjecture 5.1 *Every element in the free group with two generators can be directly reduced.*

In all of our experiences this conjecture appears to be true.

Theorem 5.1 *The fifteen words presented in Theorem 4.2 are each minimal.*

proof: Any given fundamental component will be equal to one of only 32 possible automorphisms. It can be seen that none of these 32 possible fundamental components will reduce the length of any of the fifteen words on the list. By the above conjecture, if any one of the fifteen words were not minimal, then there would exist a fundamental component that would reduce its length. Thus by contradiction, each word on the list of fifteen is minimal.♡

Lemma 5.1 *If two minimal words w and v are such that $L(w) \neq L(v)$ then w and v are distinct up to automorphism.*

proof: Assume that w and v are minimal such that $L(w) \neq L(v)$ and that w and v are elements of the same automorphism class. WLOG let $L(w) > L(v)$. Then there exists a word $v \in [g(w)]$ such that $L(v) < L(w)$, thus w is not minimal. $\Rightarrow\Leftarrow$

Applying this result to the above five categories reduces the argument for total distinctness to that of showing distinctness between the elements of $\{[aabab^{-1}aba^{-1}b^{-1}], [aaba^{-1}b^{-1}abab^{-1}], [aabab^{-1}a^{-1}bab^{-1}]\}$ and $\{[aaabab^{-1}], [aabaab^{-1}]\}$.

Lemma 5.2 *Let $w_1=[aabab^{-1}aba^{-1}]$, $w_2=[aaba^{-1}b^{-1}aba^{-1}b^{-1}]$, $w_3=[aabab^{-1}a^{-1}bab^{-1}]$, $u_1=[aaabab^{-1}]$, and $u_2=[aabaab^{-1}]$. Then the elements of $\{w_1,w_2,w_3\}$ and $\{u_1,u_2\}$ are distinct up to automorphism.*

proof: Citing a result from Whitehead [Wh], two minimal words are in the same automorphism class if one is the image of the other after applying any finite combination of level transformations. In our context of a free group with only two generators, Whitehead's level transformations for any word w are the fundamental components such that the length of the image of w equals the length of w.

So consider w_1. An exhaustive set of level transformations for this word is $\{p_1 : (a,b) \mapsto (a, a^{-1}b), p_2 : (a,b) \mapsto (a, ba^{-1}), p_3 : (a,b) \mapsto (a, ba), p_4 : (a,b) \mapsto (a, ab), p_5 : (a,b) \mapsto (a, a^{-1}b^{-1}), p_6 : (a,b) \mapsto (a, b^{-1}a^{-1}), p_7 : (a,b) \mapsto (a, b^{-1}a), p_8 : (a,b) \mapsto (a, ab^{-1})\}$. Each of these level transformations maps w_1 to a word that is equivalent to w_1 up to inconsequential automorphism. Say p_i has been applied to w_1. Let q be the inconsequential automorphism such that $w_1 = qp_i(w_1)$. Then for $p_i(w_1)$, $\{p_1q, p_2q, \cdots, p_8q\}$ constitutes a complete set of level transformations. Note that $p_jqp_i(w_1) = p_j(w_1)$, which is, as before, equivalent to w_1 up to inconsequential automorphism. Continuing inductively, the image of w_1, after applying any finite combination of level transformations is equivalent to w_1 up to inconsequential automorphism. Since w_2 and w_3 are not equivalent to w_1 up to inconsequential automorphism, no image of w_1 after repeated level tranformations will ever be equal to w_2 or w_3. Thus w_1 is distinct from w_2 and w_3 up to automorphism. This analysis can be duplicated for the other four words.♡

Theorem 5.2 *The fifteen words present in Theorem 4.2 are distinct up to automorphism and thus the list is minimal.*

proof: This result follows directly from the preceding discussion.♡

Each of the fifteen words from Theorem 4.2 are illustrated on the once-punctured torus in the following Figures 9.1-9.15.

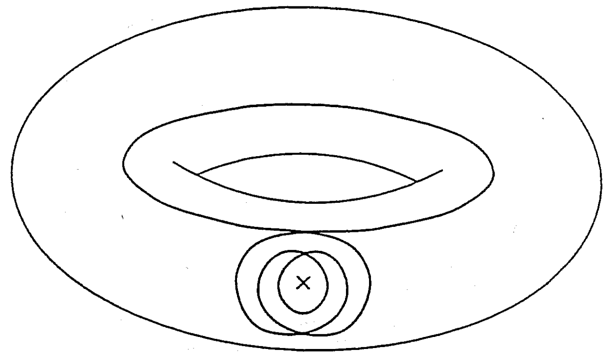


Figure 9.1 A loop on T in the free homotopy class $[aaba^{-1}b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1}]$

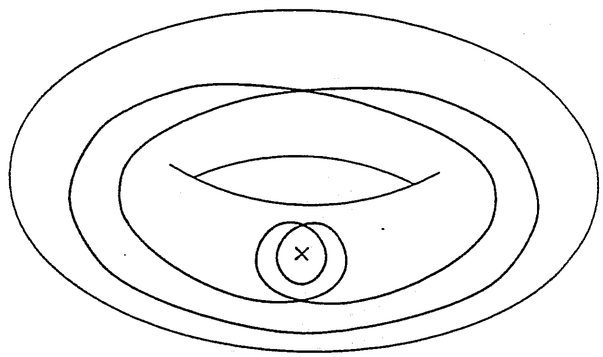


Figure 9.2 A loop on T in the free homotopy class $[aaaba^{-1}b^{-1}aba^{-1}b^{-1}]$

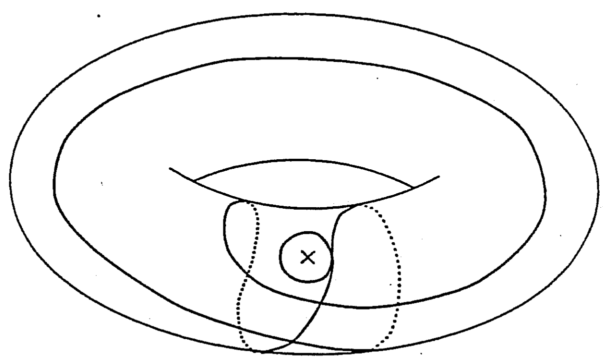


Figure 9.3 A loop on T in the free homotopy class $[aaba^{-1}bab^{-1}a^{-1}b]$

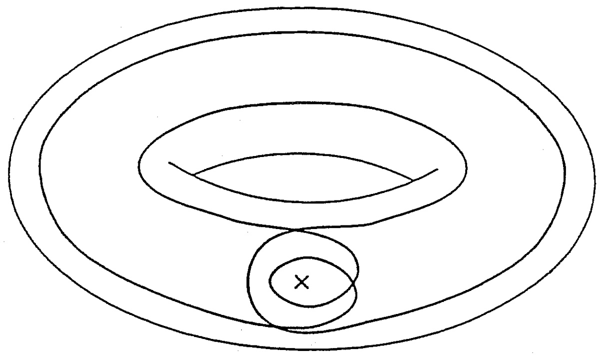


Figure 9.4 A loop on T in the free homotopy class $[aaba^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}]$

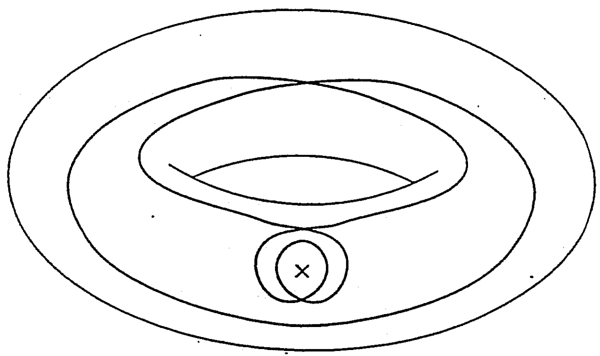


Figure 9.5 A loop on T in the free homotopy class $[abab^{-1}a^{-1}bab^{-1}a^{-1}bab^{-1}]$

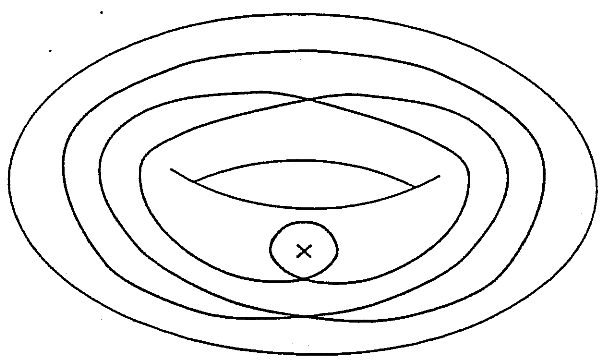


Figure 9.6 A loop on T in the free homotopy class $[aaaaba^{-1}b^{-1}]$

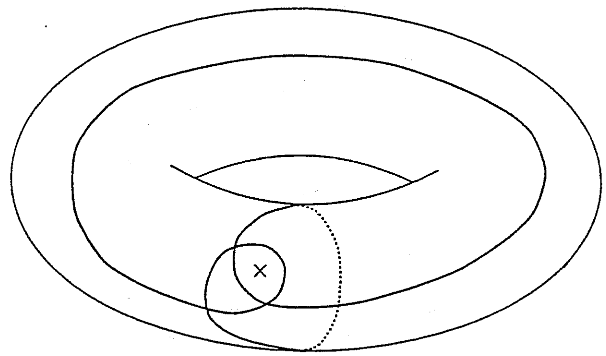


Figure 9.7 A loop on T in the free homotopy class $[aabba^{-1}b^{-1}]$

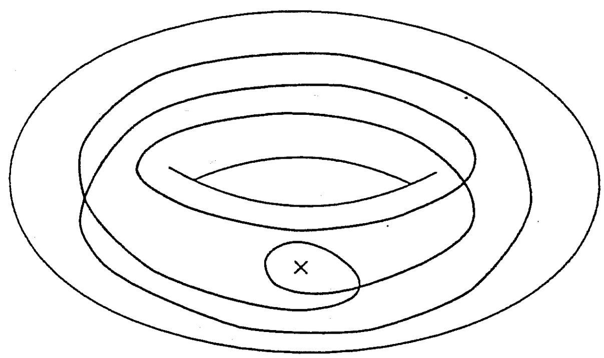


Figure 9.8 A loop on T in the free homotopy class $[aabab^{-1}aba^{-1}b^{-1}]$

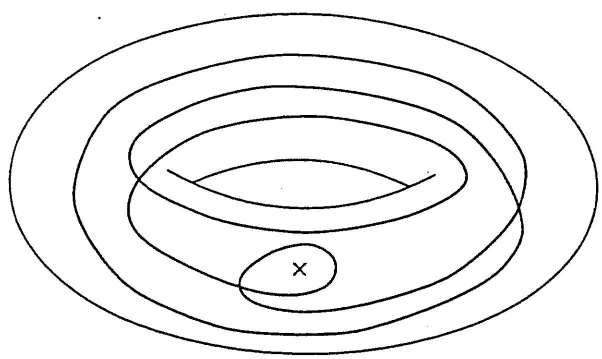


Figure 9.9 A loop on T in the free homotopy class $[aaba^{-1}b^{-1}abab^{-1}]$

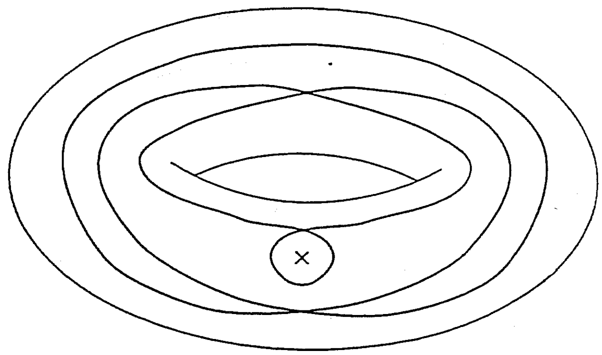


Figure 9.10 A loop on T in the free homotopy class $[aabab^{-1}a^{-1}bab^{-1}]$

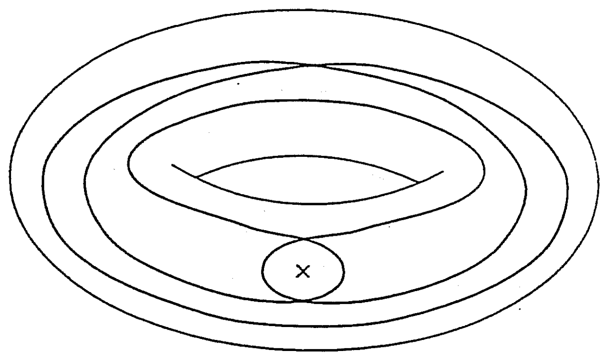


Figure 9.11 A loop on T in the free homotopy class $[aaaba^{-1}a^{-1}b^{-1}]$

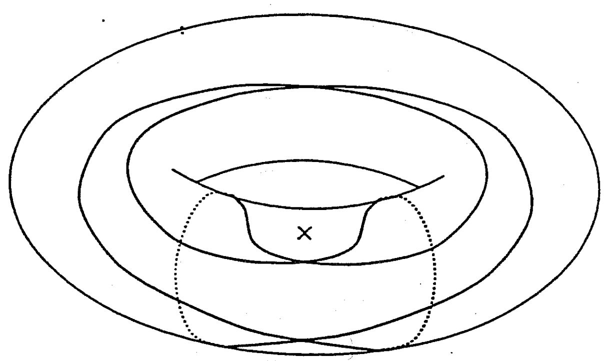


Figure 9.12 A loop on T in the free homotopy class $[aaaabb]$

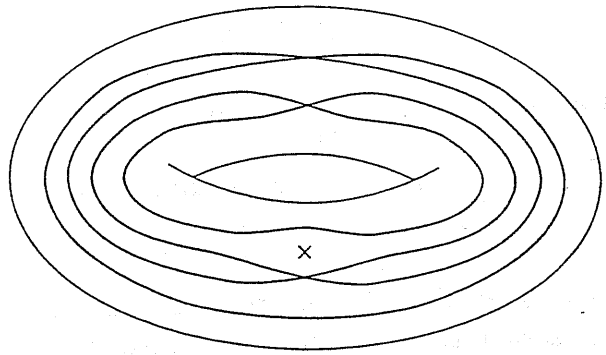


Figure 9.13 A loop on T in the free homotopy class $[aaabab^{-1}]$

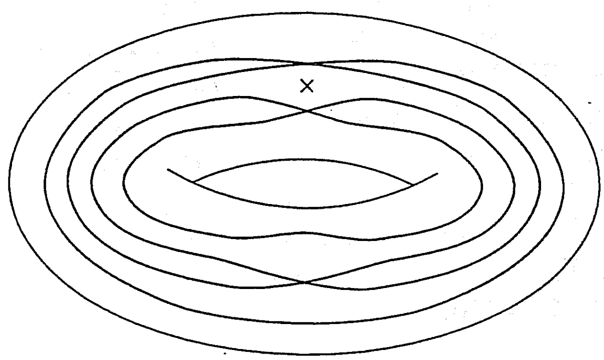


Figure 9.14 A loop on T in the free homotopy class $[aabaab^{-1}]$

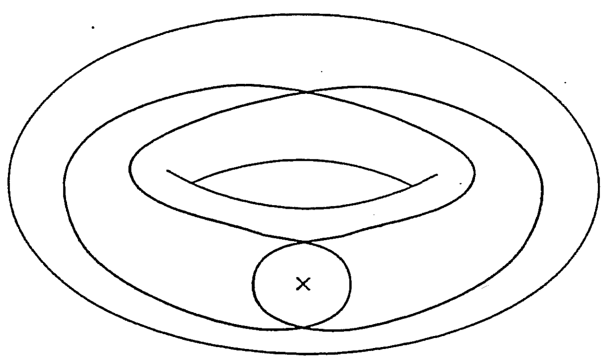


Figure 9.15 A loop on T in the free homotopy class $[abab^{-1}a^{-1}ba^{-1}b^{-1}]$

REFERENCES

[BS] J. Birman and C. Series, *AnAlgorithmforSimpleCurvesonSurfaces*, J. London Math Soc. (2) **29** (1984), 331-342.

[C] D. Crisp, *The Markoff Spectrum and Geodesics on the Punctured Torus*, Ph.D. thesis, Univ. Adelaide, 1993.

[CM] D. Crisp and W. Moran, *Single self-intersection geodesics and the Markoff spectrum*, The Markoff Spectrum, Diophantine Analysis and Analytic Number Theory (A. Pollington and W. Moran, eds.), Marcel Dekker, New York, 1993, pp. 83-94.

[CDGISW] D. Crisp, S. Dziadosz, D. Garity, T. Insel, T. Schmidt, and P. Wiles, *Closed Curves and Geodesics with Two Self-Intersections on the Punctured Torus*, REU Project follow-up paper, Oregon St. Univ., 1995.

[DIW] S. Dziadosz, T. Insel, and P. Wiles, *GeodesicswithTwoSelf − IntersectionsonthePuncturedTorus*, REU Project, Oregon St. Univ., 1994.

[N] J. Nielsen, *Die Isomorphismen der allgemeinen unendlichen Gruppe mit zwei Erzeugenden*, Math. Ann. **78** (1918), 385-397.

[Wh] J.H.C. Whitehead, *Onequivelentsetsofelementsinafreegroup*, Ann. Math. **37** (1936), 782-800.