

Explorations in Fractal Percolation

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August 11, 1995

Abstract

Fractals, as defined by Dekking, were investigated for properties of connectivity, percolation, and maximal decay. We extended the proof of x_3 Mandelbrot percolation as defined in Falconer to x_4 magnification. Using the dimension of a random Cantor set, we calculated p such that there is maximal decay along a column in neighbor interaction fractals. We showed that the neighbor interaction fractal presented by Dekking is infinitely disconnected, contrary to popular opinion.

1 Introduction

A fractal is a set with the following properties: It has detail on an arbitrarily fine scale. It is irregular enough to not be effectively described by regular geometry. It may be self-similar, either approximately or statistically. It has a larger "fractal dimension" than the topological one. Finally, it may be simply defined, perhaps recursively.

In our initial investigation into fractals, we encountered Mandelbrot Percolation. This is a method of generating a fractal based on the unit square that is indeed simple and defined recursively.

Begin with a unit square. Divide the square into fourths. Each of these fourths will have a probability, p , of remaining. So our initial square may only have three quarters left. Next divide each of the remaining squares into fourths. Each of these new squares has a probability p of remaining. Continue this process forever.

We will examine this fractal further and explore others with similar constructions. We will develop simple ways to generate these fractals and study properties that pertain to them.

2 Previous Work

The first random fractal we studied was generated by Mandelbrot Percolation. According to Falconer, in a fractal of this type where a one goes to nine ones instead of four, there is a positive probability (in fact ≥ 0.9999) that the fractal joins the left and right sides of the initial square if $0.999 \leq p \leq 1$. This simply says that the fractal will percolate from left to right. However, this model is totally disconnected if it does not percolate. Therefore, it is desirable to look for another model that may be more connected.

In the paper, "Quad-trees, Mandelbrot percolation and the modelling of random sets", Dekking discusses this shortcoming of Mandelbrot Percolation. He presents a random fractal with neighbor interactions, and on the surface it appears to definitely be more connected than the Mandelbrot technique.

3 Mandelbrot Percolation

Mandelbrot Percolation (Fractal Percolation), as described by Falconer, is defined by starting with a square and then subdividing it into nine sub-squares. Each of these sub-squares has, independently of the others, probability p of staying. The process is repeated with the remaining sub-squares. Each sub-square is divided into nine yet smaller sub-squares, the squares remaining with probability p . F_p is the resulting fractal, and E_0 is the initial square. Falconer's result for this fractal is as follows:

Theorem: Suppose that $0.999 < p < 1$. Then there is a positive probability (in fact bigger than 0.9999) that the random fractal F_p joins the left and right sides of E_0 .

The proof of this result is fairly interesting. First a p is picked. Then a sure-fire way to maintain percolation in m steps is figured. With this in mind, all the possible ways this can be done are calculated. Because of the self similar properties of this fractal, an iterative function is fairly easily developed. A fixed point, t_0 , for this function is then calculated. This number represents a lower bound on the probability of maintaining percolation for any number of steps. As $m \rightarrow \infty$ the probability settles down to this t_0 value. Therefore \exists a positive probability that the random fractal joins the left and right sides (i.e. percolates).

Now, when we programmed this on a computer, we started by dividing each square into four sub-squares instead of nine. This was done in preparation for neighbor interactions where squares would be divided by four. Finding a surefire way to maintain percolation in m steps must now be figured. This proof follows Falconer's exactly and differs only in the numbers. E_0 is the starting configuration (i.e. a single 1). E_1 is the first iteration and so on.

$$F_p = \bigcap_{k=0}^{\infty} E_k.$$

THEOREM: Suppose that $0.999 < p < 1$. Then there is a positive probability that the random fractal F_p joins the left and right sides of E_0 .

PROOF: If I_1 and I_2 are abutting squares of E_k and both contain either 15 or 16 sub-squares of E_{k+1} , then \exists a sub-square in I_1 and one in I_2 that abut with the squares of E_{k+1} in I_1 and I_2 forming a connected unit.

Say E_k is **full** if it contains either 15 or 16 squares of E_{k+1} . Say E_k is **2-full** if it contains 15 or 16 full squares of E_{k+1} . Say E_k is **m-full** if it contains 15 or 16 $(m-1)$ -full squares of E_{k+1} . If E_0 is m -full then the sides of E_0 are joined by a sequence of abutting squares of E_m .

The square E_0 is m -full ($m \geq 1$) if either:

- a) E_1 contains 16 squares all of which are $(m-1)$ -full, or
- b) E_1 contains 16 squares 15 of which are $(m-1)$ -full, or
- c) E_1 contains 15 squares all of which are $(m-1)$ -full.

If p_m is the probability that E_0 is m -full then summing the three possibilities,

$$\begin{aligned} p_m &= p^{16}p_{m-1}^{16} + p^{16}\binom{16}{15}p_{m-1}^{15}(1-p_{m-1}) + \binom{16}{15}p^{15}(1-p)p_{m-1}^{15} \\ p_m &= p^{16}p_{m-1}^{16} + 16p^{16}p_{m-1}^{15} - 16p^{16}p_{m-1}^{16} + 16p^{15}p_{m-1}^{15} - 16p^{16}p_{m-1}^{15} \\ p_m &= 16p^{15}p_{m-1}^{15} - 15p^{16}p_{m-1}^{16} \quad (\text{for } m \geq 2) \end{aligned}$$

Furthermore, $p_1 = p^{16} + 16p^{15}(1-p) = 16p^{15} - 15p^{16}$, so we have an iterative scheme $p_m = f(p_{m-1})$ for $m \geq 1$, where $p_0 = 1$ and

$$f(t) = 16p^{15}t^{15} - 15p^{16}t^{16}$$

Suppose $p = 0.999$ then $f(t) \approx 15.762t^{15} - 14.762t^{16}$.

And a little calculation shows that $t_0 = 0.9977$ is a fixed point of f which is stable in the sense that $0 < f(t) - t_0 \leq \frac{1}{2}(t - t_0)$ if $t_0 < t \leq 1$. It follows that p_m is decreasing and converges to t_0 as $m \rightarrow \infty$, so there is a probability $t_0 > 0$ that E_0 is m -full for all m . When this happens, opposite sides of E_0 are joined by a sequence of squares in E_m for each m , so the intersection $F_p = \bigcap_{k=0}^{\infty} E_k$ joins opposite sides of E_0 . Thus, there is a positive probability

of percolation occurring if $p = 0.999$, and consequently for larger values of p .

Q.E.D.

4 Fractal Dimension

The concept of dimension is usually fairly vague. Sometimes it refers to whether the object fits in \mathbb{R} or \mathbb{R}^2 or some \mathbb{R}^n . Other times it refers to how many independent variables the object requires to construct it. There are many objects that defy these classifications. How would one decide what dimension the fractals in the previous section are? They fit in \mathbb{R}^2 , but how many independent variables are required to construct them?

First consider a straight line and a plane. The line is of dimension 1 and the plane is of dimension 2. If the line is bent at some point, it then sits in the plane, but it does not take up the whole plane. It should not, then, be given a dimension of 2. The more kinky the line becomes, the more it locally looks *like* the plane, so the closer its dimension should be to 2. This is exactly what Hausdorff dimension quantifies. Unfortunately Hausdorff dimension is very complicated mathematically both to understand and to implement. Fortunately, there is a box dimension that approximates Hausdorff dimension, and it is both easy to understand and implement.

The calculation of box dimension starts by putting a box around the image and rescaling to a box with side length 1. This may be in \mathbb{R}^2 or some \mathbb{R}^n . We will suppose for the moment that this is in \mathbb{R}^2 . Count how many boxes contain part of the image (i.e. 1). Now divide each side of the box in half. For \mathbb{R}^2 this will result in 4 boxes. Again, count how many of the boxes contain part of the image (e.g. 2). We now want to calculate the slope of the line between these two points on a log-log plot. Let s be the size of the boxes. So in our example, $s = 1$ and then $s = \frac{1}{2}$. Let N be the number of boxes that are filled for a given size of the boxes. Let k be the number of cuts that have been made.

$$\frac{\log(N(2^{-(k+1)})) - \log(N(2^{-k}))}{\log(2^{k+1}) - \log(2^k)} = \log_2 \left(\frac{N(2^{-(k+1)})}{N(2^{-k})} \right)$$

e.g. $\log_2 \left(\frac{N(\frac{1}{2})}{N(1)} \right)$ *e.g.* $\log_2 \left(\frac{2}{1} \right) = 1$

These numbers will converge to the box dimension of the image, $D \in [0, 2]$, as $k \rightarrow \infty$.

This type of procedure can be easily programmed on a computer. The computer program we wrote uses a least squares method for determining the slope of the line in the log-log plot after as many iterations as the resolution allows. This gives a number that is approximate for the dimension of the image.

These dimension ideas have some very strong applications when combined with two other concepts. The first is the **(random) Cantor set**. This is a set constructed by starting with the unit interval and removing middle thirds. Then repeating this step for each interval remaining. This leads to a countable union of disjoint closed intervals (dust). Randomness is introduced by varying both the length of the interval removed and the number of intervals removed at each iteration level.

This kind of behavior can be modeled somewhat by the use of a **binary tree**. In a binary tree, each branch represents the remaining intervals in the Cantor set. So, on the first iteration of the regular Cantor set, the binary tree has a branch into two limbs. On the second iteration, each limb then branches into two more. Repeat forever. With randomness introduced, each branch can lead to any number of limbs. This analogy ignores the length of the intervals removed but preserves the information of presence of intervals.

These two ideas (the Cantor set and the binary tree), along with dimension, lead to a new tool. The boundary of a fractal column that doubles in resolution at each iteration can be thought of as a binary tree. A limb is removed if the boundary has a block on the boundary at that iteration. A limb stays if the block on the boundary is removed. The binary tree can then be thought of as a random Cantor set. With these analogies, the dimension of the resulting random Cantor set can be calculated. If the dimension is

non-negative then there is something left, and the fractal has decayed extensively. If the dimension is zero then there is nothing left, and the fractal has not decayed very much. This way of thinking about boundaries can be an extremely powerful tool.

5 Neighbor Interactions

Neighbor interaction consists simply of changing the probability of staying a one based upon the ones and zeros nearby. In the case most studied by us this picture sums up the information:

$$\begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & 0 & \end{array} \Rightarrow \begin{vmatrix} 1 & p \\ p & 0 \end{vmatrix} \text{ and } \begin{array}{ccc} & 1 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \Rightarrow \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$$

In this basic model, each square will go to four new squares. Whether the new squares contain zeros or ones depends both upon whether the original square was a one or a zero and upon the neighbors. Neighbor interaction is accomplished by looking at the values adjacent to a corner of the square. This picture details exactly which values are important.

$$\begin{array}{ccc} & 1 & \\ 1 & 1 & \\ & & \end{array} \Rightarrow \begin{vmatrix} 1 & \square \\ \square & \square \end{vmatrix}$$

When we first wrote a computer program to model neighbor interactions, it was extremely simple and we even had p fixed at $\frac{1}{2}$. With later programs it became apparent that many different situations could easily be handled. Out of this context came the way we define fractals now:

$$\begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & 0 & \end{array} \Rightarrow \begin{vmatrix} p_2^1 & p_1^1 \\ p_1^1 & p_0^1 \end{vmatrix} \text{ and } \begin{array}{ccc} & 1 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \Rightarrow \begin{vmatrix} p_2^0 & p_1^0 \\ p_1^0 & p_0^0 \end{vmatrix}$$

The superscript denotes whether the probability refers to a starting one or a starting zero. The subscript denotes the sum of the two neighbors involved in the calculation. For example, if the neighbor above is a 1, the neighbor to the left is a 0, and the original square was a one, then the sum is 1 and the

probability of a one in the upper left sub-square would be p_1^1 . Note in this case the result (a sum of 1) is independent of where the neighboring zeros and ones are placed.

Once this way of looking at the fractal was discovered, we quickly wrote a program to implement the possibilities. The program was called growthmovie8. This program allows input of six different probabilities, written $[p_0^0, p_1^0, p_2^0], [p_0^1, p_1^1, p_2^1]$, to cover the three possible sums for either a zero or a one in the original square. These pictures were useful for looking at models that might percolate or be connected.

5.1 Photo Album

After developing growthmovie8, we decided that the best way to view the models would be in a photo album or catalog of models. This allowed us to look at the fractals in relation to each other. The organization of the photo album is simple. Each model is first categorized by initial culture (see Appendix). They are then placed in numerical order based on the first entry of $[x, x, x], [x, x, x]$. Once that is done they are placed in numerical order by the second entry and so on. Any model that is a progression of iterations, instead of a final picture at, for instance, iteration level 7, is placed at the end with others of this type.

The photo album is an excellent way of "seeing" what changing a parameter in growthmovie8 does. For example in $[0, 0, 0], [0, 0.5, X]$, look at the picture when $X = 1$. Now look at the picture for $X = 0.75$. Changing that parameter only slightly, disconnects the model considerably. Look at $[0, 0, 0], [0, 0.5, 0.75]$ again. Flip to $[0, 0, 0], [0, 0.75, 0.75]$. This parameter also increases the number of ones, as would be expected, since the fifth parameter is the probability of remaining a one if surrounded by a mixed signal. Picture $[0, 0, 0], [0, 1, 0.75]$ substantially increases the number of ones present. We suggest that you look at other models to explore how changing the probabilities of different parameters changes the model.

As has been shown, trends in percolation and connectivity can be explored by looking at the various pictures. For a given initial condition, such as culture A with parameters $[0, 0, 0], [0, p, 1]$, we can easily see what happens to

the model as p varies from 0 to 1 by small increments. The first picture, $[0,0,0],[0,0,1]$, appears to be a minimal case. From then on the center of the fractal seems to "grow" as a one has an increased probability of remaining a one. All of these models appear to percolate.

There are infinitely many models that could be produced. The photo album is only a tool, however. Each of the pictures is only an example of what a fractal with a particular p -value *might* look like. To make conjectures about percolation based on the pictures without proof would be foolish. The most important reason for the creation of the album is it is a useful tool that allows us to categorize the fractals more effectively.

6 Maximal Decay

The $[0,0,0],[0,p,1]$ fractal with the A culture is interesting for two reasons: The column in the center remains iteration after iteration, and $\exists p \in [0,1]$ such that the fractal will decay maximally to reach that inner column on both sides in the same place. The proof of this requires the concepts of the random Cantor set from before and a theorem from Falconer.

THEOREM: For a random fractal F of type $[0,0,0], [0,p,1]$ with starting

condition $\begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{matrix} \exists p \in [0,1]$ such that the Hausdorff dimension of the

column boundary on one side is strictly positive (i.e. \exists points where the random fractal F reduces to a minimal state on at least one side).

PROOF: The boundary of the fractal F on one side of the column is converted to a random cantor set as was done in the section on Dimension. Falconer's theorem uses some variables to derive its results: C_0 , C_1 , and N . C_0 and C_1 represent the ratio of the current interval length to the previous interval length. C_0 is for the left and C_1 is for the right. N is the random number representing the number of C_i that are positive (i.e. present). K is the resulting random Cantor set derived from F .

Theorem 15.2: The set K described above has probability q of being empty, where q is the smaller non-negative root of the polynomial equation:

$$f(t) = \sum_{j=0}^m P(N = j)t^j = t$$

With probability $1 - q$ the set K has Hausdorff and box dimension given by the solution s of

$$E \left(\sum_{j=0}^m C_j^s \right) = 1$$

In this case, $m = 1$, and we solve $f(t) = t$ where $f(t) = \sum_{j=0}^1 P(N = j)t^j$ which is equal to

$$P(N = 0) + P(N = 1)t.$$

$$\text{So } p^2 + \binom{2}{1}p(1-p)t = t \Leftrightarrow t = \frac{p^2}{1-2p(1-p)}.$$

At $p = 0$, $t = 0$ so $q = 0$.

With probability $1 - 0 = 1$ the set K has Hausdorff dimension s given by:

$$E \left(\sum_{j=0}^1 C_j^s \right) = 1$$

Note that in this case:

$$C_0 = C_1 = \begin{cases} \frac{1}{2} & \text{with probability } (1-p) \\ 0 & \text{with probability } p \end{cases}$$

$$E(C_0^s + C_1^s) = 1 \Leftrightarrow$$

$$E(2C_0^s) = 1 \Leftrightarrow$$

$$E(C_0^s) = \frac{1}{2} \Leftrightarrow$$

$$E(C_0^s) = \frac{1}{2}^s(1-p) + 0^s p = \frac{1}{2} \Leftrightarrow$$

$$(1-p) = 2^{s-1} \Leftrightarrow$$

$$\log_2(1-p) = s - 1 \Leftrightarrow$$

$$s = \log_2(1-p) + 1 \quad \text{for } p < \frac{1}{2}$$

For all $p < \frac{1}{2}$, s is strictly greater than 0.

Now, in order to achieve maximal decay in the same place on both sides of the column, the intersection of two of these sets is taken together as a new random Cantor set.

With this new Cantor set C_0 and C_1 become:

$$C_0 = C_1 = \begin{cases} \frac{1}{2} & \text{with probability } (1-p)^2 \\ 0 & \text{with probability } p^2 \end{cases}$$

$$E(C_0^s + C_1^s) = 1 \Leftrightarrow$$

$$\frac{1}{2}^s (1-p)^2 = \frac{1}{2} \Leftrightarrow$$

$$(1-p)^2 = 2^{s-1} \Leftrightarrow$$

$$2\log_2(1-p) = s - 1 \Leftrightarrow$$

$$s = 2\log_2(1-p) + 1$$

For $s > 0$, find p such that $2\log_2(1-p) + 1 > 0$, so $\log_2(1-p) > -\frac{1}{2}$. Now, $(1-p) > 2^{-\frac{1}{2}}$, so $p < 1 - 2^{-\frac{1}{2}} \approx 0.2929$.

Therefore, for $p < 0.2929$, s is strictly greater than zero. This implies that there will exist points where the column is defined, on both sides, by its minimal width.

Q.E.D.

The power of this result truly lies in the use of the random Cantor set to convert the problem to one where a formula exists for calculating the dimension. Figuring out that the Cantor set intervals are spaces in the fractal was a key step that seems obvious after the fact.

6.1 Connectivity

A set is connected if it cannot be decomposed into two disjoint non-empty subsets. For our purposes a **connected fractal** is one where a line can connect any two points in the given fractal. This line may have right angles, but only horizontal and vertical components (no diagonal). A **connected component** is one in which the elements of that component are connected.

The fractal studied in the Maximal Decay section, $[0,0,0],[0,p,1]$ beginning with the A culture, appears to be connected for some values of p . Another fractal using the same parameters but beginning with the F culture is clearly not connected for some values of p . (see `growthmovie8(F,7,[0,0,0],[0,0.25,1])`) In fact neither of these models will be connected for $p \in (0,1)$. To prove this, first it must be known when a component will not be removed.

LEMMA 1: For a fractal generated by $[0,0,0],[0,p,1]$, if iteration level n contains a 4-block (i.e. $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$), then iteration level $n+1$ contains a 4-block.

PROOF: Suppose at iteration level $n \ni$ a 4-block.

$$\begin{array}{cccc} & * & * & \\ * & 1 & 1 & * \\ * & 1 & 1 & * \\ & * & * & \end{array} \Rightarrow \begin{array}{cccc} & * & * & * & * \\ * & 1 & 1 & * & \\ * & 1 & 1 & * & \\ & * & * & * & * \end{array}$$

$\text{iteration } n \Rightarrow \text{iteration } n+1$

(Where each $*$ is a zero or a one as defined by the fractal.)

NOTE: \ni a 4-block in iteration level $n+1$.

Q.E.D.

To prove that the fractal will disconnect, we will show that there exists a positive probability that at each iteration the random fractal will disconnect in a finite number of steps. Then we will apply the following theorem which says: an infinitely occurring event that has a positive probability of disconnecting in a finite number of steps will eventually disconnect.

THEOREM 2: If \forall iteration levels \ni a greater than zero probability of

disconnection in a finite number of steps, then the fractal, K , will disconnect with probability 1.

PROOF: Let X_n be the set of elements that lead to disconnection in a finite number of steps. Let A_i be the state of the fractal, K , at iteration level i . Therefore $K = \bigcap_{i=1}^{\infty} A_i$. Let ϵ be the positive probability that disconnection occurs in a finite number of steps. Let M be the maximum number of steps necessary to disconnect in a finite number of steps.

NOTE: $P(X_n | A_i : i \leq M) \geq \epsilon \quad \forall n$

The probability of always staying connected is:

$$\begin{aligned} P\left(\bigcap_{n=1}^{\infty} X_n^c\right) &\leq P\left(\bigcap_{n=1}^N X_n^c\right) \\ &= \prod_{i=1}^N P(X_i^c | X_{i-1}^c, \dots, X_1^c) \\ &\leq \prod_{i=1}^N (1 - \epsilon) \\ &= (1 - \epsilon)^N \end{aligned}$$

[The probability of becoming disconnected is $\geq \epsilon$ and the probability of staying connected is $\leq (1 - \epsilon)$.]

$$\lim_{N \rightarrow \infty} P\left(\bigcap_{n=1}^N X_n^c\right) = \lim_{N \rightarrow \infty} (1 - \epsilon)^N = 0$$

$$P\left[\left(\bigcap_{i=1}^{\infty} X_n^c\right)^c\right] = P\left(\bigcup_{n=1}^{\infty} X_n\right) = 1$$

and $P\left(\bigcup_{n=1}^{\infty} X_n\right)$ is the probability of becoming disconnected.

Q.E.D.

Before we prove that the fractal with neighbor interactions will disconnect, we present the following:

Corollary to Theorem 2: The fractal, K , will disconnect an infinite number of times.

Proof: Apply Theorem 2 to find m such that the fractal, K , is disconnected at iteration level m . Repeat.

Q.E.D.

And now proof that a fractal with neighbor interactions as presented by Dekking is disconnected.

THEOREM 3: For the following two initial conditions:

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \text{ and } \begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}$$

a fractal, K , with neighbor interaction (i.e. generated by $[0,0,0],[0,p,1]$) where $p \in (0,1)$ is disconnected with probability 1.

PROOF: Let A_i be the state of fractal, K , at iteration level i . It is sufficient to show

$$\forall n \exists m : n < m < \infty \text{ such that } \bigcap_{i=0}^m A_i$$

is disconnected with positive probability and apply Theorem 2.

Suppose the boundary of A_i has a structure (or rotation of):

$$\begin{array}{ccccccc}
 & & & & * & & \\
 & & & & * & 1 & 0 \\
 & & & * & 1 & 1 & 1 & 0 = B_1 \\
 & & & * & 1 & 0 & & \\
 & & & & * & & &
 \end{array}$$

Where each * is a zero or a one as defined by A_i .

The next iteration:

$$\begin{array}{ccccccc}
 & & & & * & * & 0 & 0 \\
 & & & & * & p & 0 & 0 \\
 * & * & 1 & 1 & p & 0 & & \\
 * & * & 1 & 1 & p & 0 & \xrightarrow{\text{e.g.}} & * & 1 & 1 & 1 & 0 \\
 & & & & * & p & 0 & 0 & * & 1 & 1 & 0 \\
 & & & & * & * & 0 & 0 & * & 0 & &
 \end{array} = B_2$$

with probability $p(1-p)^3 > 0$

(note, some *'s and 0's have been dropped for simplicity)

The next iteration:

$$\begin{array}{ccccccc}
 & & & & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 \\
 * & * & p & p & p & 0 & 0 & 0 \\
 * & 1 & 1 & 1 & p & 0 & 0 & 0 \xrightarrow{\text{e.g.}} & * & 0 & 1 & 1 & 0 \\
 * & 1 & 1 & p & 0 & 0 & & & * & 1 & 1 & 1 & 1 & 0 = B_3 \\
 * & * & p & 0 & 0 & 0 & & & * & 1 & 1 & 0 & 0 \\
 & & & & & & & & * & * & 0 & 0 &
 \end{array}$$

with probability $p^3(1-p)^2 > 0$

The next iteration:

$$\begin{array}{cccccccc}
& & & & 0 & p & p & 0 & 0 \\
& & & & p & 1 & 1 & p & 0 \\
* & * & p & p & 1 & 1 & 1 & p & 0 \\
* & 1 & 1 & 1 & p & p & p & 0 & \xrightarrow{\text{e.g.}} \\
* & 1 & 1 & p & 0 & 0 & 0 & & \\
* & * & * & * & & & & & \\
& & & & & * & * & & \\
& & & & & * & 1 & 1 & * \\
& & * & * & 0 & 1 & 1 & 1 & * \\
* & 1 & 1 & 1 & 0 & * & * & & = B_4 \\
* & 1 & 1 & * & & & & & \\
& & * & * & & & & &
\end{array}$$

with probability $(1 - p)^2 > 0$

(NOTE: At this iteration, the set is disconnected.)

The probability of becoming disconnected is:

$$\begin{aligned}
P(B_1 \cap B_2 \cap B_3 \cap B_4) &= P(B_1)P(B_2|B_1)P(B_3|B_2 \cap B_1)P(B_4|B_3 \cap B_2 \cap B_1) \\
&= 1 \cdot p(1 - p)^3 \cdot p^3(1 - p)^2 \cdot (1 - p)^2 > 0
\end{aligned}$$

Now suppose the boundary of A_i does not contain any rotations of B_1 .

In this case the fractal *must* contain a four block that is near the border.

It must look like either:

$$\begin{array}{cccc}
* & * & & \\
* & 1 & 1 & 0 \\
* & 1 & 1 & 0 \\
* & * & &
\end{array}
\quad \text{or} \quad
\begin{array}{ccccccc}
* & * & & & & & \\
* & 1 & 1 & 1 & \dots & 1 & 0 \\
* & 1 & 1 & & & & \\
* & * & & & & &
\end{array}$$

There are now 5 cases to examine.

CASE 1:

$$\begin{array}{cccc}
& * & * & \\
* & 1 & 1 & 0 \\
* & 1 & 1 & 0 \\
& * & * &
\end{array}
\begin{array}{c}
\Rightarrow \\
\text{either}
\end{array}
\begin{array}{cccc}
& * & * & \\
* & 1 & 1 & 0 \\
* & 1 & 1 & 0 \\
& * & 1 & * \\
& & * &
\end{array}
= 1.a \text{ or }
\begin{array}{cccc}
& * & * & \\
* & 1 & 1 & 0 \\
* & 1 & 1 & 0 \\
& * & 0 &
\end{array}
= 1.b$$

1.a

$$\begin{array}{cccc}
& * & * & \\
* & 1 & 1 & 0 \\
* & 1 & 1 & 0 \\
& * & 1 & * \\
& & * &
\end{array}
\Rightarrow
\begin{array}{cccc}
& * & * & * \\
* & 1 & 1 & p \ 0 \\
* & 1 & 1 & p \ 0 \\
& * & 1 & p \ 0 \\
& & * & *
\end{array}
=
\begin{array}{cccc}
& * & * & \\
* & 1 & 1 & 0 \\
* & 1 & 1 & 1 \ 0 \\
& * & 1 & 0 \\
& & * &
\end{array}$$

with probability $p(1-p)^2 > 0$ and with positive probability this becomes disconnected in a finite number of steps

1.b

$$\begin{array}{cccc}
& * & * & \\
* & 1 & 1 & 0 \\
* & 1 & 1 & 0 \\
& * & 0 &
\end{array}
\Rightarrow
\begin{array}{cccc}
& * & * & \\
* & 1 & 1 & p \ 0 \\
* & 1 & 1 & p \ 0 \\
& * & p & 0 \\
& & 0 &
\end{array}
=
\begin{array}{cccc}
& * & * & \\
* & 1 & 1 & 0 \\
* & 1 & 1 & 1 \ 0 \\
& * & 1 & 0 \\
& & 0 &
\end{array}$$

with probability $p^2(1-p) > 0$ and with positive probability this becomes disconnected in a finite number of steps

CASE 2:

$$\begin{array}{cccc}
& * & * & \\
* & 1 & 1 & 1 \ \dots \\
* & 1 & 1 & \\
& * & * &
\end{array}
\begin{array}{c}
\Rightarrow \\
\Rightarrow
\end{array}
\begin{array}{cccc}
& * & 0 & \\
* & 1 & 1 & 0 \\
* & 0 & & \\
& & &
\end{array}
\Rightarrow
\begin{array}{cccc}
& * & 0 & \\
& p^+ & p & 0 \\
& p^+ & p & 0 \\
& * & 0 & \\
& * & 0 & \\
& \dots & 1 & 1 \ 0 \\
& & 1 & 1 \ 0 \\
& & * & 0
\end{array}$$

with probability $\geq p^4 > 0$ and with positive probability this becomes disconnected in a finite number of steps

(note p^+ can be either 1 or p)

CASE 3:

$$\begin{array}{ccccccc}
 * & * & & & * & 0 & \\
 * & 1 & 1 & 1 & \dots & 1 & 1 & 0 \\
 * & 1 & 1 & & & * & 1 & \\
 & * & * & & & & &
 \end{array}
 \Rightarrow
 \begin{array}{ccccccc}
 * & & & & * & 0 & \\
 \dots & p^+ & p & & 0 & & \\
 & p^+ & 1 & & p & & \\
 & * & p^+ & & p^+ & & \\
 & * & 0 & & & & \\
 \dots & 1 & 1 & & 0 & & \\
 & 1 & 1 & & 0 & & \\
 & * & * & & & &
 \end{array}$$

with probability $\geq p^3(1-p) > 0$ and with positive probability this becomes disconnected in a finite number of steps

CASE 4:

$$\begin{array}{ccccccc}
 * & * & & & * & 1 & \\
 * & 1 & 1 & 1 & \dots & 1 & 1 & 0 \\
 * & 1 & 1 & & & * & 0 & \\
 & * & * & & & & &
 \end{array}
 \Rightarrow
 \begin{array}{ccccccc}
 * & p^+ & p^+ & & & & \\
 \dots & p^+ & 1 & & p & & \\
 & p^+ & p & & 0 & & \\
 & * & 0 & & 0 & & \\
 & * & * & & & & \\
 \dots & 1 & 1 & & 0 & & \\
 & 1 & 1 & & 0 & & \\
 & * & 0 & & & &
 \end{array}$$

with probability $\geq p^3(1-p) > 0$ and with positive probability this becomes disconnected in a finite number of steps

CASE 5:

$$\begin{array}{ccccccc}
 * & * & & & * & 1 & \\
 * & 1 & 1 & 1 & \dots & 1 & 1 & 0 \\
 * & 1 & 1 & & & * & 1 & \\
 & * & * & & & & &
 \end{array}
 \Rightarrow
 \begin{array}{ccccccc}
 & & p^+ & p^+ & & & \\
 \dots & p^+ & 1 & & p & & 0 \\
 & p^+ & 1 & & p & & 0 \\
 & & p^+ & & p^+ & & \\
 & * & * & & & & \\
 \dots & 1 & 1 & & 0 & & \\
 & 1 & 1 & & 0 & & \\
 & * & * & & & &
 \end{array}$$

with probability $\geq p^2 > 0$ and with positive probability this becomes disconnected in a finite number of steps

We have now shown that every iteration has a positive probability of becoming disconnected in a finite number of steps. Apply Theorem 2 to obtain the result:

The fractal, K , is disconnected with probability 1.

Q.E.D.

By applying the Corollary to Theorem 2, it is easily shown that a fractal of the type $[0,0,0],[0,p,1]$ with starting culture A or F becomes disconnected infinitely often. This implies that the fractal is indeed disconnected, contradicting the purpose for its creation, to increase connectivity.

7 Future Topics

There are a number of topics that we consider interesting that we did not have an opportunity to explore. The first relates to the Minimal Distance proof. Instead of beginning with culture A, use F. For the case, $[0,0,0],[0,1,1]$, the fractal forms a diamond shape. We believe it can be shown that as p increases, the boundary of the fractal increases until it touches the outside of the diamond. This proof could incorporate the Cantor set.

Another topic that we were unable to study in depth was dimension. The dimension of the boundaries of many of these fractals would be nice to know.

This leads into the pictures of fractals that we did not have an opportunity to study. Most of these begin with the G culture and many oscillate between zeros and ones. Questions concerning connectivity, percolation and dimension immediately come to mind.

We did not study models where the ones were constant and the zeros could change. This would be a dual of the decay model, except the initial culture would be different. It seems redundant to study this model in this manner. A more efficient method would be to change the initial culture.

The last topic that requires additional research is the idea of stochastic monotonicity. This pertains to the idea that the fractals can be ranked according to the number of zeros and ones they contain. If, for instance, a fractal contains all the ones that another fractal does and they are in the exact same locations, then that fractal would be the same size. If it contained all the ones that another fractal did, in the same locations, plus a few more, then it would be larger. This topic is much more intense than it appears and requires more study to understand it.

8 Conclusion

Our research yielded four results. The first is the proof that a fractal generated by Mandelbrot Percolation will percolate for p large enough. The second is that there exists a p such that maximal decay occurs. The third is that if there exists a positive probability of a fractal disconnecting in a finite number of steps, it will disconnect. Finally, we have shown that a random fractal generated by $[0,0,0],[0,p,1]$ with initial culture A or F will disconnect infinitely many times.

We enjoyed researching this topic a great deal. We learned what research in mathematics really means and have a greater appreciation for it. We found that communication between each other and our advisor was the key to not becoming unnecessarily idle. On many occasions we became frustrated by our lack of progress; this was the best time to talk to our advisor. He really helped us to focus on a particular fractal with a property that we previously overlooked. We enjoyed exploring this topic and wish we had more time.

9 Bibliography

Dekking, F.M. Quadrees, Mandelbrot percolation and the modelling of random sets.
Delft University of Technology.

Falconer, Kenneth. Fractal Geometry Mathematical Foundations and Applications.
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Peitgen, Heinz-Otto, Hartmut Jürgens, Deitmar Saupe. Chaos and Fractals New
Frontiers of Science. Springer-Verlag: New York, 1992.

Appendix

Cultures

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}_{4 \times 4} = [0000111111110000]$$

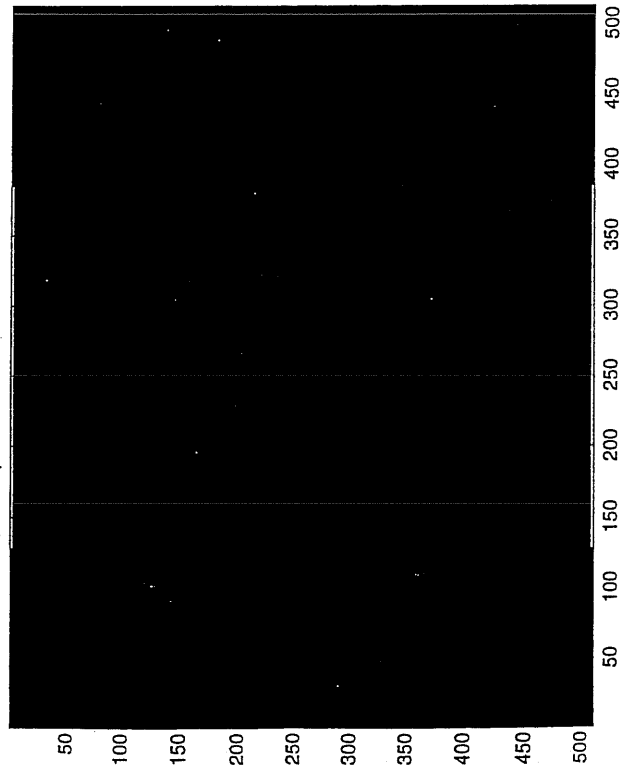
$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4 \times 4} = [0000011001100000]$$

$$G = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}_{8 \times 8}$$

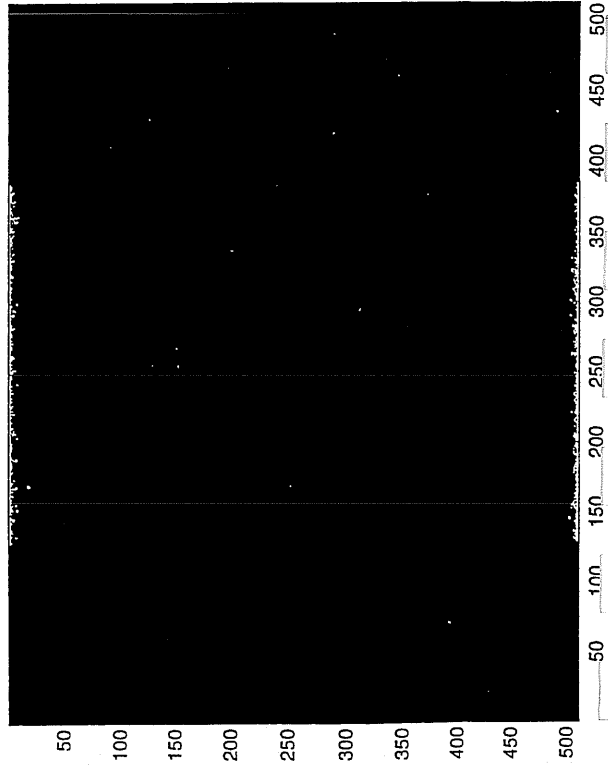
NOTE: command line notation:

`growthmovie8(culture,iterations,[p00,p10,p20],[p01,p11,p21]);`

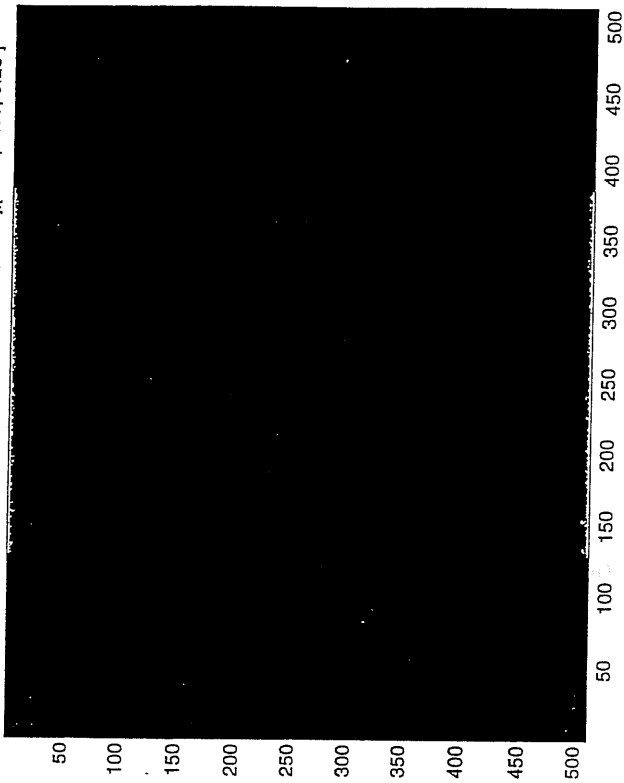
growthmovie8(A,7,[0,0,0],[0,0,0]);



growthmovie8(000011111110000,7,[0.00,0.00,0.00],[0.00,0.00,0.50])



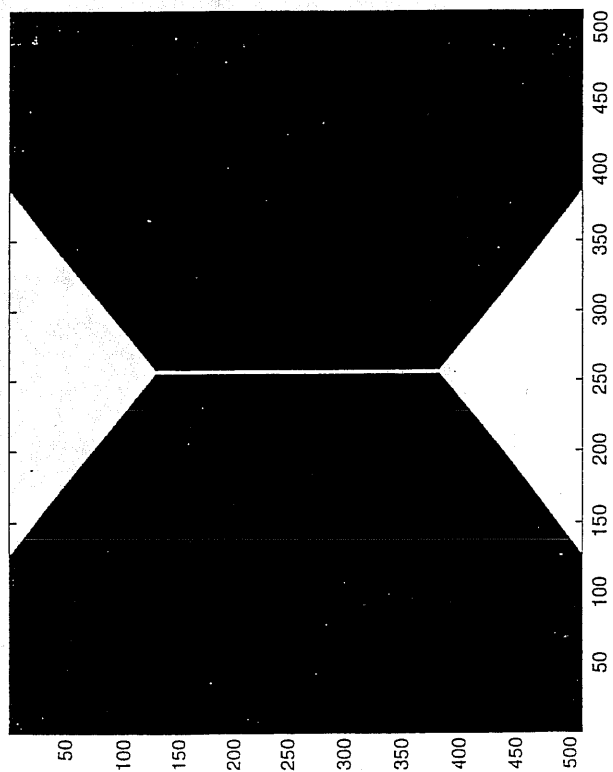
growthmovie8(000011111110000,7,[0.00,0.00,0.00],[0.00,0.00,0.25])



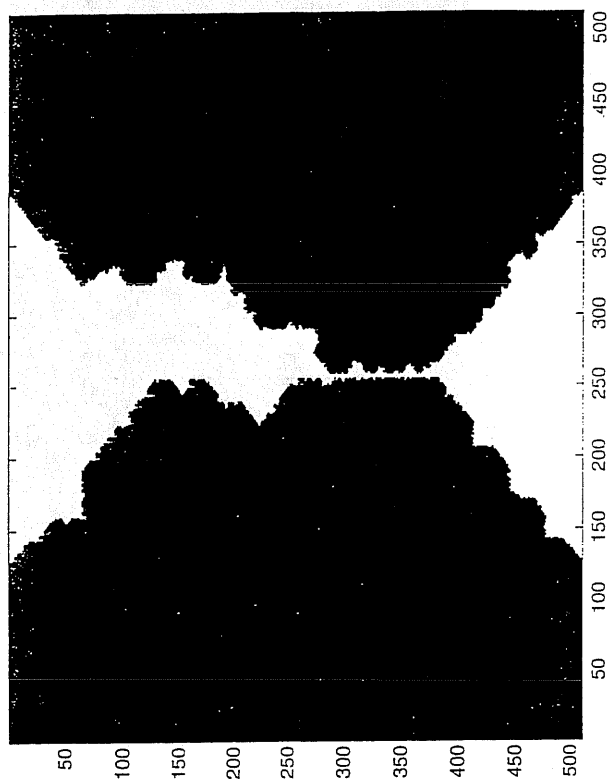
growthmovie8(000011111110000,7,[0.00,0.00,0.00],[0.00,0.00,0.75])



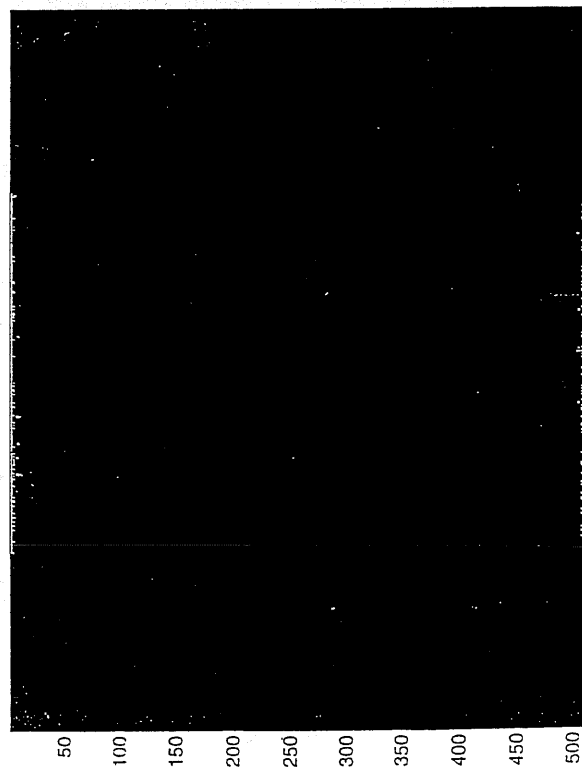

```
growthmovie8(A,7,[0,0,0],[0,0,1]);
```



```
growthmovie8(A,7,[0,0,0],[0,0.2,1]);
```



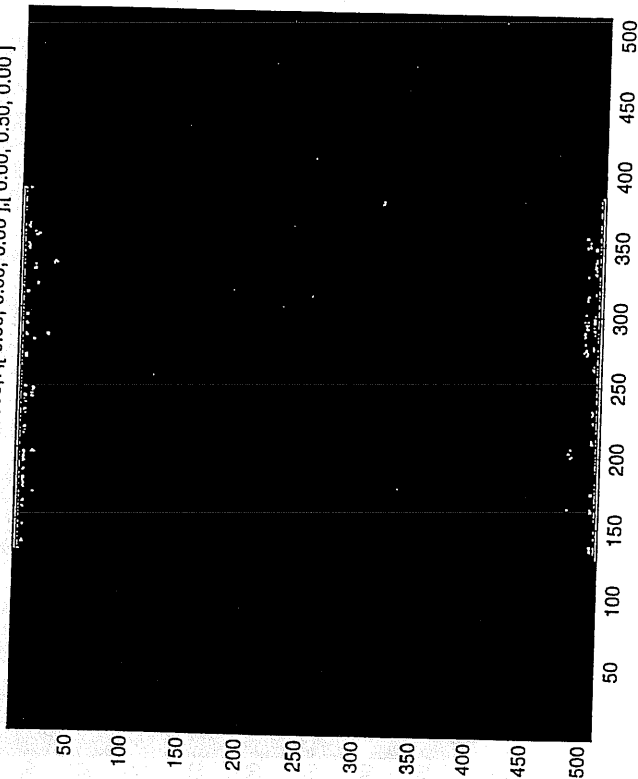
```
growthmovie8(000011111110000,7,[0.00,0.00,0.00],[0.00,0.25,0.00])
```



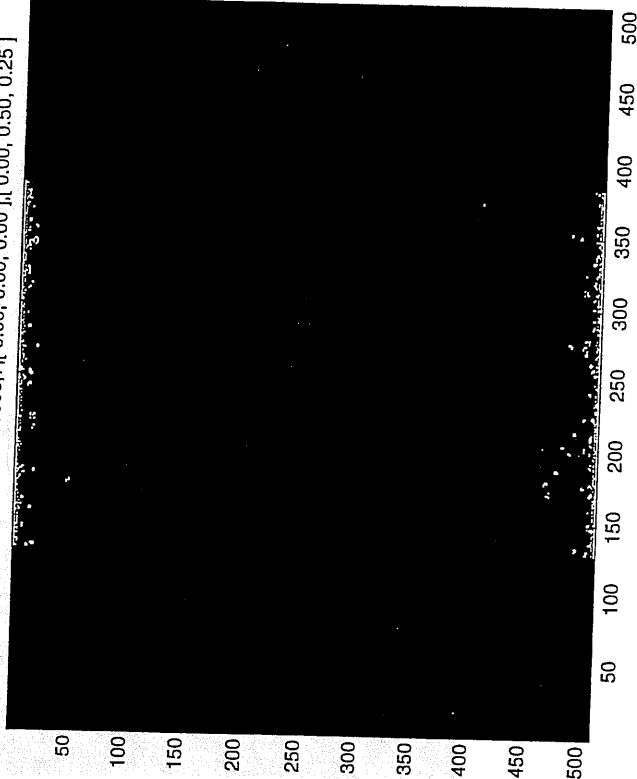
```
growthmovie8(000011111110000,7,[0.00,0.00,0.00],[0.00,0.25,0.25])
```



growthmovie8(000011111110000,7,[0.00,0.00,0.00],[0.00,0.50,0.00])

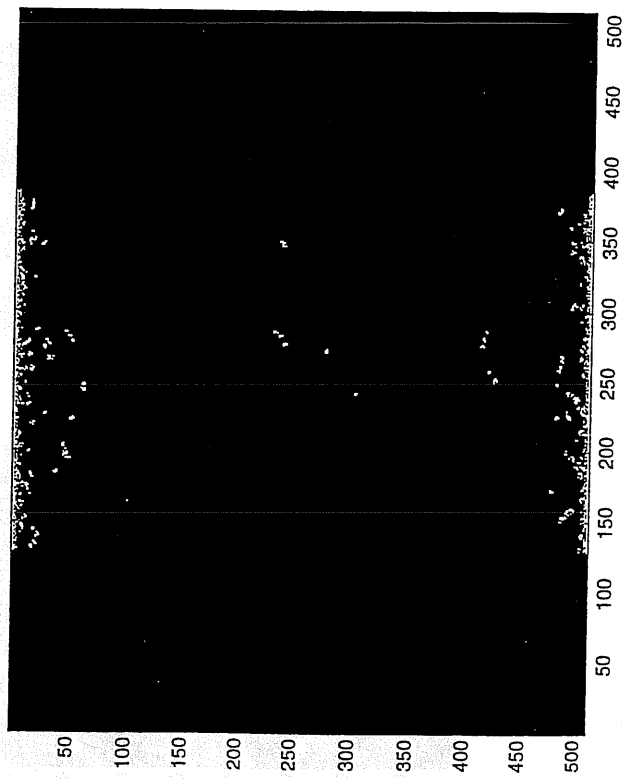


growthmovie8(000011111110000,7,[0.00,0.00,0.00],[0.00,0.50,0.25])

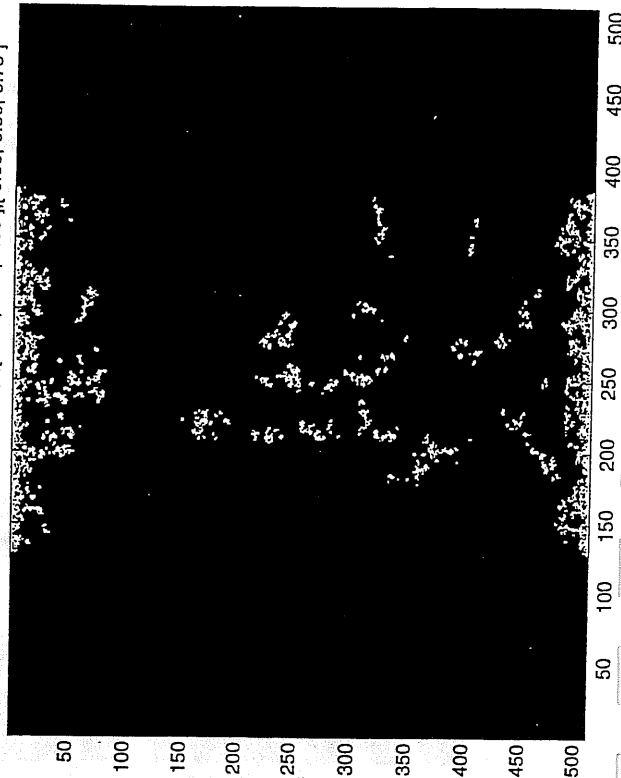


88

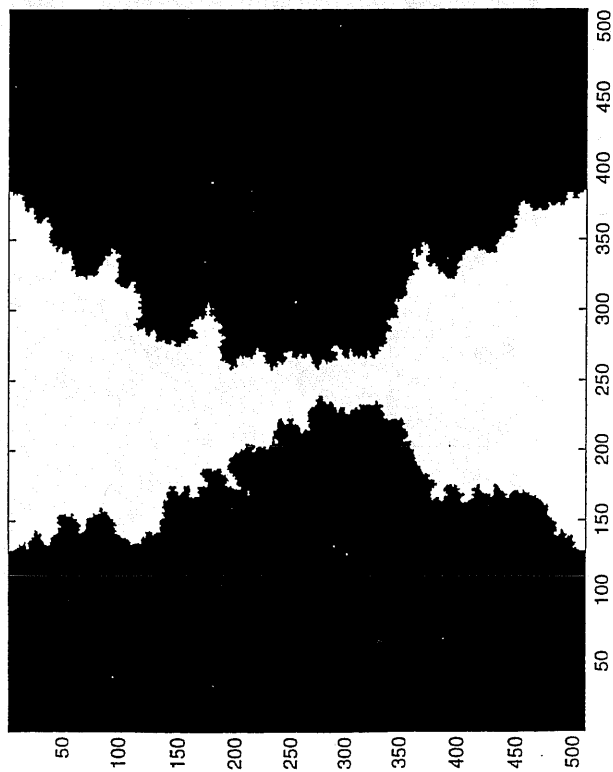
growthmovie8(A,7,[0,0,0],[0,0.5,0.5]);



growthmovie8(000011111110000,7,[0.00,0.00,0.00],[0.00,0.50,0.75])

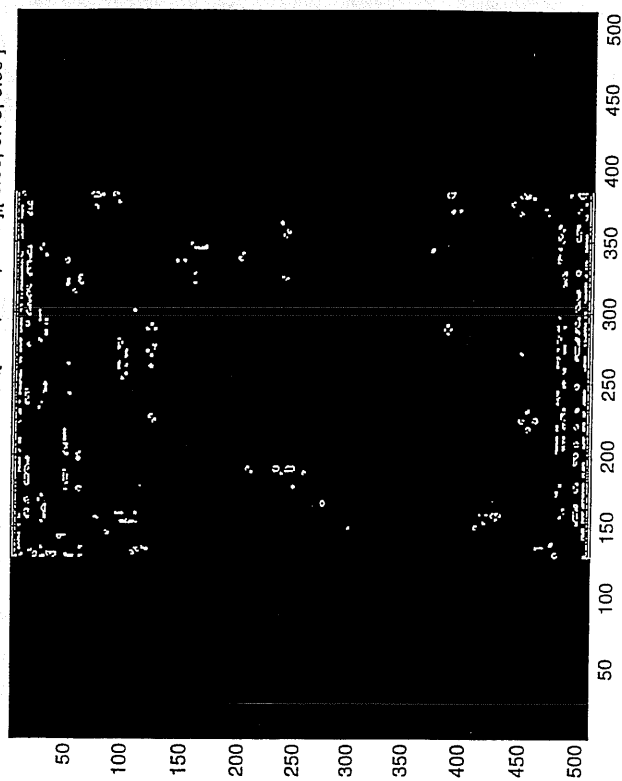


growthmovie8(A,7,[0,0,0],[0,0.5,1]);

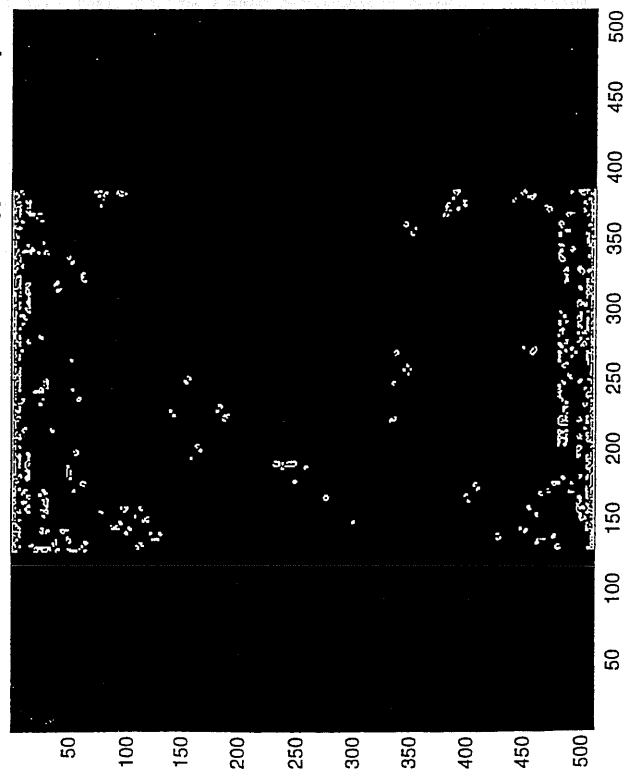


89

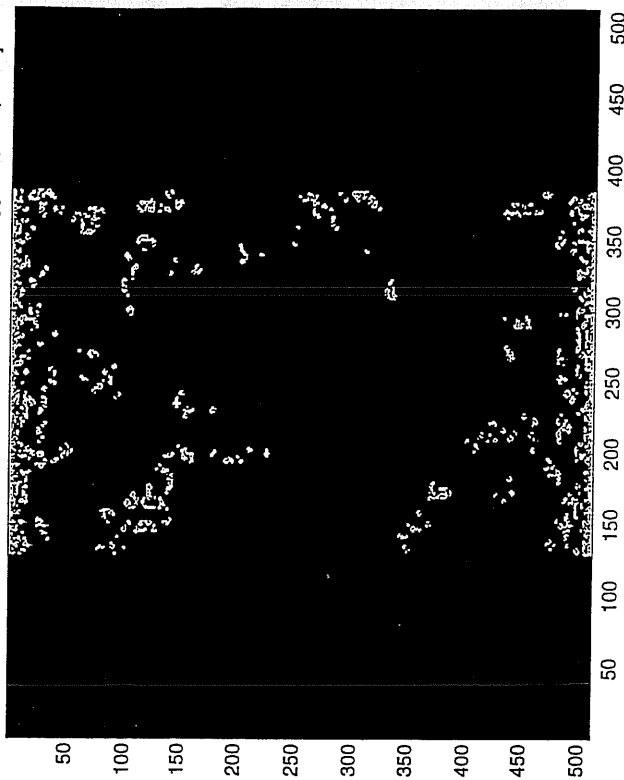
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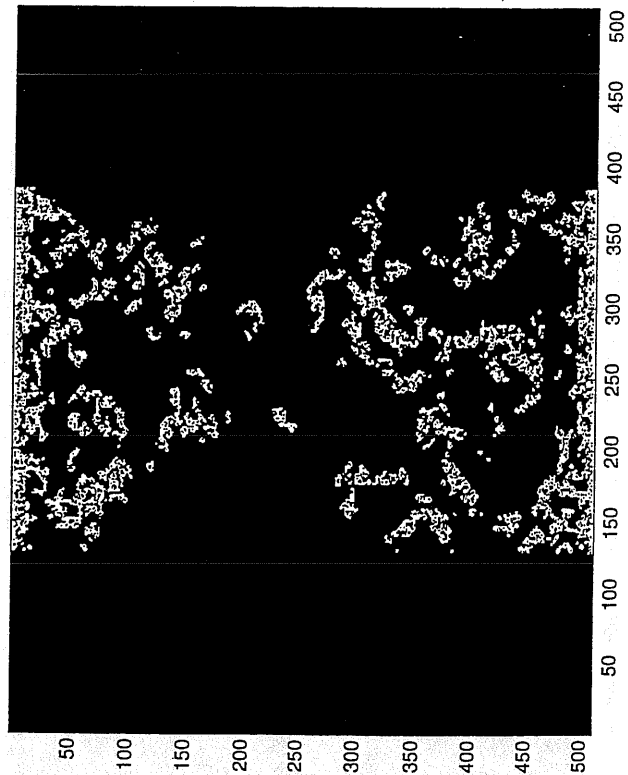
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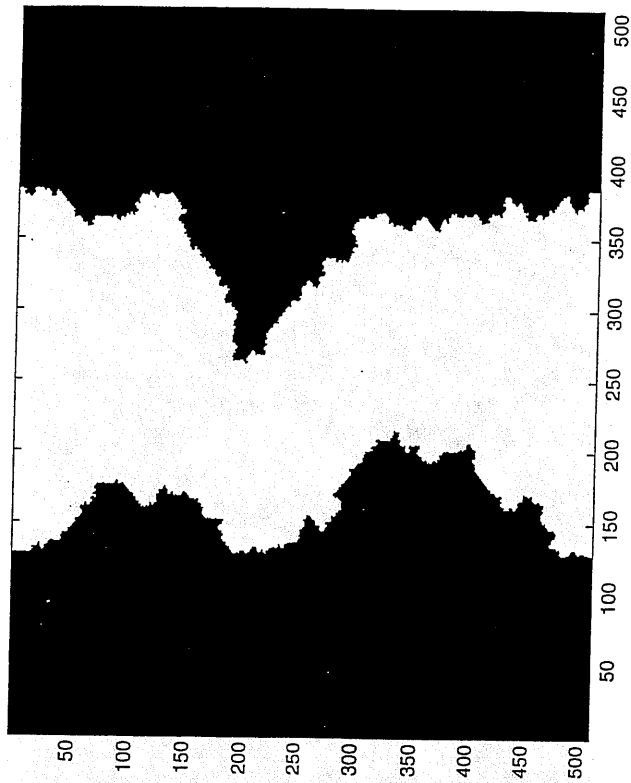
growthmovie8(0000111111110000,7,[0.00,0.00,0.00],[0.00,0.75,0.50])



```
growthmovie8(A,7,[0,0,0],[0,0.75,0.75]);
```

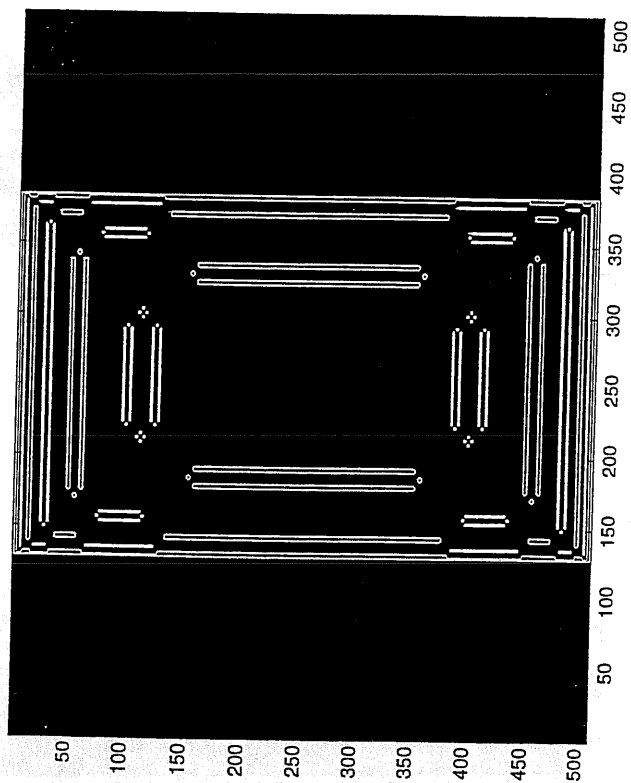


```
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```

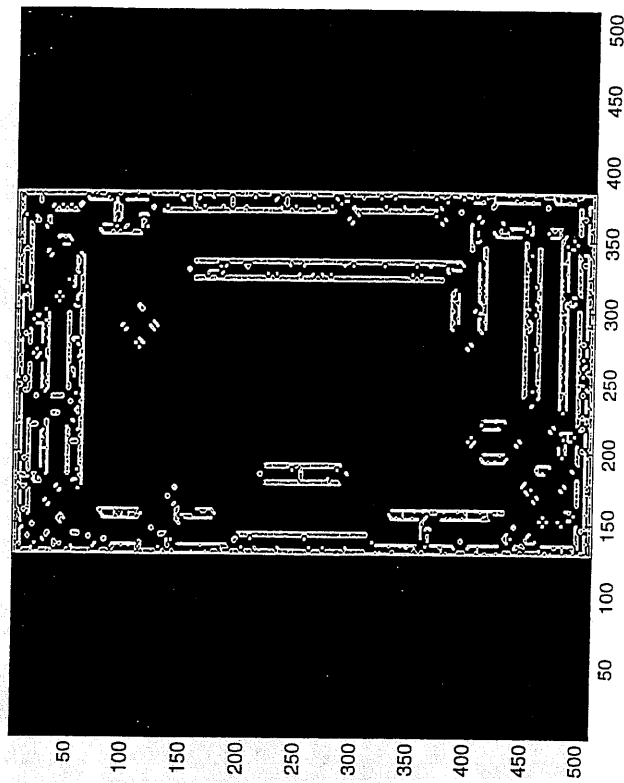


90

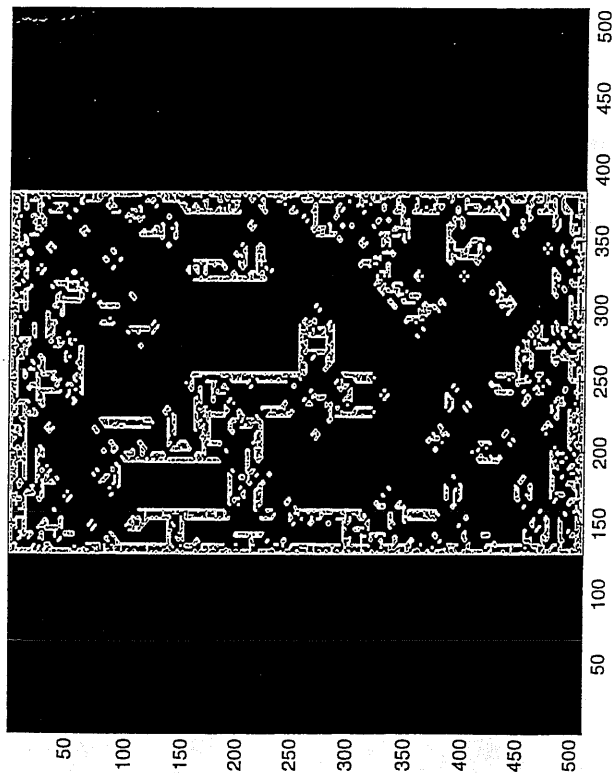
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growthmovie8(A,7,[0,0,0],[0,1,0]);
```



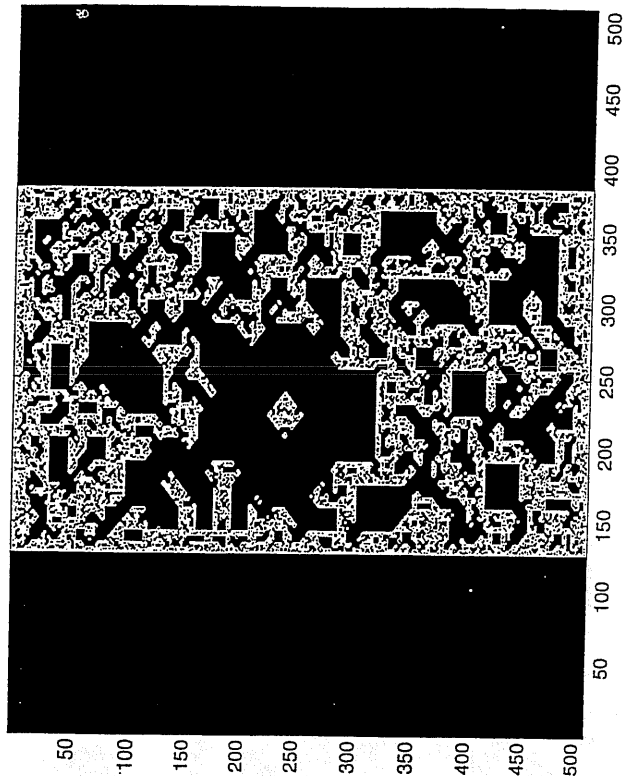
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```



```
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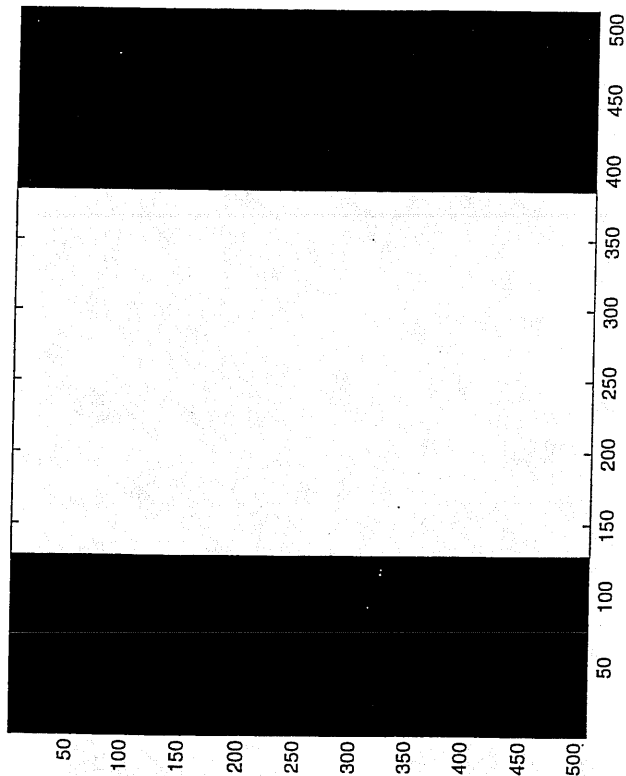


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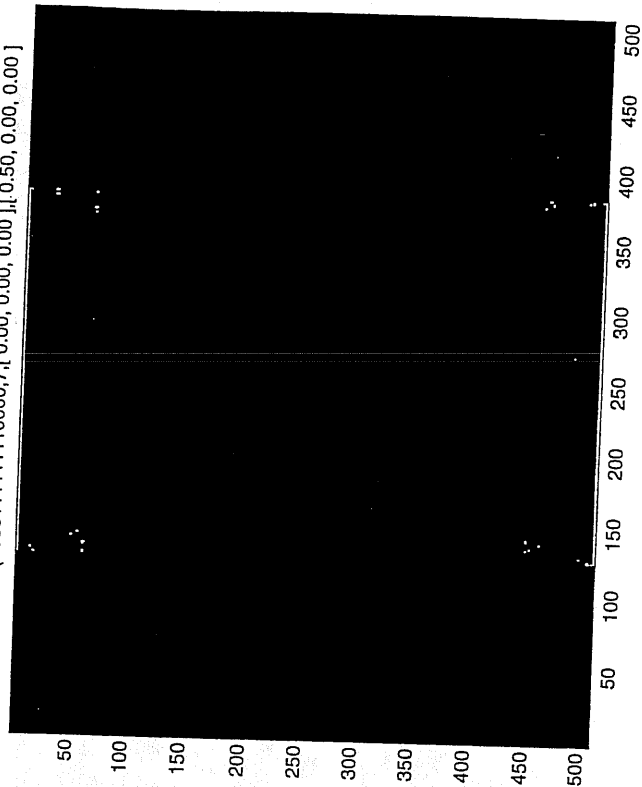


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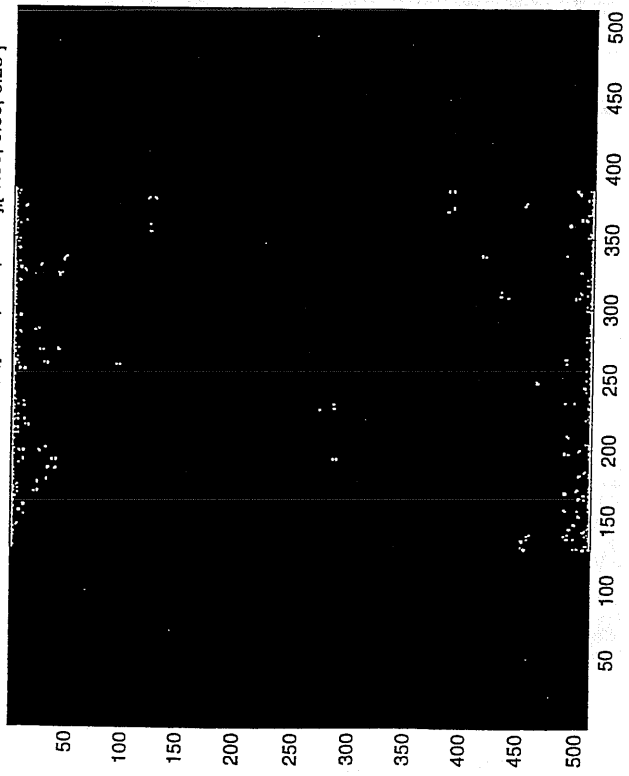
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```



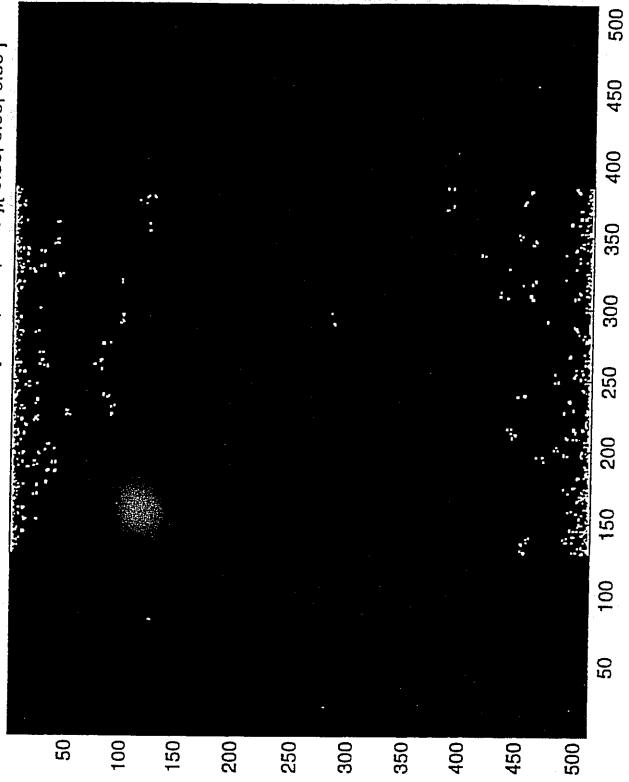
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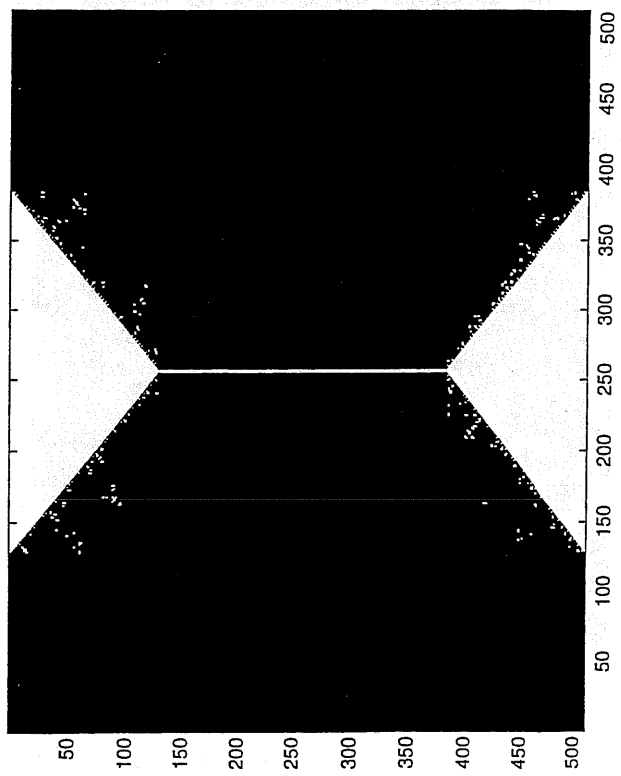
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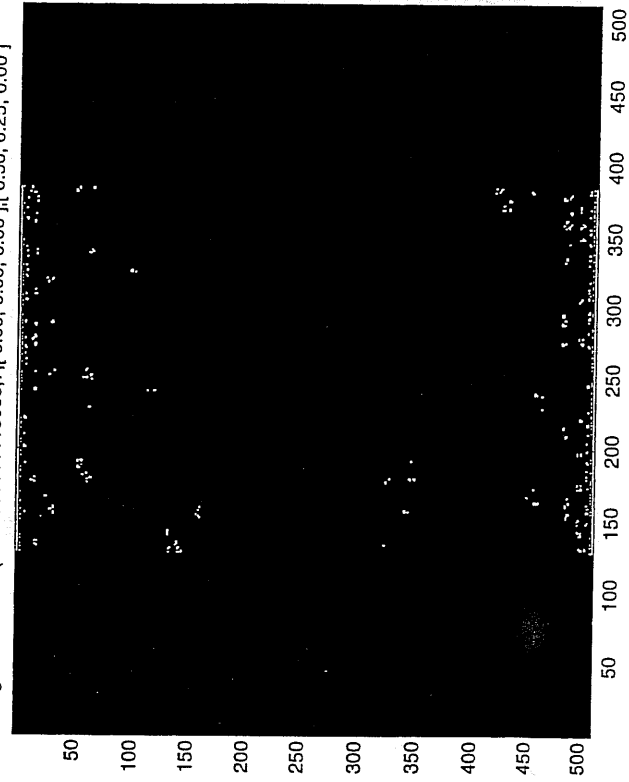
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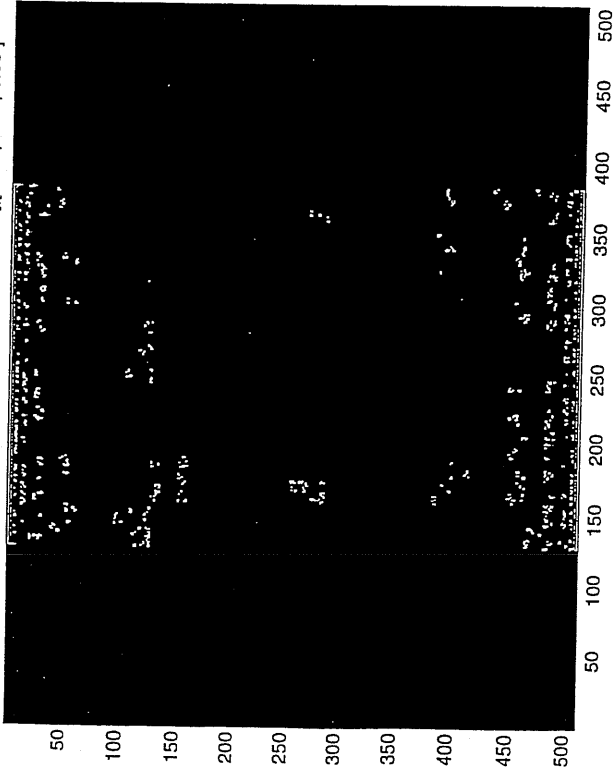
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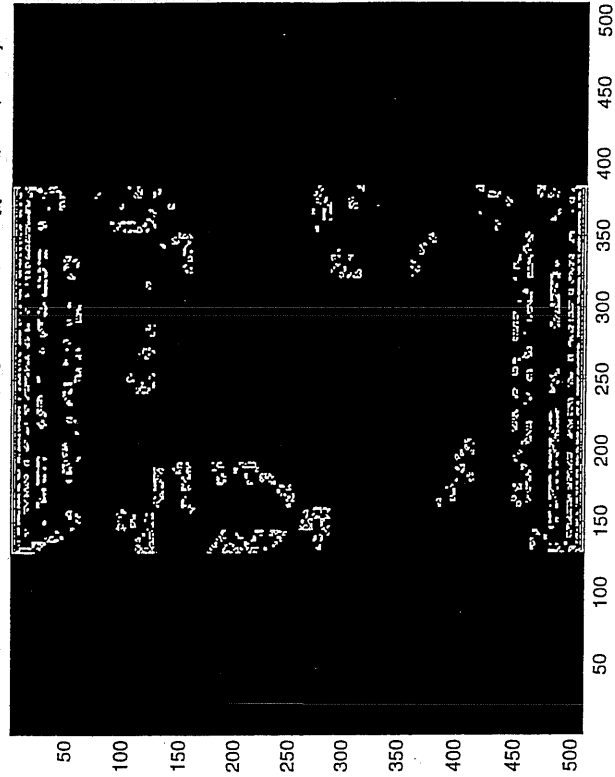
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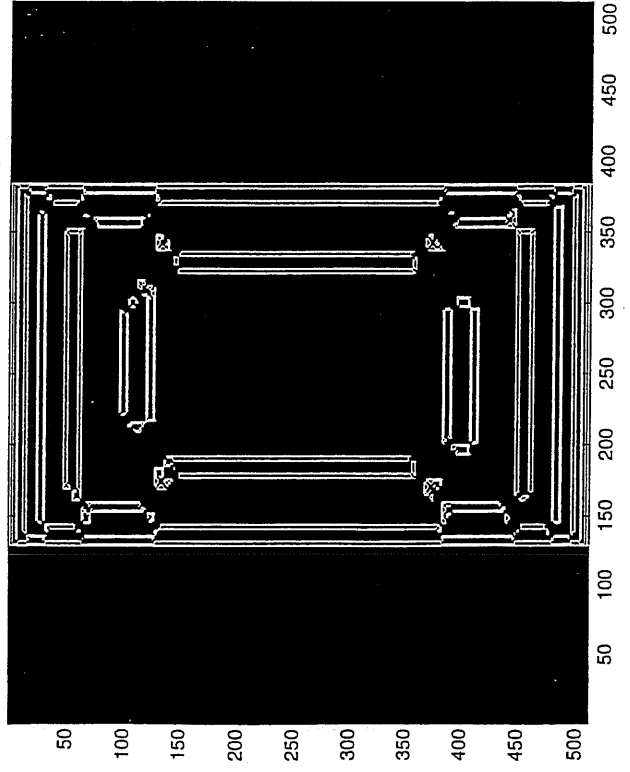
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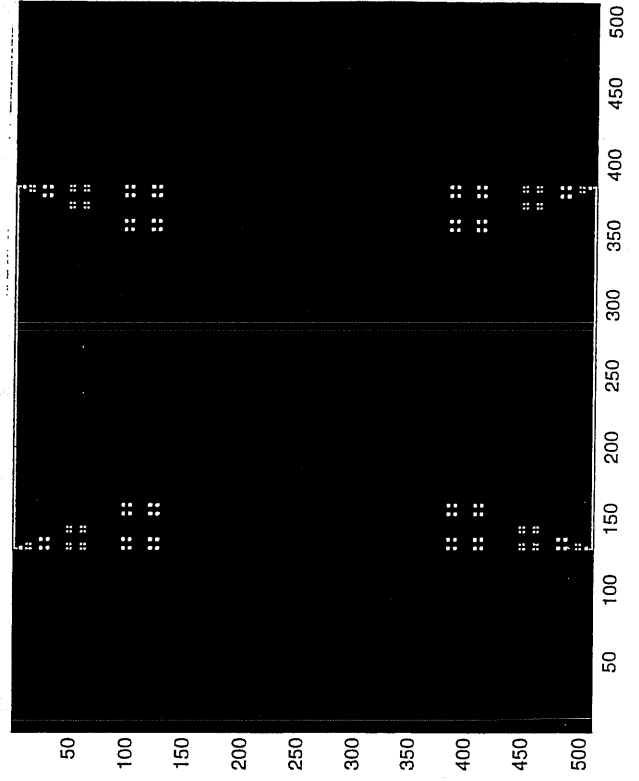
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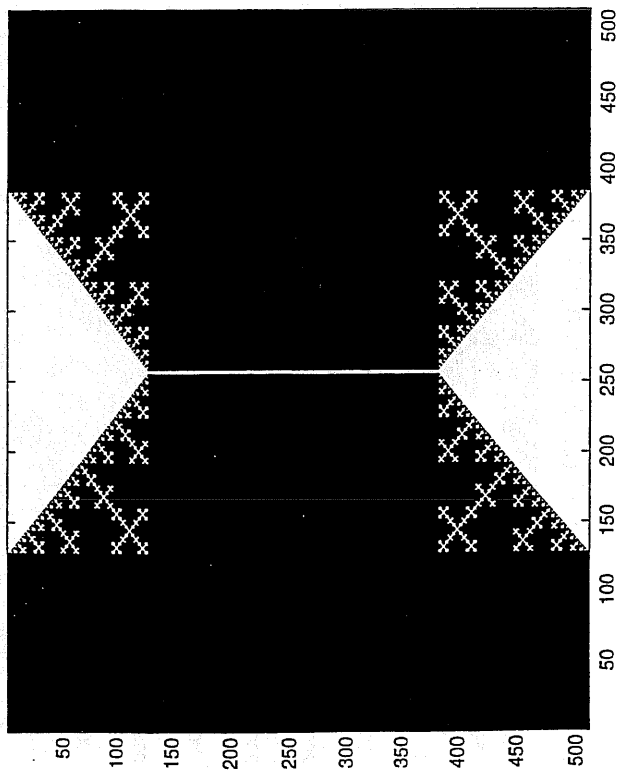
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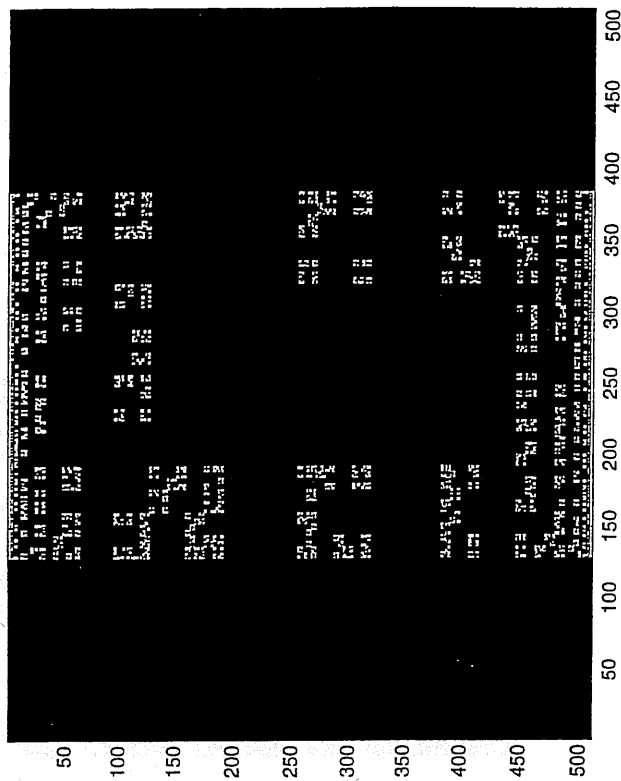
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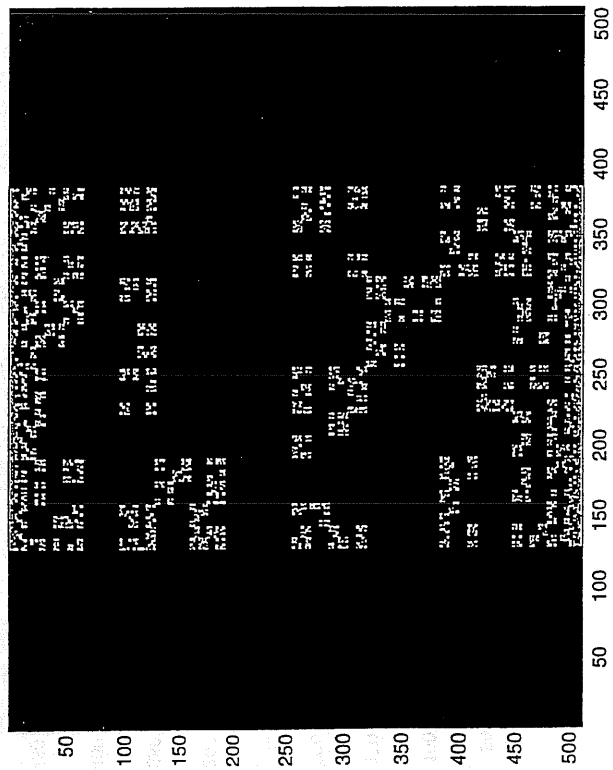
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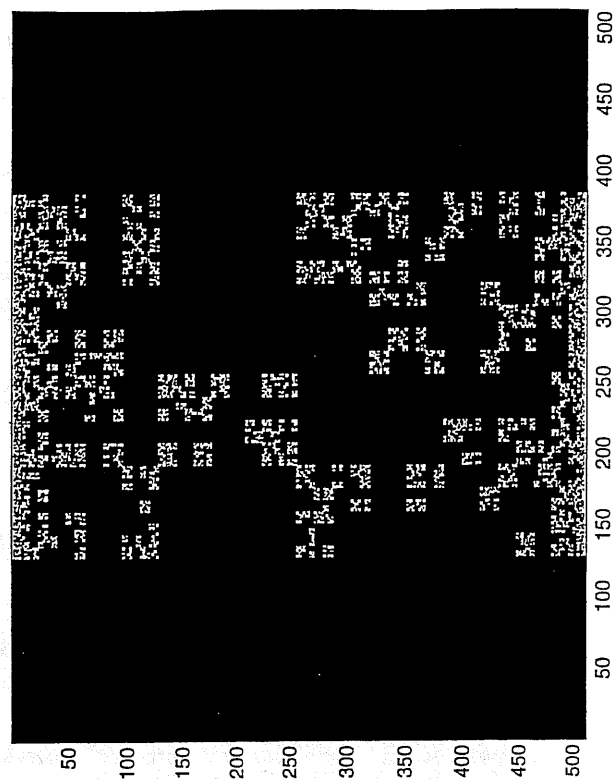
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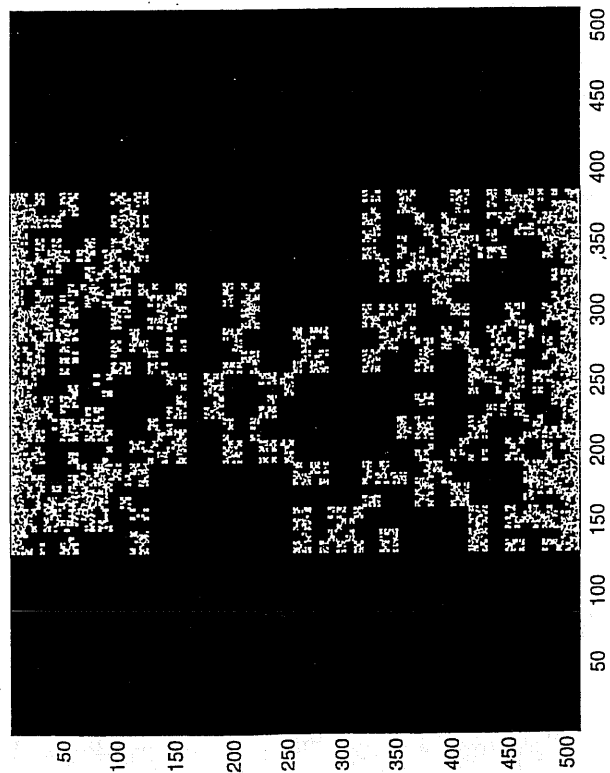
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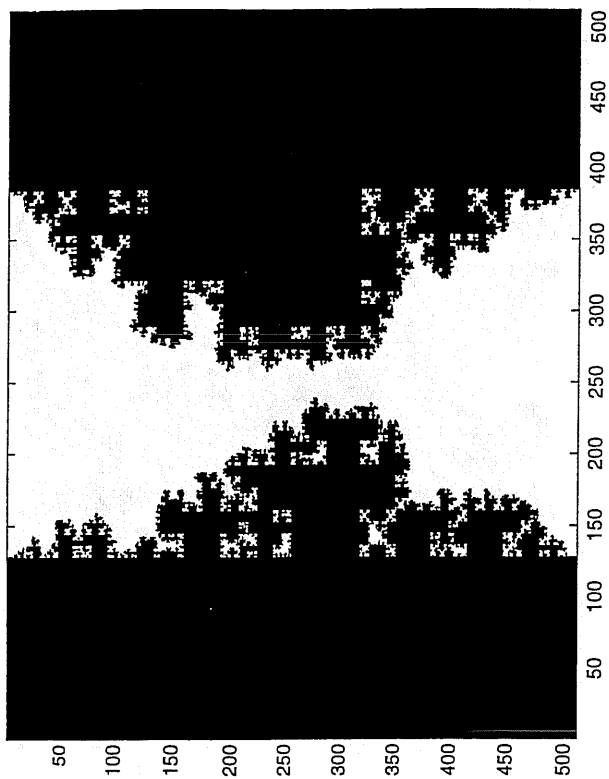



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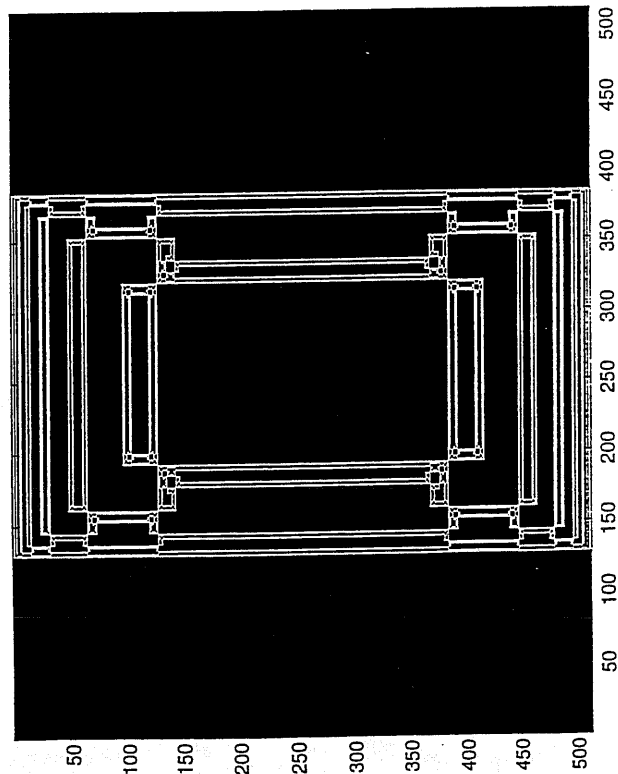


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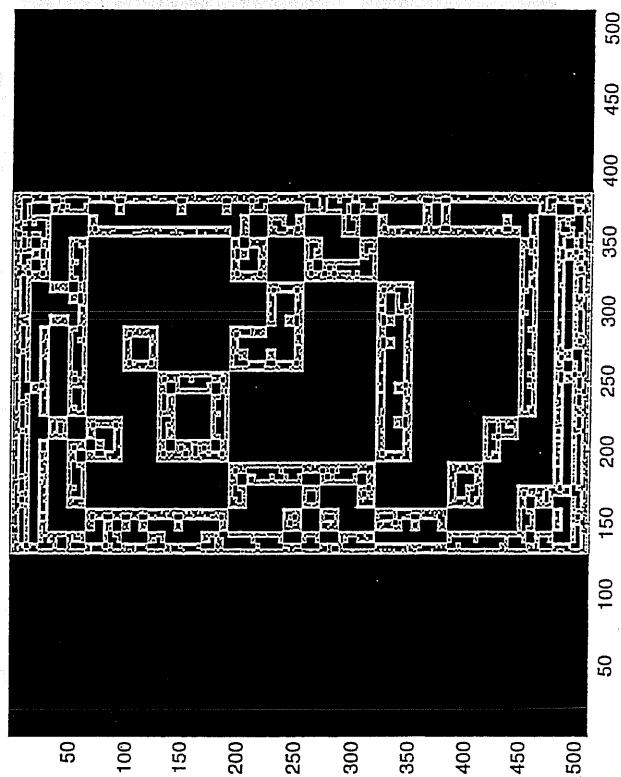
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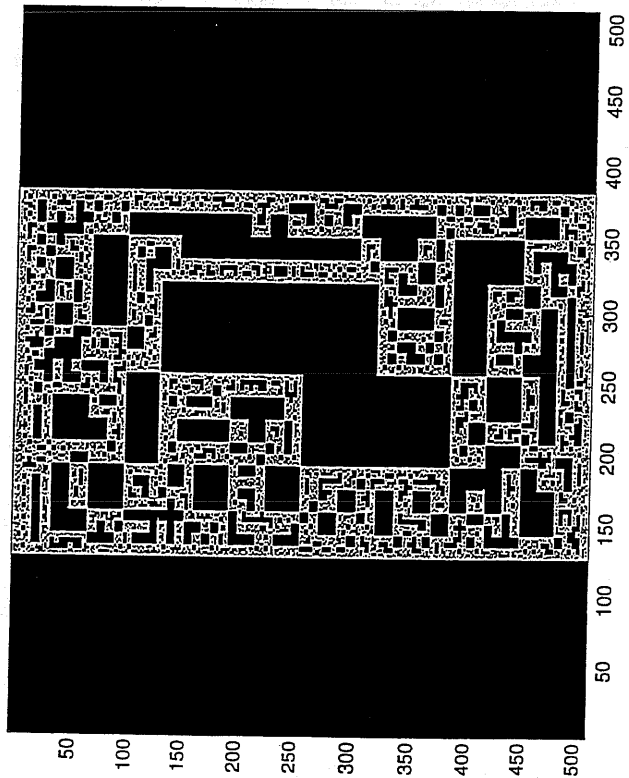
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```
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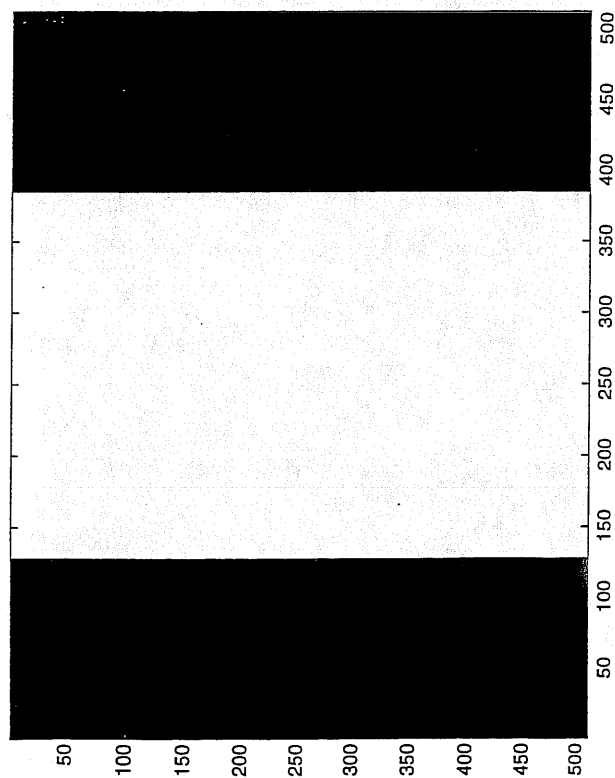


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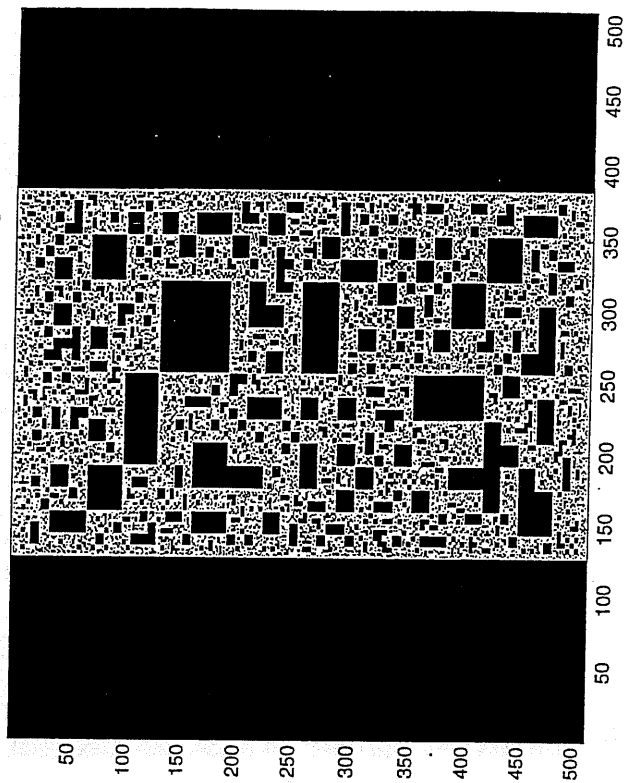


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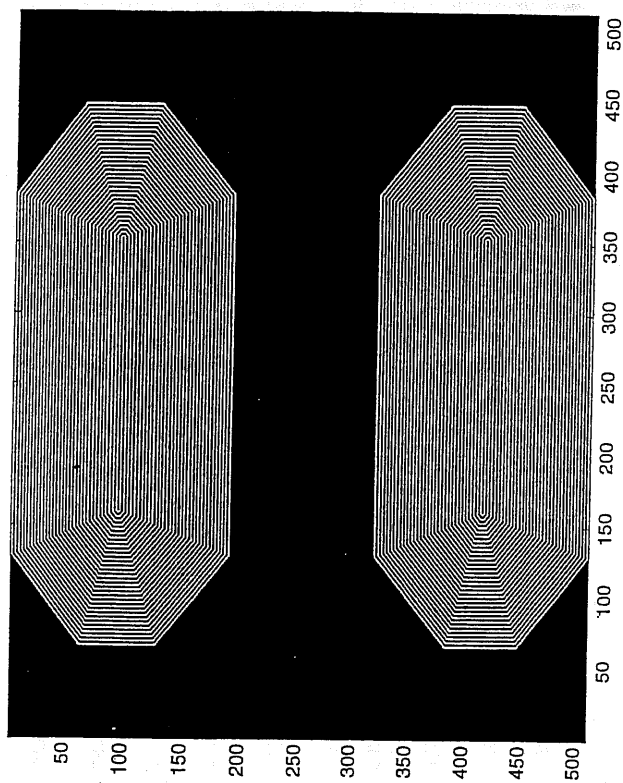
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growthmovie8(A,7,[0,0,0],[1,1,1]);
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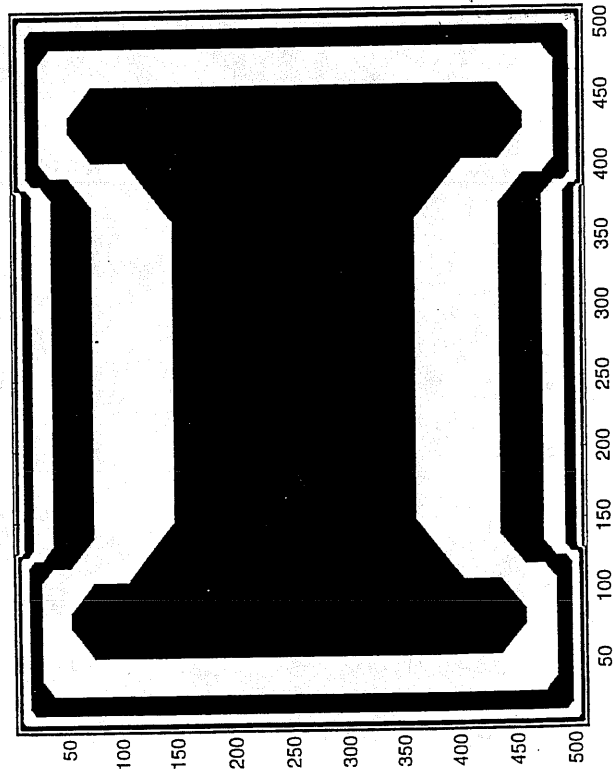
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growthmovie8(A,7,[0,0,0],[1,1,0.75]);
```



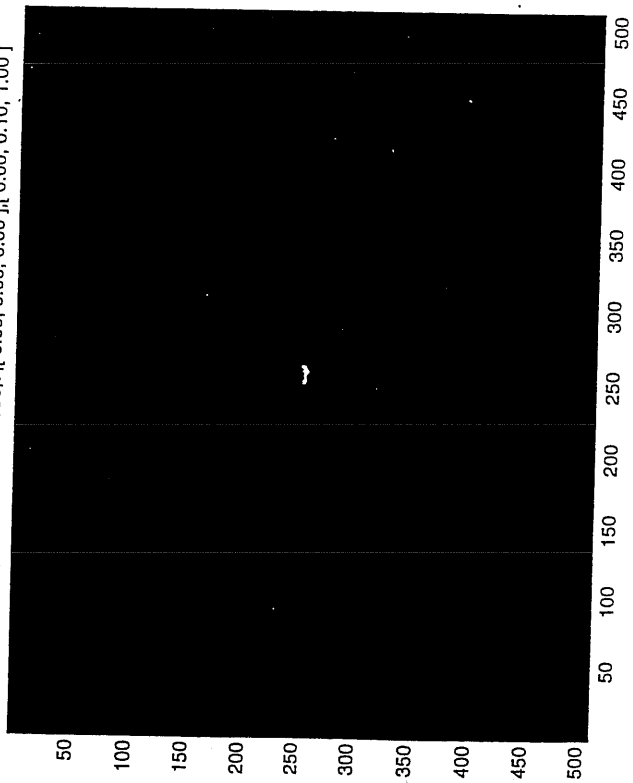
```
growthmovie8(A,7,[0,1,0],[0,0,0]);
```



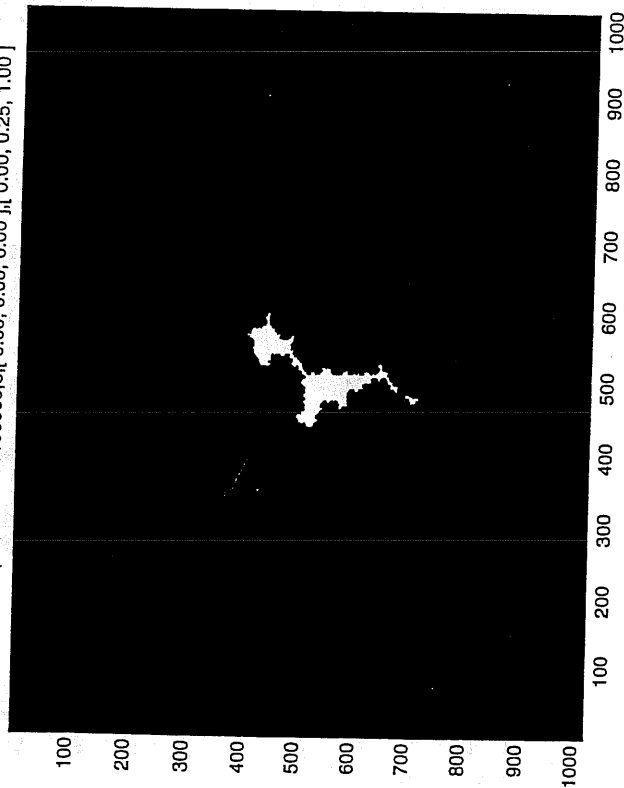
```
growthmovie8(A,7,[1,0,0],[0,0,0]);
```



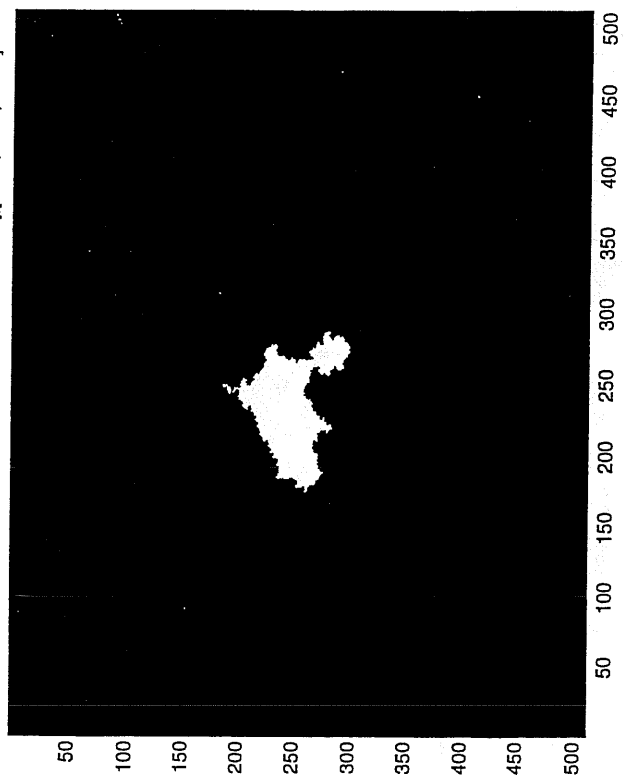
growthmovie8(00000011001100000,7,[0.00,0.00,0.00],[0.00,0.10,1.00])



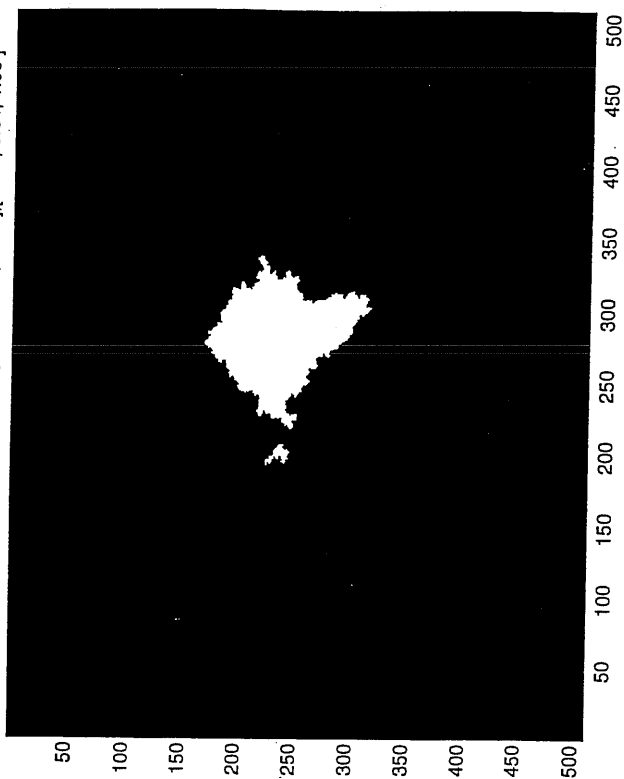
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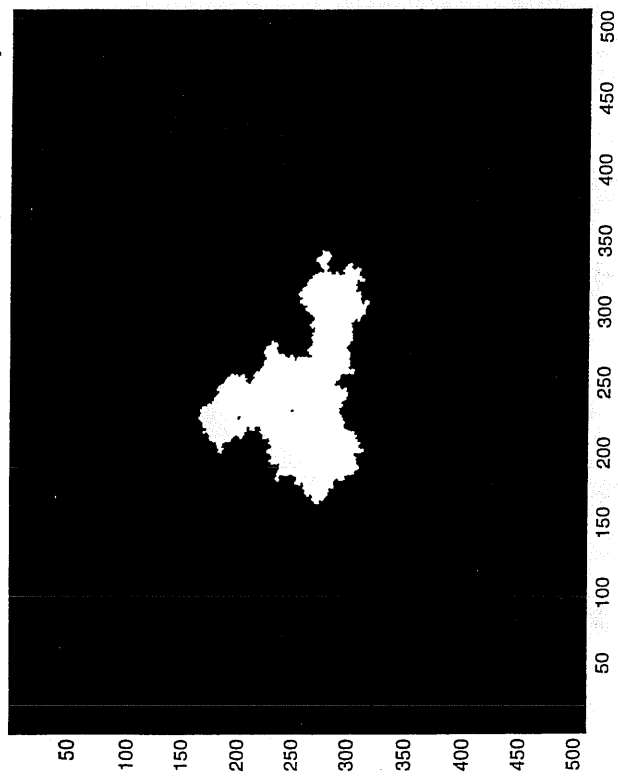
growthmovie8(00000011001100000,7,[0.00, 0.00, 0.00],[0.00, 0.60, 1.00]



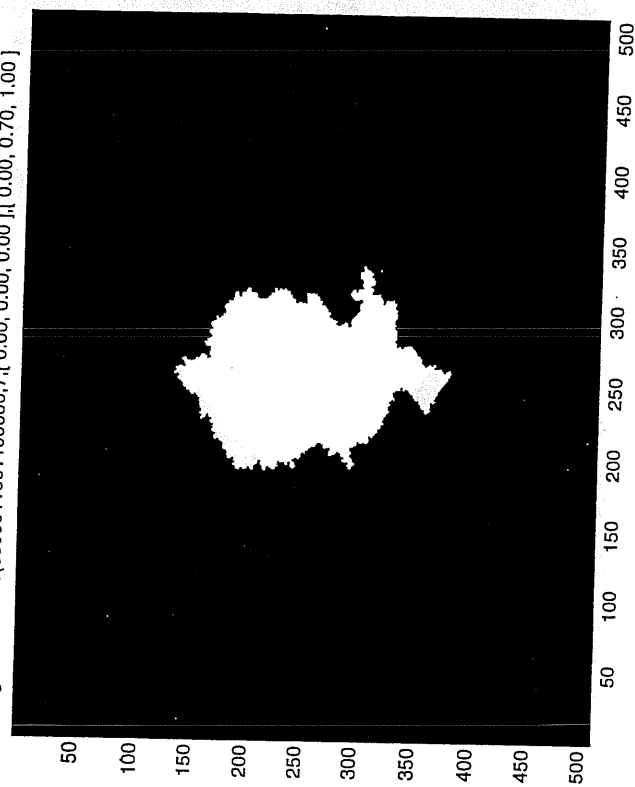
growthmovie8(00000011001100000,7,[0.00, 0.00, 0.00],[0.00, 0.64, 1.00]



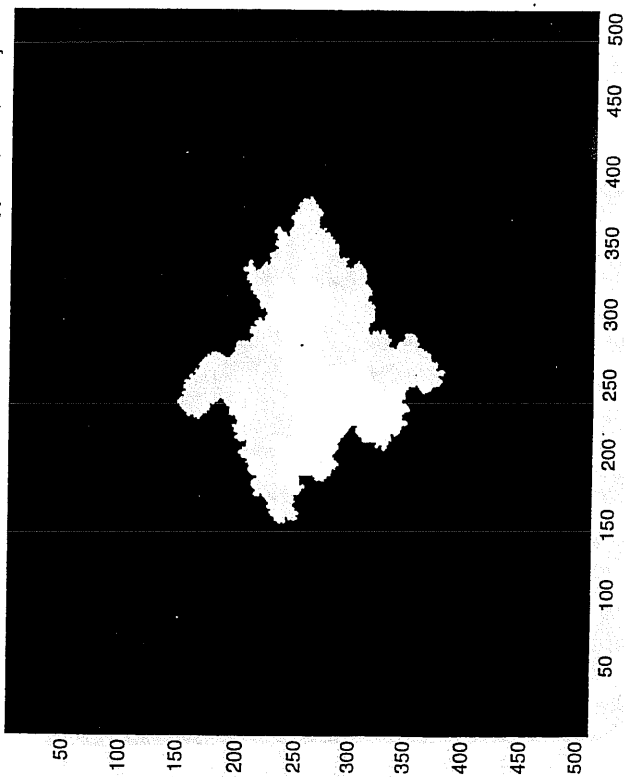
growthmovie8(00000011001100000,7,[0.00, 0.00, 0.00],[0.00, 0.65, 1.00]



growthmovie8(00000011001100000,7,[0.00, 0.00, 0.00],[0.00, 0.70, 1.00]

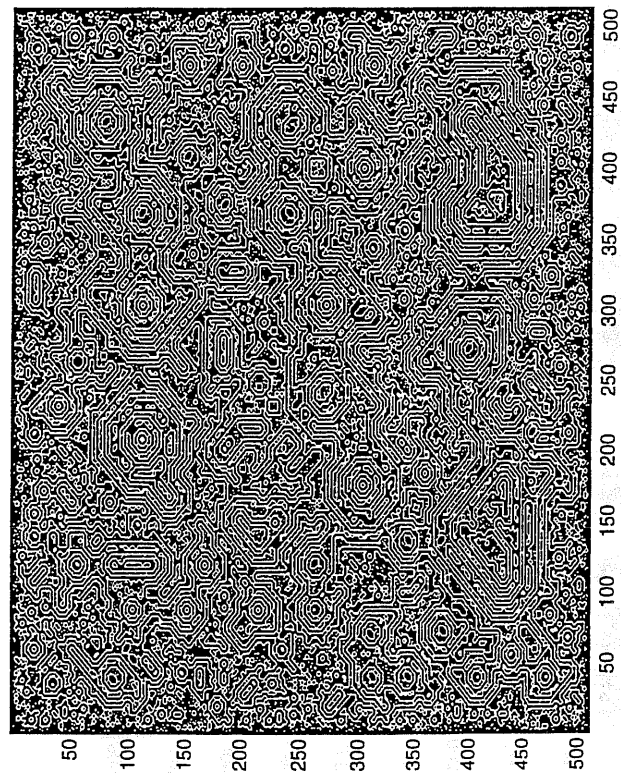


growthmovie8(0000011001100000,7,[0.00, 0.00, 0.00],[0.00, 0.75, 1.00]

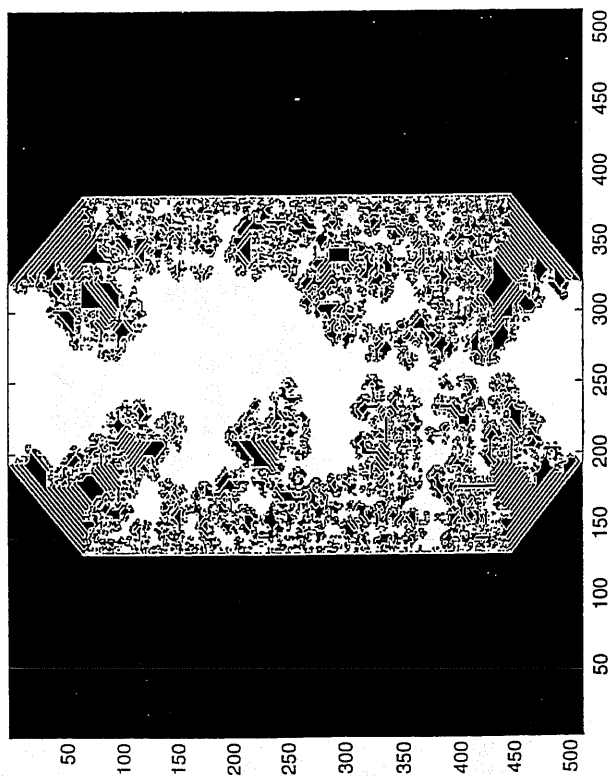


100

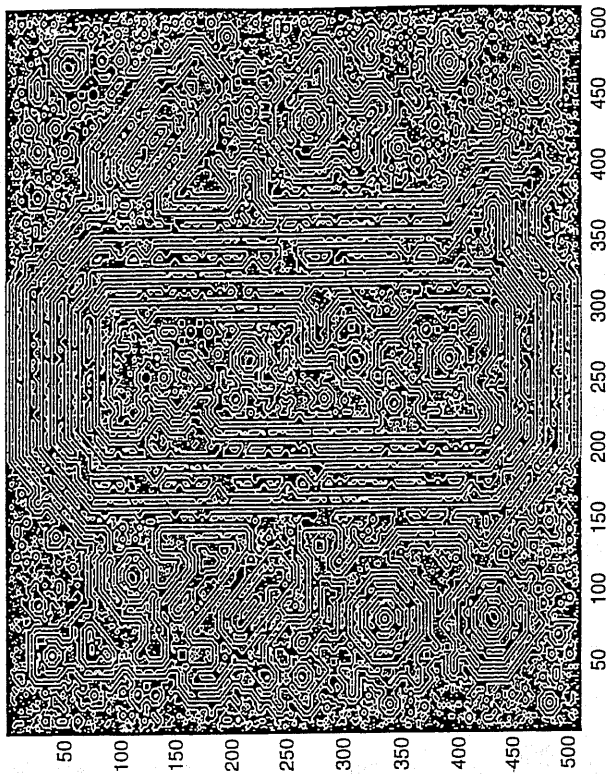
growthmovie8(F,7,[0.1,1,0],[0,0,0]);



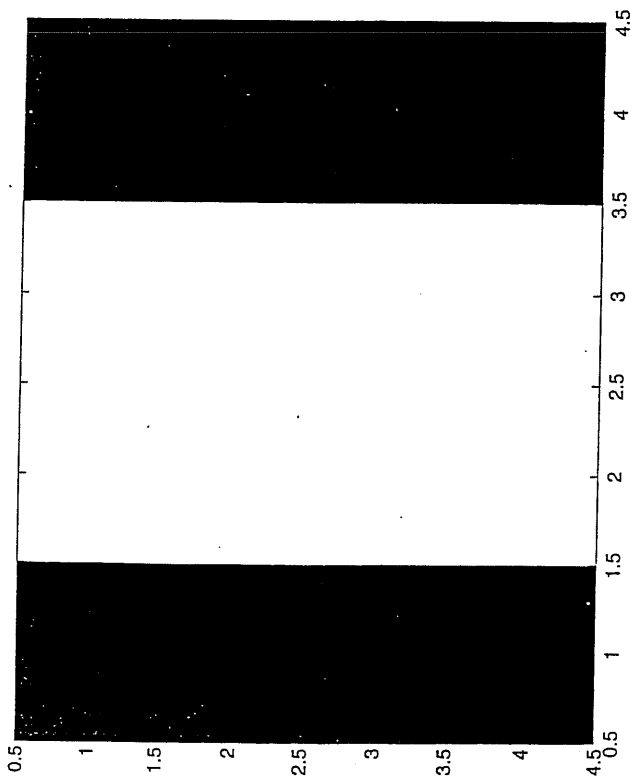
```
growthmovie8(G,6,[0,1,0],[0,0.5,1]);
```



```
growthmovie8(G,6,[0.1,1,0],[0,0,0]);
```

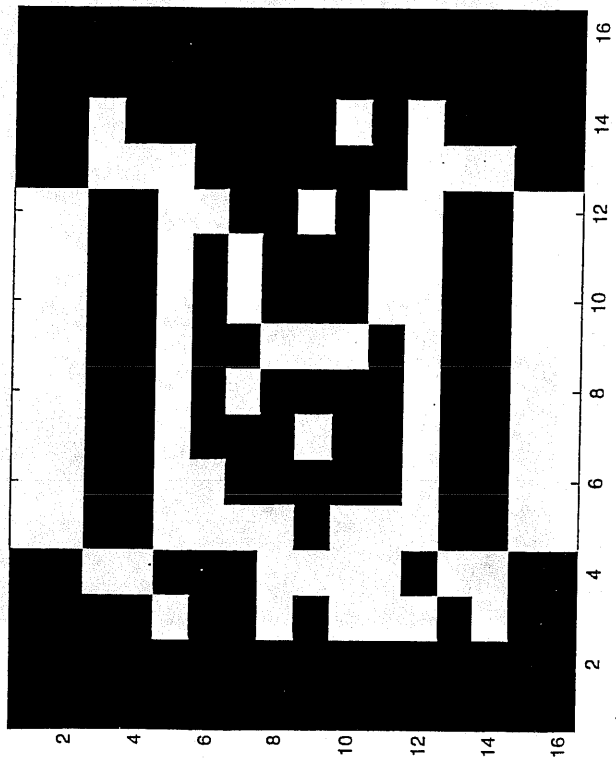


```
growthmovie8(A,7,[0.5,1,0],[0,0,0]); 1
```

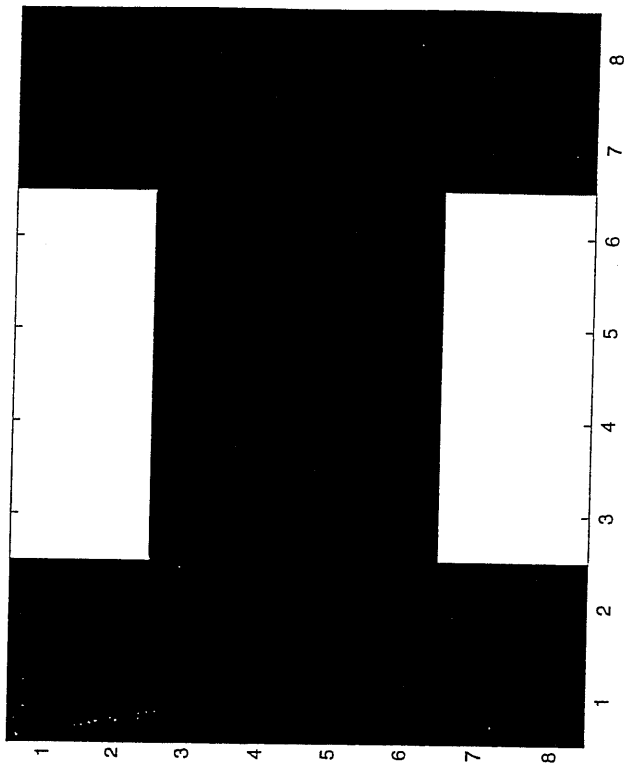


102

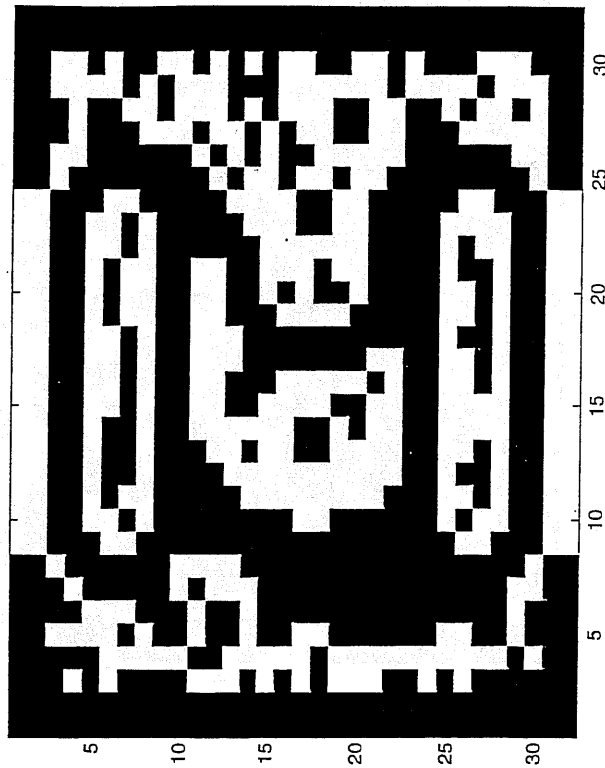
```
growthmovie8(A,7,[0.5,1,0],[0,0,0]); 3
```



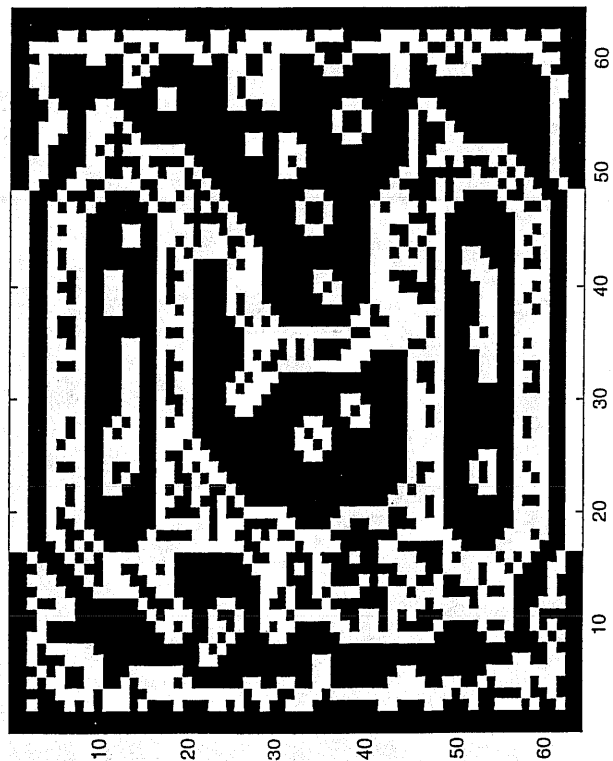
```
growthmovie8(A,7,[0.5,1,0],[0,0,0]); 2
```



```
growthmovie8(A,7,[0.5,1,0],[0,0,0]); 4
```

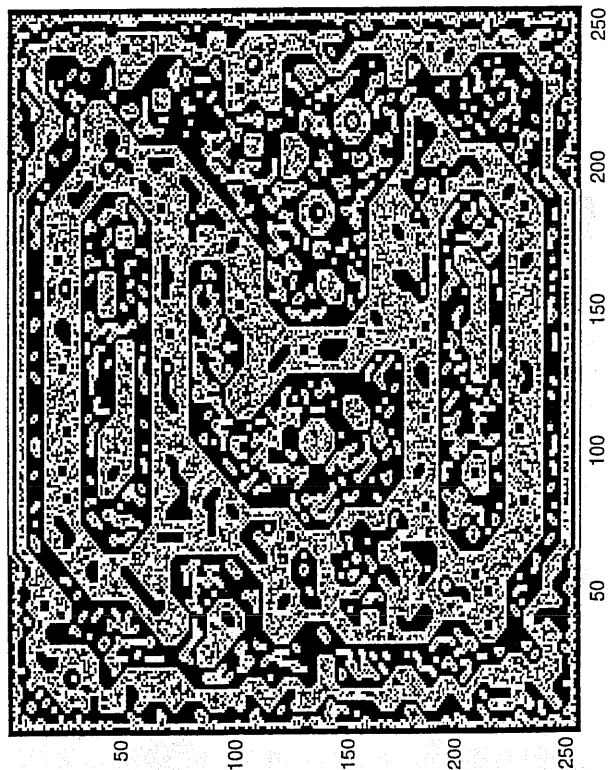


growthmovie8(A,7,[0.5,1,0],[0,0,0]); 5

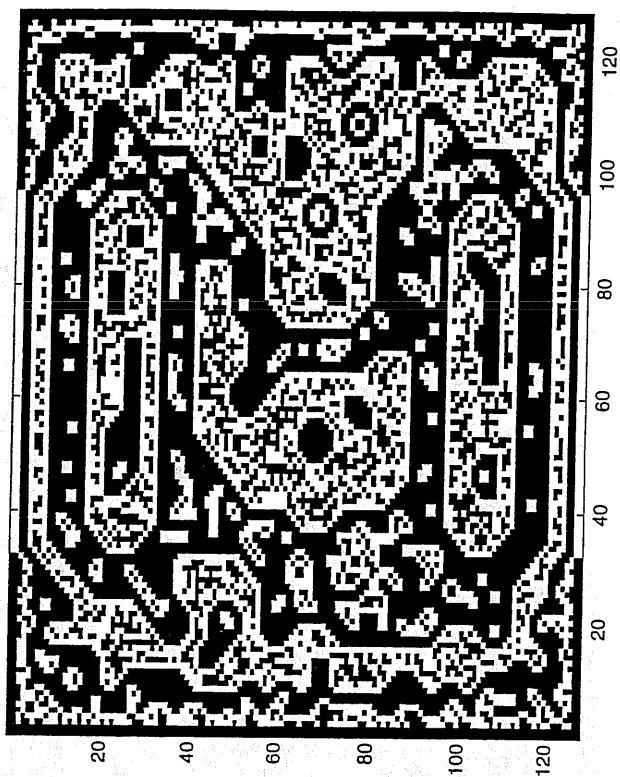


103

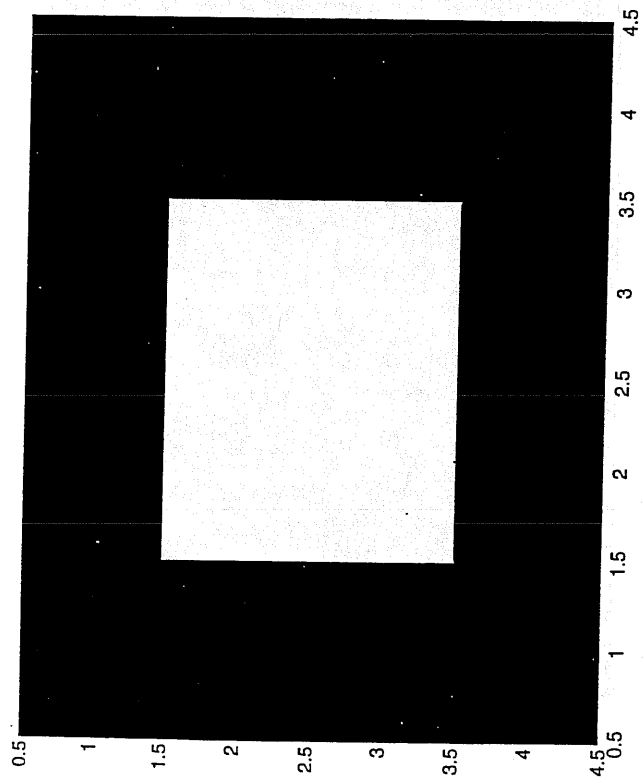
growthmovie8(A,7,[0.5,1,0],[0,0,0]); 7



growthmovie8(A,7,[0.5,1,0],[0,0,0]); 6

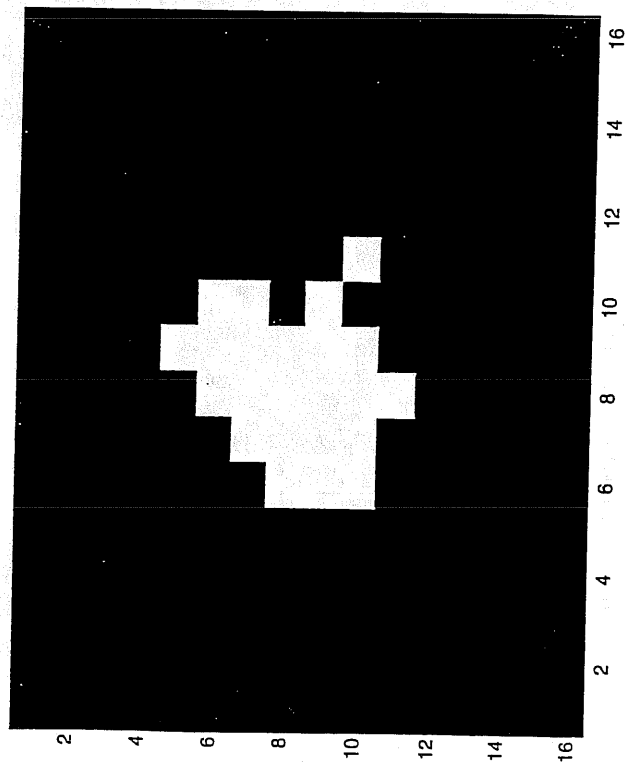


```
growthmovie8(F,7,[0,0,0],[0,0.45,1]); 1
```

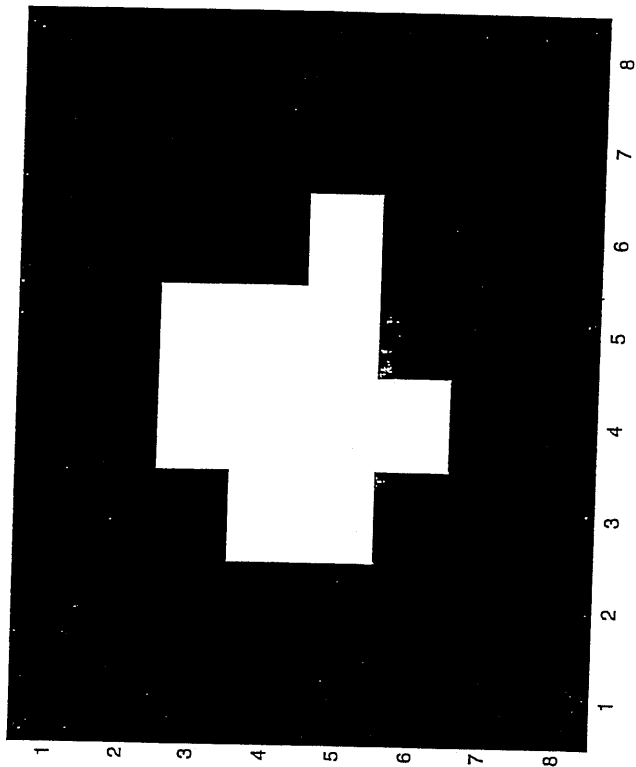


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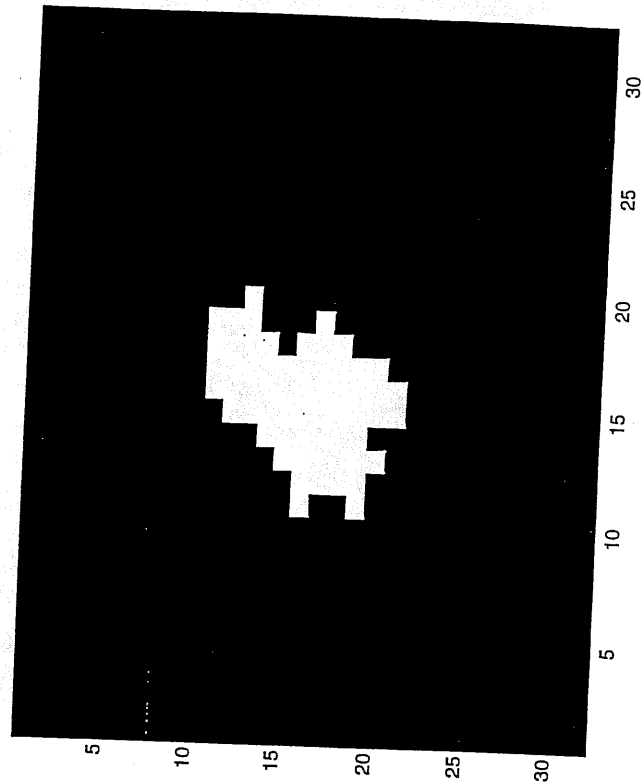
```
growthmovie8(F,7,[0,0,0],[0,0.45,1]); 3
```



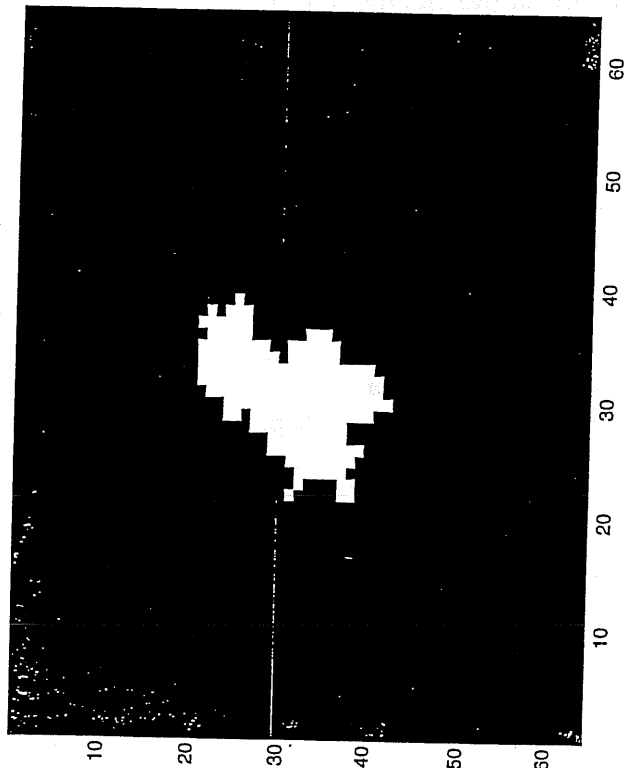
```
growthmovie8(F,7,[0,0,0],[0,0.45,1]); 2
```



```
growthmovie8(F,7,[0,0,0],[0,0.45,1]); 4
```

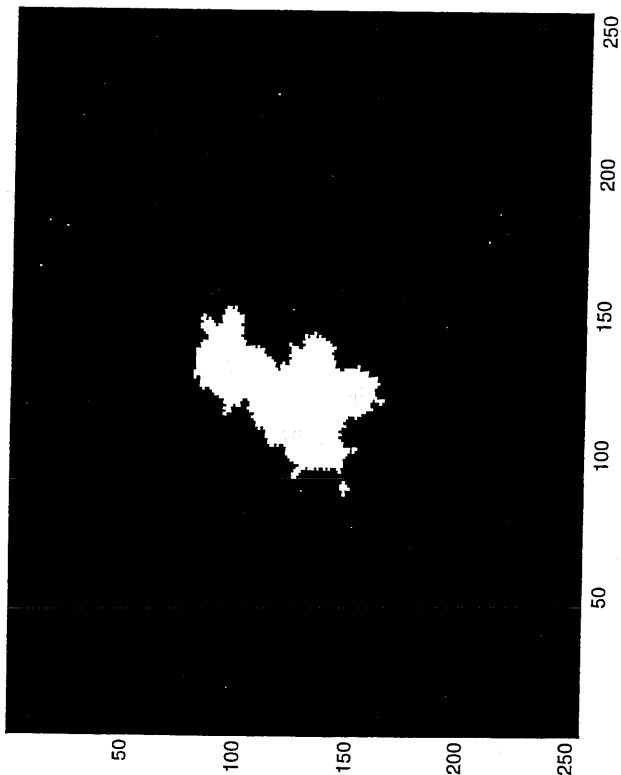


growthmovie8(F,7,[0,0,0],[0,0.45,1]); 5



105

growthmovie8(F,7,[0,0,0],[0,0.45,1]); 7



growthmovie8(F,7,[0,0,0],[0,0.45,1]); 6

