Two Independent Hamiltonian Circuits and Embedding Mesh of Trees on the Möbius Cube or: What I did on my Summer Vacation.

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Abstract

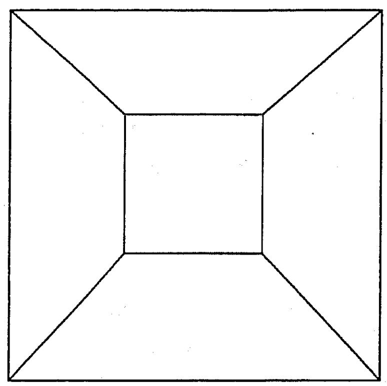
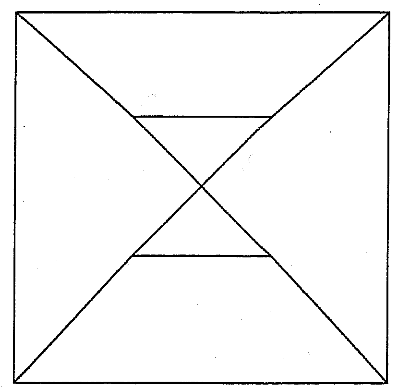
The Möbius cube is a twisted variant of the Hypercube with half the diameter and two thirds the expected distance. This paper shows the existence of two disjoint Hamiltonian circuits for dimensions 4 and how to construct them through dimension n. It also gives suggestions on how to embed mesh of trees into the Möbius cubes.

keywords-Möbius Cube, Hamiltonian circuits, mesh of trees, Hypercubes, interconnected networks, graph theory

1 Introduction

1.1 The Hypercube and its Variants

The Hypercube is a structure for parallel computing. Its relatively small diameter, small amount of connections between nodes and easy routing algorithm makes its topology a popular choice. The diameter of the three cube can be improved upon by changing the topology and twisting two edges. Attributed to Hillis [3] the twisted three cube reduces the diameter from 3 to 2.

The discovery of the twisted three cube lead to a desire to abstract the cube to higher dimensions. Different designs of this sort include the Twisted Cube, the crossed cubes , the multiply twisted cube, the bent cube, the mcube, and the Möbius cube. The Variant Cubes' topologies, like that of the Hypercube's, consists of 2^n nodes with n connections per node. The variant cubes differ with the Hypercube in diameter and expected distance. The Möbius Cube In particular, has approximately half the diameter and two thirds the expected distance of the Hypercube.[1]

1.2 Embeddings of Interconnected Networks

There is also great interest in seeing what interconnected networks can be embedded into the Hypercube and its variants. Among these are the Binary Tree, the Mesh of Trees, and Hamiltonian Circuits.

1.3 Contents

In section 2 will present The definition of the Möbius Cube and define other relevant terms. Section 3 will discus previous works. The discovery and construction of path independent Hamiltonian circuits will be explained in section 4. The proof of these circuits will be presented in section 5. Section 6 will show some examples of embedded mesh Trees and suggest posable Ideas for abstracting them.

2 Definitions

Diameter: let $P = d(x, y)$ where d is the minimum number of edges it takes to get from a node x to a node y in an interconnected graph G. The diameter of a connected graph G is defend to be the $\min(P)$.

2.1 Möbius Cubes

A Möbius Cube is made up of 2^n nodes with n connections at each node. A node X is represented by a unique binary vector of length n $(X_1, X_2, \ldots, X_n)$. The ith neighbor of X is a node Y_i that is connected to X by the following equations where $\bar{X}_i$ is defined to be the compliment, or opposite, of X_i:

if $X_{i-1} = 0$

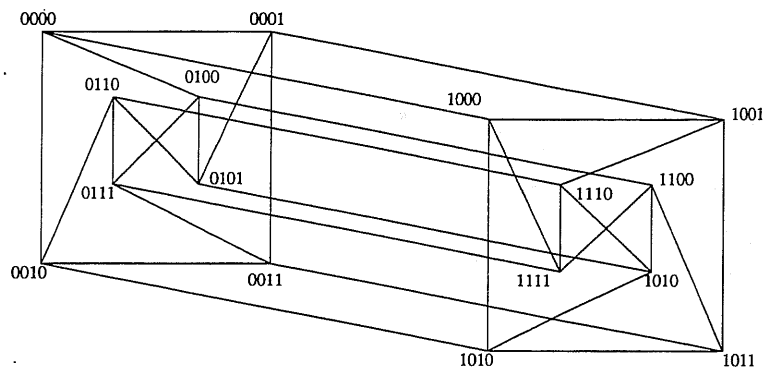
$$Y_i = (X_1, \ldots, X_{i-1}, \bar{X}_i, X_{i+1}, \ldots, X_n) \tag{1}$$

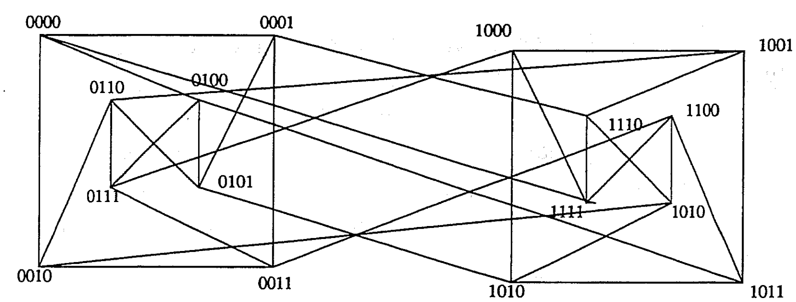
or if $X_{i-1} = 1$

$$Y_i = (X_1, \ldots, X_{i-1}, \bar{X}_i, \bar{X}_{i+1}, \ldots, \bar{X}_n) \tag{2}$$

The Neighbors are represented by $(X, X + e_i)$ where e_i is the n dimensional $(0, 1)$ vector with all 0's and a 1 in the ith component iff it is determined by (1). Likewise $(X, X + E_i)$ determines the neighbor by (2) where E_i is the n dimensional vector represented with 0 in the $1, \ldots, i-1$th components and 1 in the $i, \ldots, n$th components. There is clearly a problem with a nodes first neighbor since X_0 is undefined. If $X_0 = 0$ then it is called a 0-Möbius cube and if $X_0 = 1$ it is said to be a 1-Möbius cube.

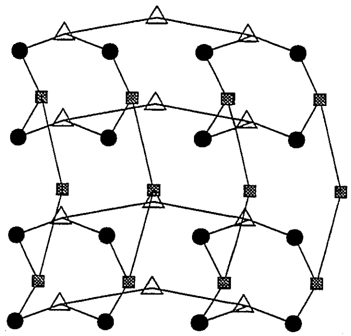
The Möbius cube of dimension $n + 1$ can be constructed from a 0 and a 1-Möbius cube of dimension n. A new address X' is created by assigning $X'_2 = X_1, X'_3 = X_2, \ldots, X'_{n+1} = X_n$ then $X'_1 = 0$ if X is in the n 0-Möbius cube or $X'_1 = 1$ if X is in the n 1-Möbius cube. The 0-cube is then formed by connecting all nodes that differ in only the 1st index and the 1 cube is formed by connecting the nodes that differ in all the indices [1]. Here are the 4d Möbius cubes:



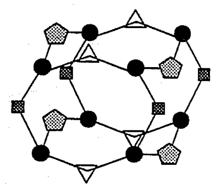


2.2 Mesh of Trees

A mesh of trees is a powerful interconnected network. The creation of an N X N mesh of trees starts with an N X N grid of nodes. Processors and connections are then added to create a binary tree in each row and column. The N X N network has $3N^2 - 2N$ processors and has a diameter of $4 \log N$. The N X N X N cube is created in a similar manor.[2] See examples of the 4 X 4, and the 2 X 2 X 2 below.



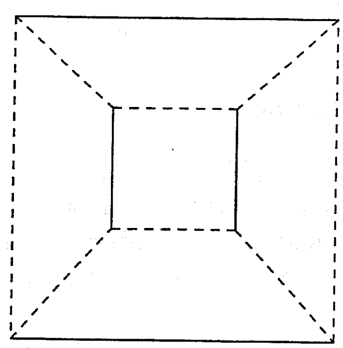
4 x 4 Mesh of Trees



2 x 2 x 2 Mesh of Trees

2.3 Hamiltonian Circuit

A Hamiltonian path is a path that hits each node of a graph once and only once. Likewise a Hamiltonian circuit is a circuit that runs through a graph such that it hits each node once and only once and returns to the starting place. In other words it is a path that goes from any node back to itself with out visiting anyplace twice. For example dotted here is a Hamiltonian circuit on the three dimensional Hypercube.



3-cube with Hamiltonian circuit

A path independent Hamiltonian Circuit is an additional Hamiltonian Circuit in the same interconnected network that doesn't share an edge with any previous Hamiltonian circuits. For the use of this paper the circuits will be referred to as H_1 and H_2.

3 Previous Work

The properties of interconnected networks have been of great interest in the computing research community. Many scholars have researched these properties. Rowely's PhD. thesis covers Embeddings of Mesh trees and edge disjoint Hamiltonian cycles in De Bruijn graphs. [4] Schwabe demonstrated an efficient embedding of the mesh of trees into De Bruijn graph, allowing simulation of arbitrary mesh of trees computations. [5] Kulasinghe and Bettayeb presented a dilation 1 embedding of binary trees into the crossed cube in the IEEE Journal. [6] Liu, Hildebrandt and Cavin, showed that it is possible to form Hamiltonian cycles on all processors in a shuffle exchange connected array. [7] M.Y. Chan and S. Lee proved the problem of detecting an n-cube with arbitrary link faults has a Hamiltonian circuit is NP Complete. [8] Paul Cull and Shawn Larson show how to embed Binary trees and give an algorithm to find a Hamilton circuit for the Möbius cube in their initial paper on the networks. [1]

4 Two Path Independent Hamiltonian Circuits In Möbius Cubes of Dimension $d \geq 4$

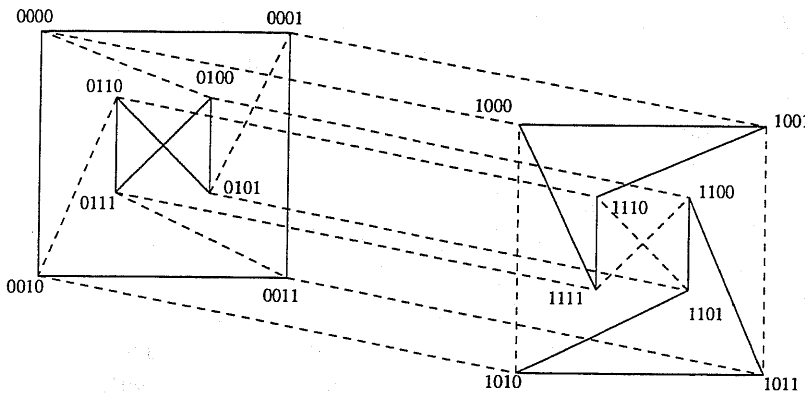
4.1 Algorithm from [1]

In [1] an algorithm is given to find a Hamiltonian circuit in any dimension of a Möbius cube. these are shown in the following two theorems.

Theorem 1 *There is a Hamiltonian Path for the n-dimensional 0-Möbius cube from node* $(111\ldots111)$ *to* $(111\ldots110)$*given by* $H(1) = e_1$ *and recursively by* $H(n) = H(n-1)e_nH(n-1)^R$, *where R means in the reverse order. So a Hamiltonian circuit exists because* $(111\ldots111)$ *is adjacent to* $(111\ldots110)$ *by* E_n.

Theorem 2 *There is a Hamiltonian Path for the n-dimensional 1-Möbius cube from node* $(000\ldots000)$ *to node* $(000\ldots001)$ *given by* $H(1) = E_1$ *and recursively by* $H(n) = H(n-1)E_nH(n-1)^R$. *So there exists a Hamiltonian Circuit because* $(000\ldots000)$ *is adjacent to* $(000\ldots001)$ *by* e_n.

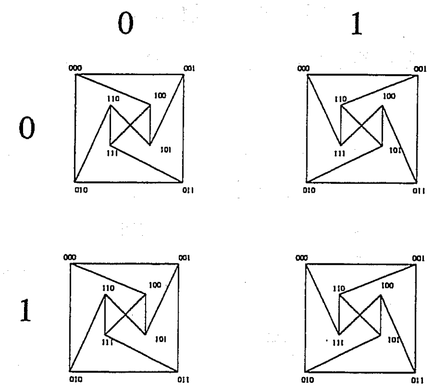
The following picture is of the 4 dimensional 0-Möbius cube. The dotted line represents the Hamilton Circuit.



If this definition is used to define H_1 it becomes impossible to find an H_2. The Formula use e_1 or E_1, relative to the 0 or 1 cube, 2^{n-1} times using all the e_1's or E_1's causing a separation. Since there is a separation of the two sub-cubes with $X_n = 0$ and $X_n = 1$. More formally the set of nodes and edges of the Orignal Möbius cube $-H_1$ is a disconnected set into at least two nonempty components A and B. Since A and B are both nonempty they contain at least one node a and b respectively. Then Since A is disjoint from B there is no path connecting a to b. Therefor there exists no Hamiltonian circuit in $A \cup B$.

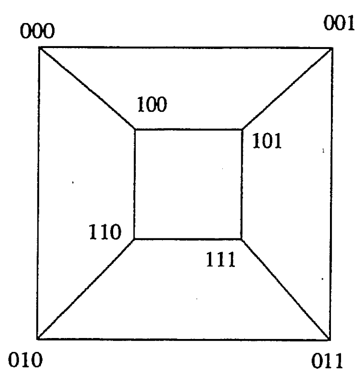
4.2 The Process Gone Through to find the Existence of H_1 and H_2.

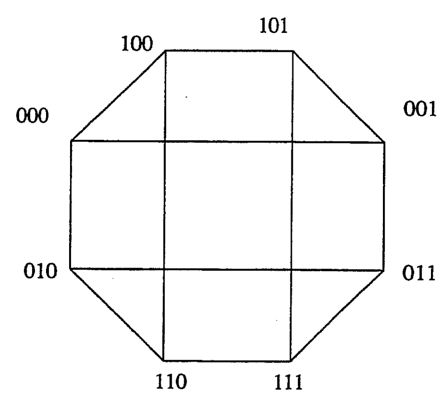
In order to get a better feel for the Möbius cube the first tactic taken on in this project was an application of something I had learned in pre-school. I constructed a five dimensional 0-Möbius cube out of pipe cleaners. This allowed me to physically feel what has happening in the structure. This lead me to think of the 5 cube as the crossproduct Z_2XZ_2 with the first index representing the four cubes and the second representing the three cubes. The cubes were then connected to make either the 5d 0 or 1-cube. The cross product methods lead me to discover the separation developed from the algorithm in [1].



The cross product idea developed into a need to construct the cubes on the computer. The computers allowed me to use Xfig to progress in a trial and error method to avoid separations caused by H_1 and find the desired Hamiltonian circuits. Since at the time I was unsure of there existence and was pretty sure that they did not exist in dimension 4, I decided on searching for them in the 5d Hypercube.

Trial and error was leading me nowhere. I decided that there had to be a better way to visualize the Hypercube. Since the 3 cube has eight nodes it is a subgraph of the complete 8 graph. I used this thought to map the 3 cube into an octagon with the vertexes of the octagon being the nodes. I then used a Hamiltonian circuit to help define the desired homeomorphism in the picture below. This method helped me to find two path independent Hamiltonian circuits on the 5 Hypercube.



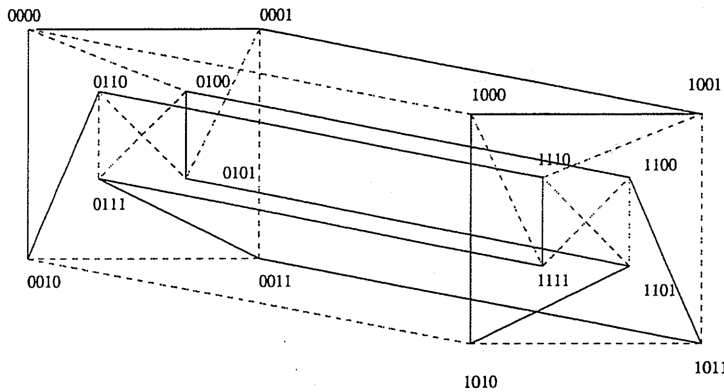


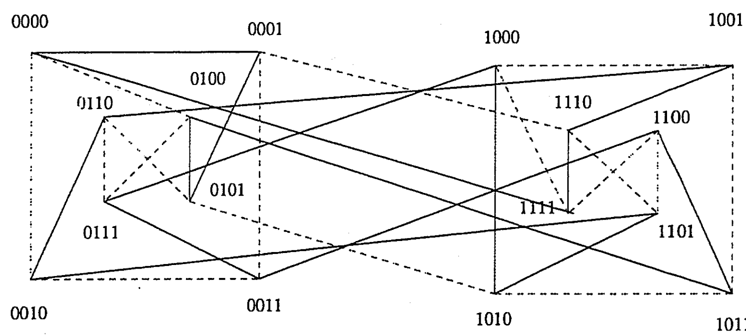
4.3 A Constructive Method of Finding H_1's and H_2's in Dimensions Four and Five

From the first five cube I came up with the obvious idea that lead me to discover the four cubes presented in the next section. Once a first Hamiltonian circuit was defined in the five cube that left three unused edges at each vertex. Of these three edges only two could be used in the second circuit. Once those two have been chosen the third unused edge should be marked so that it won't mistakenly be used. Once the unused edge has been marked the node at the other end of the edge will only have two remaining edges which therefore must be included in the second Hamiltonian Circuit. This helps to avoid the creation of a sub circuit. In the four cube it is now possible to create a H_1 and H_2 simultaneously by using a method similar to the one stated above. In the case of the four cubes once two edges at a node are chosen to be in one circuit, the other two edges must be in the other circuit. By using this method and looking ahead to avoid making sub Hamiltonian circuits it is easy to Find an H_1 and H_2.

4.4 Existence and Recursive Construction of H_1 and H_2 in Dimension n

Once the proper four cubes have been chosen a recursive algorithm can give the H_1 and H_2 in any dimension. The following 4-cubes posses the desired properties. H_1 is represented by the doted line and H_2 is represented by the solid line.

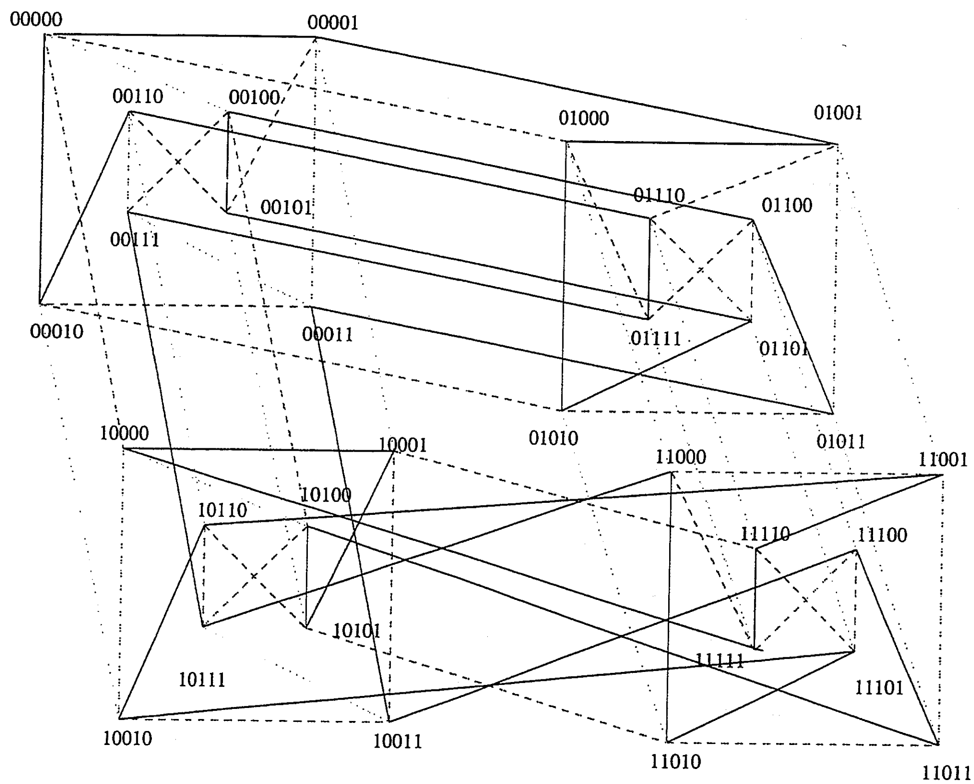




4.4.1 0-Möbius Cube

Algorithm 1 *To Recursively create the 0-cube of dimension* $n+1$, H_1 *and* H_2 *of the* n *cubes must be placed in their corresponding nodes and edges of the* $n+1$ *cube. Then to create the new* H_1 *Break the connections between* $(X_1 00\ldots000)$ *and* $(X_1 01\ldots000)$. *Then connect* $(000\ldots000)$ *to* $(100\ldots000)$ *and connect*$(001\ldots000)$ *to* $(101\ldots000)$.

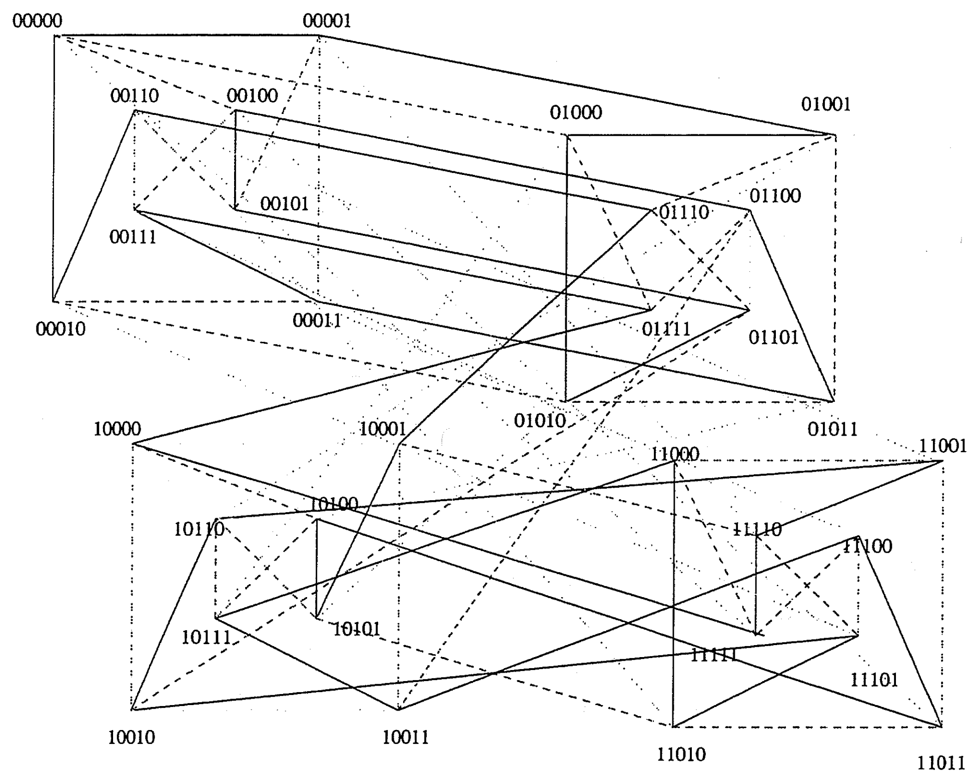
Algorithm 2 *The new* H_2 *is similarly created by breaking the connections between* $(X_1 00\ldots011)$ *and* $(X_1 01\ldots0011)$. *The new circuit is then created by connecting* $(000\ldots011)$ *to* $(100\ldots011)$ *and* $(001\ldots011)$ *to* $(101\ldots011)$.



4.4.2 1-Möbius Cube

Algorithm 3 *The 1-cube is recursively made in dimension $n+1$ in a similar manner to the 0-cube. First H_1 and H_2 should be placed into dimension $n+1$ from dimension n as above. Then to create H_1 the connection between* (100...010) *and* (100...011) *is broken. Likewise break the connection between* (011...101) *and* (011...100)*. The new circuit is then created by connecting* (100...010) *and* (100...011) *to their respective compliments.*

Algorithm 4 *To construct H_2 start by the embedding done in the previous lemmas. Then break the connection between* (100...000) *and* (100...001)*. Also break the connection between there compliments. The new circuit is then created by joining* (100...000) *and* (100...001) *to their respective compliments.*



5 Proof of Algorithm for H_1 and H_2 in Möbius Cubes of Dimension $d \geq 4$

The following two lemmas show the existence of the circuits. The last theorem will show that they are path independent.

5.1 Why Not for $d \leq 3$?

A Möbius cube has 2^n nodes with n connections at each node. This means that there are $n2^{n-1}$ edges. There must be n edges used in a Hamiltonian circuit. So in order for there to be a second Hamiltonian Circuit $n2^{n-1} - 2^n \geq 2^n$. For $n \leq 3$ this is not true.A Möbius cube has 2^n nodes with n connections at each node. This means that there are $n2^{n-1}$ edges. There must be 2^n edges used in a Hamiltonian circuit. So in order for there to be a second Hamiltonian Circuit $n2^{n-1} - 2^n \geq 2^n$.

For $n \leq 3$ this is not true.

5.2 H_1

Theorem 3 *There exists an H_1 on the 0 and 1-Möbius cubes of dimension $d \geq 4$ such that there exist connections between the following pairs of nodes:*

$$\begin{array}{lllll} (0\ldots011) & to & (0\ldots010) & in\ the & 0\ and\ 1-cubes \\ (1\ldots100) & to & (1\ldots101) & in\ the & 0\ and\ 1-cubes \\ (0\ldots000) & to & (010\ldots0) & in\ the & 0\ and\ 1-cubes \\ (0\ldots000) & to & (100\ldots0) & in\ the & 0-cube \end{array}$$

Proof: From the picture in section 4.4 it is clear that this holds for $d = 4$. now assume this holds for $d = n$, now show it is true for $d = n + 1$.

The 0-Möbius cube: Use algorithm 1. When the n cube is embedded into the $n + 1$ cube all the connections are maintained and:

$$\begin{array}{lllll} (0\ldots011) & of\ the & 0-cube & goes\ to & (0\ldots011) \\ (0\ldots010) & of\ the & 0-cube & goes\ to & (0\ldots010) \\ (0\ldots000) & of\ the & 0-cube & goes\ to & (0\ldots000) \\ (010\ldots0) & of\ the & 0-cube & goes\ to & (0010\ldots0) \\ (100\ldots0) & of\ the & 0-cube & goes\ to & (010\ldots0) \\ (0\ldots000) & of\ the & 1-cube & goes\ to & (100\ldots0) \\ (010\ldots0) & of\ the & 1-cube & goes\ to & (1010\ldots0) \\ (1\ldots100) & of\ the & 1-cube & goes\ to & (1\ldots100) \\ (1\ldots101) & of\ the & 1-cube & goes\ to & (1\ldots101) \end{array}$$

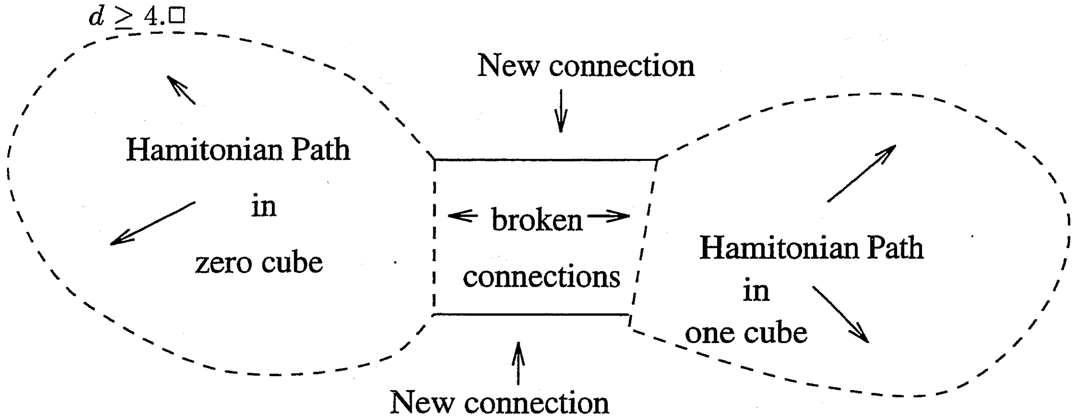
The connection between $(0\ldots000)$ and $(0010\ldots0)$ and the connection between $(100\ldots0)$ and $(1010\ldots0)$ are broken. Then there exists a Hamiltonian paths from $(0\ldots000)$ to $(0010\ldots0)$ and from $(100\ldots0)$ to $(1010\ldots0)$. So when $(0\ldots000)$ is joined to $(100\ldots0)$ and $(0010\ldots0)$ is joined to $(1010\ldots0)$ by e_1 edges then there exists a new H_1 that posses all the desired connections. See picture after proof.

The 1-Möbius cube. Use algorithm 3. When the n cube is embedded into the $n+1$ cube all the connections are maintained and:

$$
\begin{array}{lllll}
(0\ldots011) & of\ the & 0-cube & goes\ to & (0\ldots011) \\
(0\ldots010) & of\ the & 0-cube & goes\ to & (0\ldots010) \\
(0\ldots0) & of\ the & 0-cube & goes\ to & (0\ldots0) \\
(100\ldots0) & of\ the & 0-cube & goes\ to & (010\ldots0) \\
(1\ldots100) & of\ the & 0-cube & goes\ to & (01\ldots100) \\
(1\ldots101) & of\ the & 0-cube & goes\ to & (01\ldots101) \\
(1\ldots100) & of\ the & 1-cube & goes\ to & (1\ldots100) \\
(1\ldots101) & of\ the & 1-cube & goes\ to & (1\ldots101) \\
(0\ldots011) & of\ the & 1-cube & goes\ to & (10\ldots011) \\
(0\ldots010) & of\ the & 1-cube & goes\ to & (10\ldots010)
\end{array}
$$

The connection between $(01\ldots100)$ and $(01\ldots101)$ is broken. Likewise the connection between $(10\ldots011)$ and $(10\ldots010)$ is broken. Then there exists a Hamiltonian path from $(01\ldots100)$ to $(01\ldots101)$ and also between $(10\ldots011)$ and $(10\ldots010)$. Then $(01\ldots100)$ is connected to $(10\ldots011)$ and $(01\ldots101)$ is connected to $(10\ldots010)$ by E_1 edges. This creates a new H_1 in the 1-Möbius cube with all the desired connections.

Since there exists H_1 in the 0 and 1-Möbius cubes with the required connections then there exists H_1 for all 0 and 1-Möbius cubes with $d \geq 4$.□



5.3 H_2

Theorem 4 *There exists an H_2 on the 0 and 1-Möbius cubes of dimension $d \geq 4$ such that there exist connections between the following* *pairs of nodes:*

$(0\ldots0)$	*to*	$(0\ldots01)$	*in the*	0 *and* $1-cubes$
$(1\ldots1)$	*to*	$(1\ldots10)$	*in the*	0 *and* $1-cubes$
$(0\ldots011)$	*to*	$(010\ldots011)$	*in the*	0 *and* $1-cubes$
$(0\ldots011)$	*to*	$(100\ldots011)$	*in the*	$0-cube$

Proof: Similar to that for theorem 3

5.4 Independence

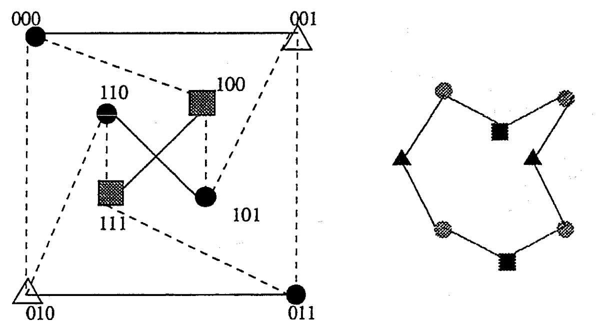
Theorem 5 *H_1 and H_2 are path independent for all dimensions $d \geq$ 4.*

Proof: From the picture in section 4.4 it is clear for the case $n = 4$. Assume it is true for the case $d = n$. Then when the n cubes are embedded into the $n+1$ cubes the sub-cubes are path independent. When the needed connections are broken, they are path independent Hamiltonian Paths. Since the New edges that create the new H_1 and H_2 are not the same then the new circuits are path independent.□

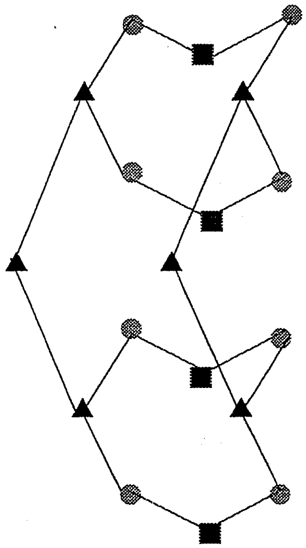
6 Embedding Mesh of Trees

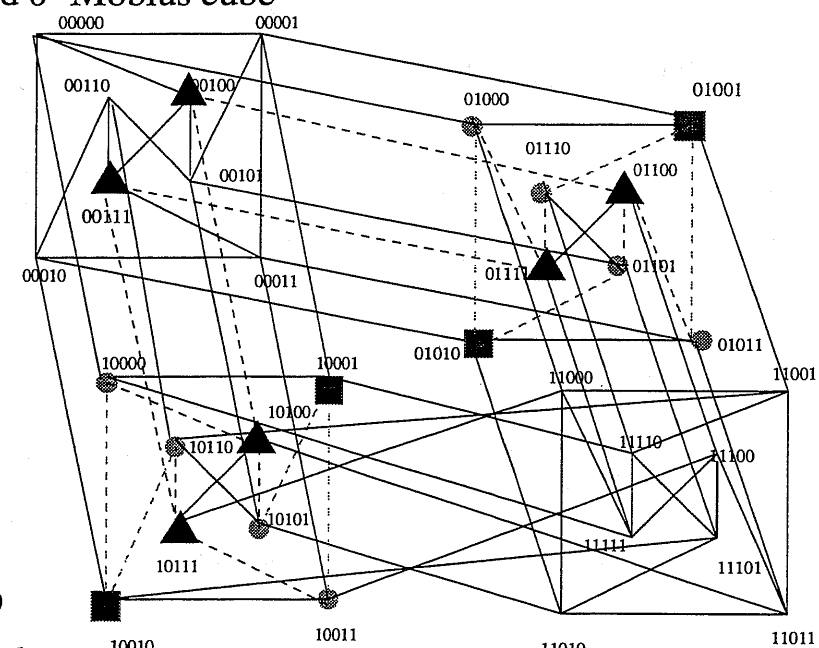
Unfortunately I did not complete a way to embed Mesh of trees into any dimension of a Möbius cube, but I did come up with some dilation 1 examples in the Five cubes. The following will be examples of the 2 X 4 and the 2 X 2 X 2 mesh of trees into the five zero cube and a suggestion of where to look for the abstraction.

The most important observation of the mesh of trees has to do with the 2 X 2 case. This case is exactly a Hamiltonian circuit with eight nodes. This allows one to embed it directly onto a Möbius 3 cube by a one to one correspondence with the cube's Hamiltonian circuit. Embedding these circuits into the cube now becomes a game of placing them properly into an appropriate three sub-cube. To do this the diameter between the sub-cubes must be 2.



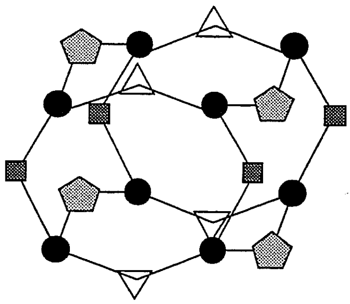
2x2 Mesh into 3d 0 -Mobius cube

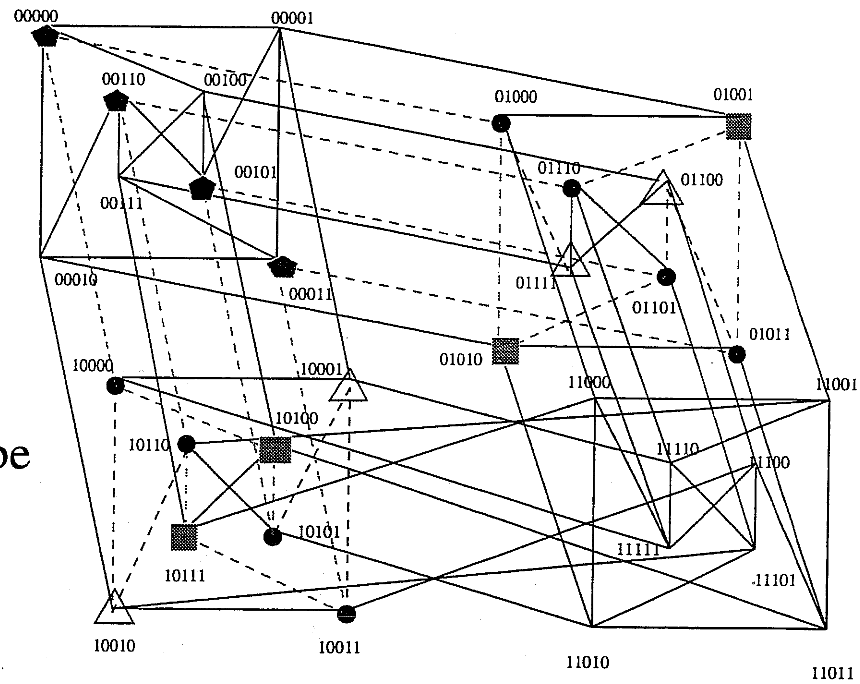




2 x 4 Mesh into

4d 0-Mobius Cube





2x2x2 Mesh into

4d 0-Mobius cube

7 Conclusion

The Möbius cube is a variant Hypercube with about half the diameter of the Hypercube. This paper suggests a way to embed mesh of trees on the Möbius cube, and gives an algorithm to construct Two path independent Hamiltonian circuits on cubes with dimension $d \geq 4$.

8 Acknowledgements

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