Explorations into Global Stability of Population Models

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Abstract

Seven common population models demonstrate the unexpected characteristic that global stability coincides with local stability. We search for an explanation that would tie the models together. We explore several modified Schwarzians, polynomials associated with the equations, and the implications of Culls theorem stating no two cycles implies global stability for our definition of population models. None of these approaches is successful, though the latter is far from exhausted.

1 Introduction

1.1 Motivation

Biologists commonly use population models to envision the dynamics of a species or ecosystem. These models are simplifications of the actual systems, since it would be impossible to include all initial conditions precisely, or even to identify all related factors. As a result it is extremely important that a model be dynamically well behaved, since poorly behaved models could exhibit dramatically varying behavior from arbitrarily close initial points. If the take of Coho Salmon is chosen based on an unstable ecological model a change of a few poached salmon could have disastrous results.

Unfortunately it can be extremely difficult to prove global stability for a model, therefore it is a common practice in biological research to prove local stability of a model and then conclude global stability. Clearly this is not desirable. Mathematicians exploring biological models noted that the commonly used models share the unexpected characteristic that local stability implies global stability. Our goal is to find some unifying theory that would explain why this occurs in the seven models we examine. The hope is that this would suggest a simple test that could be applied to new population models to determine the presence of global stability.

1.2 Previous Work

The traditional method for establishing global stability is to construct Liapunov functions as Fisher et al. (1979) and Goh (1979) do. This is a laborious undertaking, and one not attempted by many biologists. We are hoping to find a result that would circumvent this approach.

Cull (1988) has developed two theorems which, together, prove global stability for the seven models under consideration. Neither is effective for all seven, and the second is not as simple as might be hoped. Cull also observes that the absence of two cycles is equivalent to global stability, by Sarkovskiis theorem combined with our definition of population model.

Singer (1978) develops several conditions dependant upon uniform negative Schwarzian, one of which forms the basis for the work of Heinschel (1994). The ones of interest to us are: negative Schwarzian implies each stable cycle has a critical point attracted to it and negative Schwarzian implies no positive minimums or negative maximums in the first derivative of the function.

Heinschel's work is based upon the theorem stating stable cycles have attracted critical points if negative Schwarzian is present. Since only a single critical point is allowed in our definition of population model, the idea is that only the known periodic point is allowed. Closer examination revealed, however, that an additional condition is necessary to eliminate non-stable two-cycles. This condition is that $f(f(x)) > x$ for x close enough to zero. Unfortunately not all the functions under consideration have uniformly negative Schwarzians. We will be testing the models as they appear in Cull's paper for several properties in the hope that a theorem may suggest itself.

1.3 Definitions

A population model is a function of the form:

$$x_{t+1} = f(x_t)$$

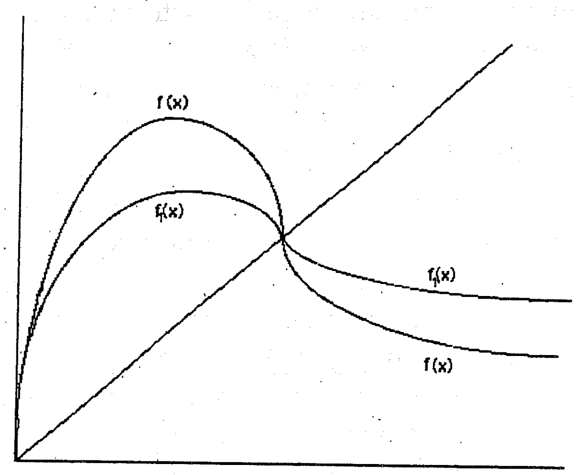
where f is a continuous function with $f(0) = 0$, and there is a unique positive equilibrium point $\bar{x}$ such that:

$$f(\bar{x}) = \bar{x}$$
$$f(x) > x \text{ for } 0 < x < \bar{x}$$
$$f(x) < x \text{ for } \bar{x} < x$$

and such that if $f(x)$ has a maximum x_m in $(0, \bar{x})$ then $f(x)$ is monotonically decreasing for all $x > x_m$ such that $f(x) > 0$. Our definition is identical to that of Cull.

An enveloping curve is one which satisfies:

$$f(x) \geq f_1(x) > x \text{ for } \overline{x} > x > 0$$
$$f(x) \leq f_1(x) < x \text{ for } x > \overline{x}.$$



We shall use globally stable to mean a function for which $\lim_{t\to\infty} x_t = \overline{x}$ for all x_o such that $f(x_o) > 0$. A population model is locally stable iff there is some small enough neighborhood of $\bar{x}$ such that for all x_o in this neighborhood,

x_t is in this neighborhood and $\lim_{t\to\infty} x_{t'} = \bar{x}$. For the class of functions we have defined as population models global stability then implies local stability, since we can take the region where $f(x) > 0$ as our neighborhood.
The Schwarzian derivative of f at a point x is given by:

$$S(f,x) = \frac{f^3(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

for any real valued function f that is at least C^3.

1.4 Models

The models we use are from Cull.

$$\begin{aligned}
one &= x\,\mathrm{e}^{\left(r\left(1-\frac{x}{k}\right)\right)}\\
two &= x\left(1+r\left(1-\frac{x}{k}\right)\right)\\
three &= x\,(1-r\ln(x))\\
four &= x\left(\frac{1}{b+c\,x}-d\right)\\
five &= \frac{x\,r}{1+\mathrm{e}^{\left(-a\left(1-\frac{x}{b}\right)\right)}}\\
six &= \frac{x\,r}{\left(1+\frac{x}{b}\right)^B}\ ;\ B>1\\
seven &= \frac{x\,r}{1+(r-1)\,x^c}
\end{aligned}$$

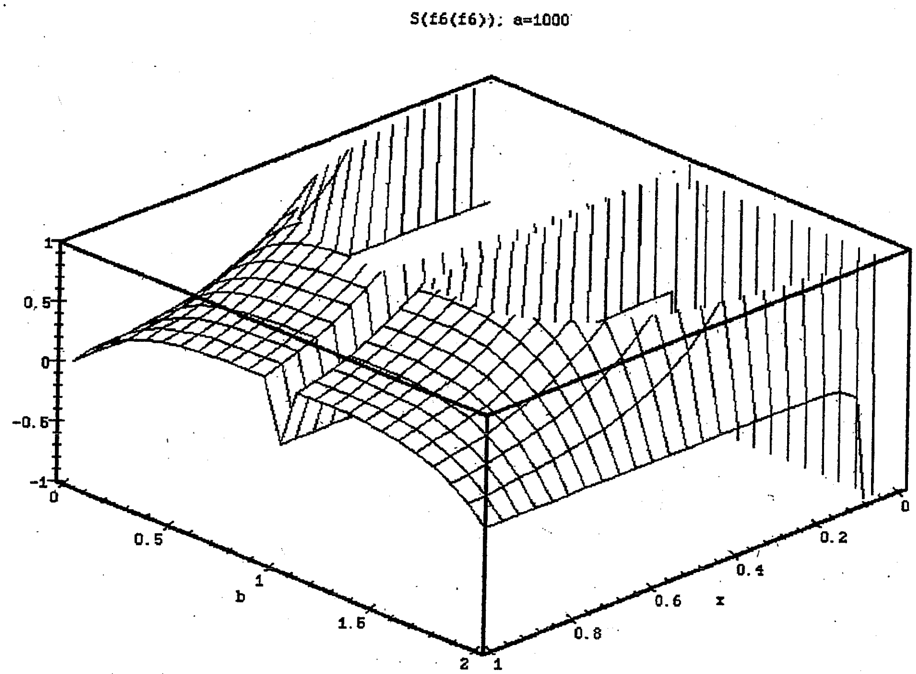
2 Schwarzian Explorations

2.1 S(f(f),x)

A theorem in Singers paper states that :

Theorem 1 *If $S(f) < 0$ for all x, then the function G' cannot have either a positive local minimum value or a negative local maximum value.*

Heinschel showed that functions 3, 6 and 7 do not have uniformly negative Schwarzian, however we hoped that the Schwarzians might be negative for the second iterations of these functions. It is known that succesive iterations of a function with negative Schwarzian have negative Schwarzians, so it was unnecessary to test functions 1, 2, 4 and 5. Unfortunately exploration quickly showed that all three functions had nonegative values. $S(f_6(f_6))$ had a particularly interesting graph:



2.2 $S_{mod}(f(x))$

We then explored a modified Schwarzian, formed by taking the Schwarzian of the integral of $\frac{f(x)}{x}$. The Schwarzian was not uniformly negative for functions 3, 4, 6 and 7. At this point we abandoned exploration into the Schwarzian.

3 Polynomials

We then looked into polynomials formed by examining $g(x) = \frac{f(x)}{x}$. We searched for simple polynomials of the form $P_1(g'(x)) = P_2(g(x)) + P_3$. The hope was that if sufficiently simple polynomials could be found we would be

able to formulate a theorem suggested by them. For equations 1, 2, 3, 4, and 6 such polynomials existed. (See appendix A) However for equations 5 and 7 we could find no suitable simplification. We were left with the choice between simple polynomials and second order differential equations or first order differential equations with relatively complicated polynomials. Neither suited our purposes.
The idea we hoped to pursue was that we could find constraints on the functions, in this case the first derivative of an associated function, and prove the constraints hold for our models.

let	$f(x) = xg(x)$
we know	$f(f(x)) > x$
so	$f(xg(x)) = xg(x)g(xg(x)) > x$
	$h \equiv g(x)g(xg(x)) > 1$ for $x < 1$
finally	$D(h) = g'(x)g(xg(x)) + g(x)g'(f)f'$

We know if $D(h) < 0$ on $(x_m, 1)$ then global stability holds. If simple polynomials had existed, this might have proven quite easy to establish. It is possible that further exploration along these lines might prove fruitful.

4 Implications of two-cycles

Our final method of exploration involves geometric approaches suggested by Culls theorem:

Theorem 2 *A population model is globally stable iff it has no cycles of period 2.*

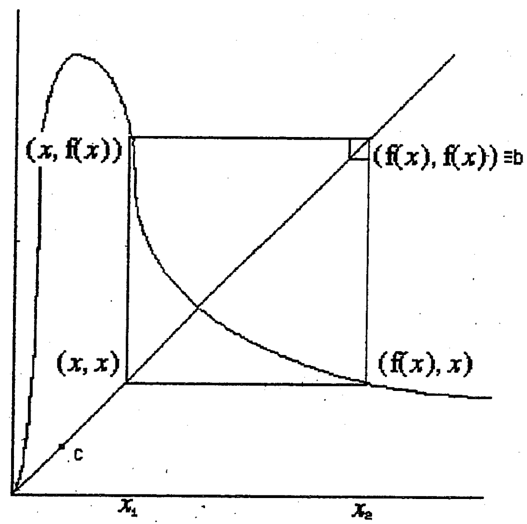
Our work is based on two observations:

Observation 1 *Let x_1, x_2 be the two values comprising a two-cycle. Then*

$$slope = -1 = \frac{f(x_1) - f(x_2)}{(x_1 - x_2)}$$

Observation 2 *the points x_1 and x_2 are equidistant from any point on the $x = y$ line.*

To demonstrate our observations, consider the geometric qualities of a two cycle:



Clearly

$$\frac{f(x_1) - f(x_2)}{(x_1 - x_2)} = \frac{f(x_1) - x_1}{x_1 - f(x_1)} = -1$$

To observe the second fact, we note that using the standard metric, $d((x_1, f(x_1)), b) = d((x_2, f(x_2)), b)$. Further, the line connecting $(x_1, f(x_1))$ and b is horizontal and the line connecting $(x_2, f(x_2))$ and b is vertical. Thus they form a right angle bisected by the line $x = y$ and by the side-angle-side theorem we have $d((x_1, f(x_1)), (c, c)) = d((x_2, (f(x_2)), (c, c))$.

4.1 Using our Observations

We recognized that functions made from our observations could be manipulated into equations of the form $g(x_1) = g(x_2)$. We hoped that these functions could then be graphed and would be found to be obviously increasing or decreasing in the area of interest to us. This area may be constrained by noting that to correspond to a two-cycle a solution pair must have one member on either side of $x = 1$, and recognizing that the point furthest from $(0, 0)$ before $x = 1$ is $(x_m, f(x_m))$ so all points $x > f(x_m)$ may be discarded. Thus we are only concerned with pairs of solutions in a limited area. Unfortunately in each case we have explored, functions which have pairs of solutions tend to exhibit this behavior around $x = 1$, the heart of our region of interest. We had hoped that perhaps by combining more than one of the functions on the same graph we would find that their areas of solution pairs did not overlap, but this is not graphically clear. We are still catalogueing which functions fail in the critical area for each equation and hope to include this in an appendix at a later time:

$g \equiv f(x_1) + x_1$ This was the first attempt, using only the fact $m = -1$.
$h \equiv f(x)^2 + x^2$ Next we use only the fact $d(x_1, c) = d(x_2, c)$
$p \equiv f(x)x$ Here we take g^2 and substitute in h. This was the first attempt to use both facts.
$q \equiv (f(x) - c)^2 + (f(x) - d)^2 + (x - c)^2 + (x - d)^2$ where $c, d \in \Re$
Here we recognize the fact that only points with the qualities we desire are equidistant from two points on the $x = y$ line. Unfortunately our equation also admits points with equal combined distance from the two points. We are currently working to rectify this.

To outline the procedure we are using, we submit our first attempt using only $m = -1$: We manipulate

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

to get

$$f(x_1) + x_1 = f(x_2) + x_2$$

(see proof) We hoped that the function $g(x) = f(x) + x$ might prove strictly increasing or decreasing for our seven functions. Unfortunately this was not the case, and indeed we doubt it would often prove to occur. For those few cases when it does, we offer:

Theorem 3 *If $g(x) = f(x) + x$ is strictly monotonic and f is a population model, then $f(x)$ is globally stable.*

Proof 1 *Assume a two cycle exists. By Observation 1 we know a two cycle consists of two points connected by a line of slope -1.*

since $\frac{f(x_1)-f(x_2)}{(x_1-x_2)} = -1$
then $f(x_1) - f(x_2) = x_2 - x_1$
so $f(x_1) + x_1 = f(x_2) + x_2$

Thus if the function $g(x) = f(x) + x$ is strictly monotonic, only $x_1 = x_2$ satisfies this equation, and no two-cycle exists. By Cull's theorem the model must be globally stable.

It became clear with further thought that we must also invoke the additional information that x_1 and x_2 are equidistant from the $x = y$ line. While it would be possible to show that our function preserves distance proportions, it does not appear useful to state that the points $(x_1, g(x_1))$ and $(x_2, g(x_2))$ are equidistant from the line $2x = y$. Similar processes have been used in the other cases.

5 Conclusion

Clearly we have failed to find a uniform method to show global stability for our models. Our methods encompassed the range from exploration with minimal motivation to theorems relieing on conditions which are neccesary, but proved insufficient for our purposes. It is our belief that further exploration into Schwarzian will prove fruitless. It is possible that further search into simple polynomials as described in section 3 may be useful. Our exploration into the conditions $m = -1$ and equidistance are ongoing. At the moment we are pursuing additional equations that might be both neccesary and sufficient and plan to catalogue which functions are useful for which models. We are also considering the idea that while measuring distance to a finite number of points on the $x = y$ line is insufficient, perhaps some method for taking the limit as additional points are added might prove useful.

A Appendix

The polynomials $P_1(g(x)) = P_2(g(x)) + P_3$ found by taking $g = \dfrac{f(x)}{x}$ for the seven models:

one	$e^{(r-rx)}$	$g' = -\dfrac{g}{r}$
two	$1 + r(1-x)$	$g' = \dfrac{1}{2}g + \left[\dfrac{1}{2}x - \dfrac{1}{2}r\right]$
three	$1 - r\ln(x)$	$xg' = -r$

four	$\dfrac{1}{b+cx} - d$	$(\frac{b+cx}{-c})g' = g + d$
five	$\dfrac{1+a\,e^{b}}{1+a\,e^{(bx)}}$	$g' = -g^2 \frac{abe^{(bx)}}{1+ae^{(bx)}}$
six	$\dfrac{(1+a)^b}{(1+a\,x)^b}$	$(1+ax)g' = -b\,a\,g$
seven	$\dfrac{a}{1+(a-1)\,x^c}$	$g' = \dfrac{g^2\,x^{(c-1)}\,c\,(a-1)}{a}$

For models 5 and 7 it is certainly possible to find polynomials that form a first order differential equation, however these polynomials are not simple:

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