Classification of Loops with Self-intersections on the Once Punctured Torus with Genus N

Ryan Gould
Jean Steiner
Beverly Steinhoff

Advisor: Professor Dennis Garity
NSF sponsored REU Program
Oregon State University

August 16, 1996

Abstract

In this paper we classify the homotopy classes of loops with one and two self-intersections on the punctured torus with genus n. We rely upon topological arguments to develop the classification of loops with a given intersection number. We then provide a partial proof of the distinctness of the homotopy classes for once and twice intersecting loops on the punctured torus of genus 2.

1 Introduction

In this paper, we study single self-intersecting closed loops of a general torus with n holes, where n is a positive integer, and classify the free homotopy classes of loops on the once punctured two holed torus with two self intersections. The idea for this paper grew out of the following studies, where the authors classify once self-intersecting loops and geodesics on the puncture one-holed torus and study their relation to the Markoff spectrum: [2], [3], [5], [11] and [12].

In section two, we will define our general terminology, background, and techniques. In addition, we classify the simple loop on T_2 in this section.

Section three classifies single self-intersecting loops on T_2, while section four classifies twice self-intersecting loops and describes the two different approaches used to do so. The generalization of the once-intersecting loops on T_n is discussed in section five.

Finally, the distinctness of the loops using an algorithm of Whitehead is shown in section seven.

2 Background

We will note the punctured n-holed torus as T_n. Furthermore, when we speak generically of the n-holed torus, we will assume we mean the punctured n-holed torus unless otherwised noted. The fundamental group of T_n, $\pi_1(T_n)$, is isomorphic to the free group on $2n$ letters, $F(a_1, b_1, a_2, b_2 ... a_n, b_n)$. There is a bijection between free homotopy classes of closed curves on T_n and conjugacy classes of elements of $F(a_1, b_1, a_2, b_2 ... a_n, b_n)$. For notation, we will define the fundamental group of T_2 as being isomorphic to the free group $F(a, b, c, d)$.

2.1 Definitions and Notations

It is convenient to think of T_n as the conected sum of n tori. For example, T_2, which is a quotient space of an octagon (see Figure 1), can be viewed as the sum of two one-tori. We let one of these tori be generated by the curves a and b, and refer to it as ab-torus, and let the other torus be generated by the curves c and d and refer to it as cd-torus. By identifying the sides of the octagon in figure 1, we fix the orientation of the generators of T_2 as drawn in figure 2. Without loss of generality, we can let the puncture lie on the cd torus, since we could define an appropriate homeomorphism of T_2 which makes this the case. In this same manner, T_n can be viewed as the connected sum of n one-tori. This figure is the quotient space of the $4n$-gon formed in the same manner as in figure 1, namely the segments will be named $a_1 b_1 \bar{a_1} \bar{b_1} a_2 b_2 \bar{a_2} \bar{b_2} ... a_n b_n \bar{a_n} \bar{b_n}$, clockwise. Therefore, when we identify the sides, we create T_n with generators oriented as in figure 3. Let the ith one-torus of this connected sum be generated by the free group $F(a_i, b_i)$. Hence, we refer the ith torus as the $a_i b_i$-torus. Without loss of generality, the puncture can be placed in the $a_n b_n$-torus.

In order to define a closed loop, l, on T_n, we will let f be a continuous function that maps the interval [0,1] to T_n, such that the initial and terminal points are the same, ie $f(0) = f(1)$. Hence, l is the image of this mapping. The loop is simple if $f(i) = f(j)$ if and only if $i = 0$ and $j = 1$. Otherwise, a loop is said to have a self-intersection when $f(i) = f(j)$ for some finite number of i's and j's where $0 < i, j < 1$. When we analyze the various loops on T, we only consider non-trivial intersecions. A loop has a single non-trivial intersection provided the intersecion is a transverse one and the loop is not homotopic to a simple loop. A loop has two non-trivial intersecions provided the intersections are transverse and the loop is not homotopic to a loop with a single non-trivial intersection or a simple loop [5].

2.2 Cutting

In this section, we will lay the foundation for exploring closed loops with self-intersections on T_n. Let the closed loop l be defined as in the introduction. Since T_n is a manifold, there exists an open interval around each point on l such that when we remove the interior, we will produce two identical copies of l, $l \times 0$ and $l \times 1$. When we refer to a boundary component of a region or a surface, we are referring to a single copy of the image of a loop. Hence, we have created two new boundary components, the edges of l, which are loops.

Furthermore, a loop l is said to be *nonseparating* if when cut, the two boundary components created lie on the same surface. A loop is said to be *separating* if it creates two new surfaces, each containing one boundary component of l.

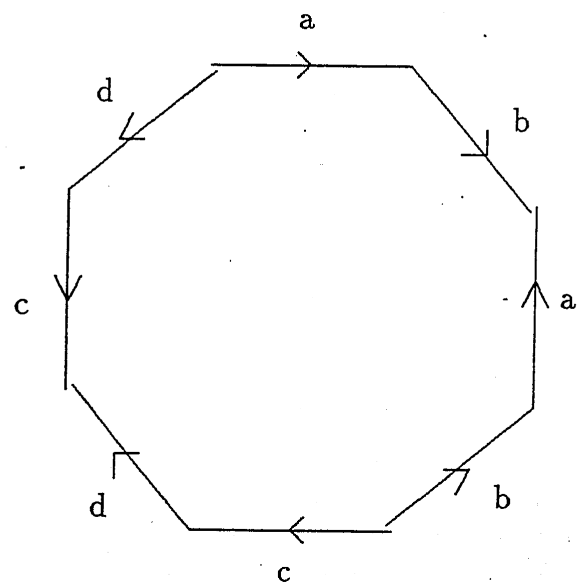


Figure 1

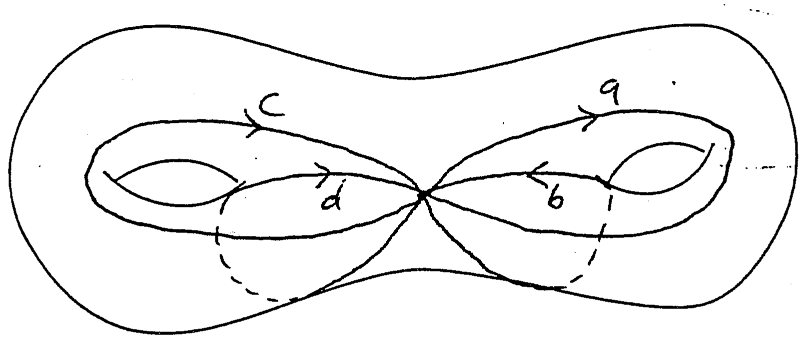


Figure 2

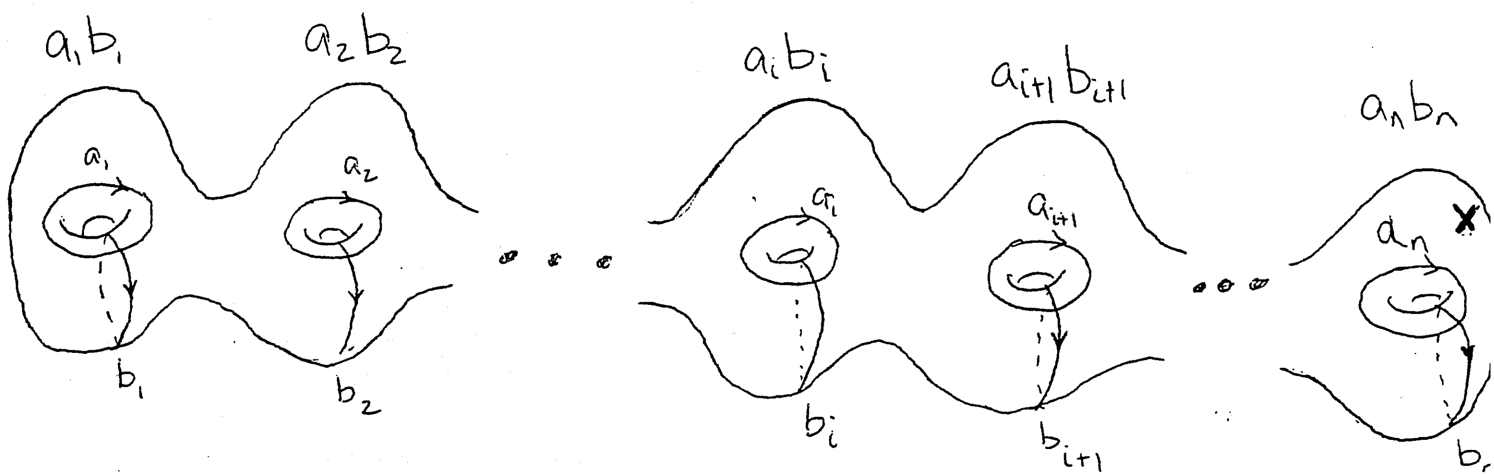


Figure 3

2.3 Simple Loops on T_2

We will now begin our study of self-intersecting loops on T_2. This is then followed by a generalization made to T_n. Before studying self-intersecting loops on T_2, however, we must establish the following theorem regarding simple loops on T_2.

Theorem 2.1 *On the once punctured, two-holed torus, T_2, there exists a homeomorphism which takes any simple loop, l, to an element in the equivalence class of one of the following:*

1. *the non-separating curve, b,*
2. *a curve, λ, which separates T_2 into two one-holed tori which can be described by the word $ab\bar{a}\bar{b}$,*
3. *a loop bounding a disc or the identity,*
4. *a loop, Δ described as $\bar{c}dc\bar{d}ba\bar{b}\bar{a}$, which bounds a punctured disc.*

Proof We will demonstrate the existence of each of the four classes in the theorem through an Euler characteristic, and we will use similar arguments throughout this paper. We will need to prove that loops which yield surfaces of the same topological type are equivalent up to homeomorphisms. In order to show that two loops on T_2 are equivalent (ie. there is a homeomorphism of T which takes one loop to the other), we will have to use the following argument [14]. We will first consider the case where l is a nun-separating loop. Let p be a nonseparating loop on T_2 such that when cut, it leaves an orientable surface F with two boundary components and an Euler characteristic, which we will denote as χ. Assume there is another nonseparating loop, q, that results in an orientable surface F' also having two boundary components and the same χ as F. Then, there exists a homeomorphism, $h : F \longrightarrow F'$.

When cutting p, let p_0 and p_1 be the two boundary components created after removing the interior (hence, $p_0 = p \times 0$ and $p_1 = p \times 1$). Each distinct point on p is mapped to both a unique point on p_0 and a unique point on p_1. Therefore, there is a 1-1 and continuous identification between p_0 and p_1. Hence, we can identify each of these two edges by a specific homeomorphism $g : p_0 \longrightarrow p_1$. Furthermore, there is a homeomorphism $hgh^{-1} : q_0 \longrightarrow q_1$ where q_0 and q_1 are the two edges of q. Hence, there is a homeomorphism $f : T_2 \longrightarrow T_2$ which maps p onto q.

Similarly, if m is a separating curve, then T_2 will be cut into two surfaces, F_1 and F_2 (only two because each surface must contain one of the two boundary components). Assume another loop n is equivalent to m. Then n will create two surfaces F_1' and F_2', where $F_1' \cong F_1$ and $F_2' \cong F_2$ and $\cong$ will denote equivalence under homeomorphism. Hence F_i will have the same Euler characteristic and number of boundary components as F_i'.

Therefore, if two simple loops create a surface or pair of surfaces of the same topological type without boundary, then there exists a homeomorphism taking one loop to the other. In order to see if two surfaces are homeomorphic, we can use the part of the Classification theorem which simply states that if two orientable surfaces without boundary have the same Euler characteristic, then they are homeomorphic [8, pages 10-11].

We have seen that a simple loop will create two boundary components which are loops. Therefore, we must glue a disc to each bounded surface, obtaining a surface, without boundary. We must glue in two such faces, so we increase the Euler characteristic by two, and now χ equals 0.

The possibilities of surfaces without boundary which a simple, closed loop on T_2 can create are therefore restricted by $\chi = 0$ and the fact that only one or two surfaces can be created since each surface must contain a boundary component before gluing on a face. The possible resulting surfaces are as follows:

1. a one-holed torus
2. two one-holed tori, or
3. a two-holed torus and a sphere.

In the first case, there is only one resulting surface, so l must be a nonseparating simple loop. Without loss of generality, let l be the loop described by the generator b, which is a nonseparating curve. Since we have shown that, when cut, any nonseparating loop will result in a one-hole torus, then for any nonseparating loop on T_2, there is a homeomorphism which maps the loop onto b. We note that each generator of the free group is a non-separating curve.

For the second case, the loop divided T_2 into two surfaces, so it must have been separating. We will define a cut which separates T into two one-hole tori, with the puncture lying in either one of them, as λ. Furthermore, we have oriented this loop such that it can be described by the word $ab\bar{a}\bar{b}$, where the puncture will lie on the one-holed torus generated by c and d. Note that if the puncture lies on the ab-torus, then the name of a loop that divides T_2 into two one-tori can be described with the word $\bar{c}dc\bar{d}$.

Finally, in the third case, we again have a separating loop l. First, assume that the puncture lies on the two-holed torus. On the sphere, there will be a boundary component corresponding to an edge of l, say l_0. Since there is no puncture on the sphere, l_0 is contractible to a single point, and therefore l is freely homotopic to the identity. Now assume the puncture is on the sphere. The loop is no longer contractible to a single point, but the cut can be contracted to a disc enclosing the puncture. The word $\bar{c}dc\bar{d}ba\bar{b}\bar{a}$ corresponds to a loop enclosing a punctured disc, and we will refer to this as Δ for convenience. Hence, any l creating a two-holed torus and a punctured sphere will be equivalent to Δ.$\diamond$

2.4 k+1 Simple Loops

Now that we have established the four simple loops on T_2, we can begin to consider self-intersecting closed loops. Using the language of [5], we can define a closed curve as having a single nontrivial self-intersection if it has a single transverse intersection and is not freely homotopic to a simple loop. In addition, a loop is said to have n non-trivial intersections if it has n

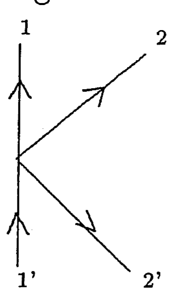
transverse self-intersection points and is not homotopic to a loop with $(n-i)$ intersections, where $i = 1, 2, ...n$. Using this definition of self-intersecting loops, we can classify such loops on T_2. First, we will establish the following lemma:

Lemma 2.1 *Up to a free homotopy, any loop, l, on T_2 with k transverse self-intersection points can be formed as the composition of $k+1$ simple loops, which intersect at only one point.*

The reader is referred to [5] for the proof of this on the one-holed, once punctured torus. The same argument holds for the two-holed once punctured torus. With this lemma, we can view the once-intersecting loop, l as the composition of two simple loops intersecting at a common point, which we will call the basepoint.

2.5 Transverse Intersections

To see if an intersection is transverse, we can simply look at the neighborhood around the basepoint. We can represent this neighborhood with the following k-diagram:

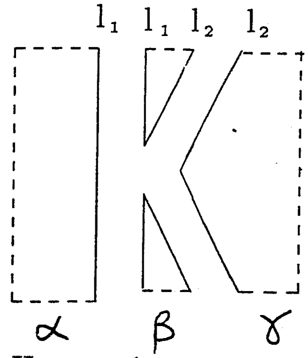


Here, the number i, where $i = 1, 2$ represents the initial segment of the ith loop, and i' the final segment. Note that the above diagram can be reflected such that the second cut can be made first.

2.6 Change in χ

Since much of our analysis relies on the values of the Euler characteristics of the srufaces which are created, we would like to determine how our cutting and gluing process affects $\chi(T_2$. Separating T_2 along two simple loops, l_1

and l_2, which share a common basepoint a divides T_2 into the following three regions:



Here, region α is bounded by l_1 and region γ by l_2. Note that both of the middle pieces are really connected as one region, namely β, since they are both bounded by the same loops, l_1 and l_2. Depending on whether or not one or both of these curves are separating, there can be one, two, or three disjoint surfaces.

Note that cutting along both of the simple loops forms three new boundary components. When we separate the regions, an additional vertex is added. Hence, after separating and removing the boundary by adding three discs, we will increase the Euler characteristic by 4. Therefore, after cutting we will have one, two or three surfaces without boundary and a total $\chi = 2$.

3 The classification of loops with a single intersection on T_2

Using the cutting and gluing technique outlined in the previous section, we can now classify the free homotopy classes of loops with one non-trivial intersection on T_2 up to homeomorphisms of T_2. Using the k-diagram and the total Euler characteristic requirement, we can determine the possible combinations of the topological types for each region.

Without loss of generality, we may assume that the first cut corresponds to one of the four cannonical representations for a simple loop. We are only looking at loops with one non-trivial intersection which are not freely homotopic to either a simple loop or a single point, so we only need to consider the cases where l_1 is a non-separating curve, a λ, or Δ. Choosing one of these three loops as l_1 restricts the possibilities for the types and numbers of surfaces that the three regions can form. We may now make some generalizations which will reduce the possibile combinations of surfaces in the three regions for our classification.

Claim 3.1 *If region α or γ yields a sphere without a puncture, the region is bounded by a loop which is freely homotopic to the identity. If region β yields a phere without a puncture, l_1 is homotopic to l_2.*

Proof Consider the case when region α or γ yields a sphere without a puncture. Prior to adding a face to the region we had a bounded disc. The boundary of this disc is comprised of a single loop. On this disc, any simple loop, including the boundary, is freely homotopic to a point. Thus the boundary of this disc must correspond to the identity in the free group.

Now consider the case where region β is a sphere without a puncture. Prior to adding the face to the region, we had a disc whose boundary is comprised of l_1 and l_2 now joined at two vertices (recall that in the separating process we made two copies of the original vertex). We know that on a disc, any path between two fixed points is homotopic to another path between the two points. On the disc in region β, l_1 and l_2 are both paths between the two fixed points which are the two copies of the vertex, and we can smoothly deform l_1 into l_2, ie. l_1 is homotopic to l_2. $\diamond$

We can now state the following theorem which classifies the loops with a single transverse intersection.

Theorem 3.1 *On the once punctured, two-holed torus, T_2, there exists a homeomorphism which taks any closed loop, l, with one transverse self-intersection to one of the following:*

1. $\lambda\lambda$,
2. $\bar{c}dc\bar{d}\lambda$,
3. $\lambda\bar{\Delta}$,
4. $\lambda\bar{b}$
5. $\bar{c}dc\overline{\bar{d}b}$,

6. $\Delta\Delta$,

7. Δb,

8. bb,

9. $b\bar{\Delta}b$,

10. bd,

11. $bab\bar{a}$,

12. $b\bar{c}dc\bar{d}b$.

Proof There are three basic cases that we must consider: cutting along λ, Δ, or a nonseparating curve as l_1. First consider the case where λ is the first cut we make. When we cut along λ we are left with two 1-tori. Our second cut, l_2 must be freely homotopic to a curve which lies completely on one of the tori so that we only have one transverse intersection. If we cannot smoothly deform l_2 so that the only intersection point is the basepoint, ie. so that the curve does not lie entirely on one torus, we would have more than one intersection. Recall that we have defined λ so that we assume the puncture lies on the cd torus, and l_2 may lie on either torus. Thus, after cutting along l_2 we will be left with a 1-torus and one or two other surfaces. The sum of the Euler characteristics of all the remaining surfaces must be 2; however, the Euler characteristic of a 1-torus is 0, so our second cut must yield surfaces which will increase the Euler characteristic by 2. We will assume that region α corresponds to the 1-torus on which the second cut is not made, so region α will be a 1-torus when we are done cutting. After cutting along l_2 we may be left with the following combinations of regions.

1. Region β is a sphere, region γ is a torus.

2. Region β is a torus, region γ is a sphere.

3. Regions β and γ are a sphere.

When case 1 arises, the puncture may be in one of the tori or in the sphere. First assume the puncture is in a torus. We have seen that when region β is an unpunctured sphere, l_1 must be homotopic to l_2. We have chosen l_1 to be a λ so l_2 must also be a λ. In the free group, the word that corresponds to case 1 with the puncture on a torus is thus $\lambda\lambda$. When case 1 arises, the puncture may also be in the sphere. Since the puncture is in the sphere, l_1 is no longer homotopic to l_2. We have chosen the puncture to be on the cd torus, so l_2 must be on the cd torus, and $\bar{c}dc\bar{d}\lambda$ the word which describes the combination of l_1 and l_2. or l.

When case 2 arises, the puncture must be in region γ. If the puncture were not in region γ, then the region would be bounded by a loop which is freely homotopic to a point, and so l_2 would be the identity; but we can ignore this situation. Prior to adding a face to region γ, we have a punctured disc which has a boundary component consisting of a single loop, l_2. Thus, l_2 is a loop which bounds a punctured disc. The cut along l_2 was made on the cd torus missing a disc. In terms of the generators of the torus and the missing disc whose boundary component is described by λ, the word corresponding to a loop which is in the free homotopy class of l_2 is $\bar{c}dc\bar{d}ba\bar{b}\bar{a}$, which we recognize to have the same name as Δ. In order to have a transverse intersection, the word which describes this loop is $\lambda\bar{\Delta}$.

When case 3 arises, the puncture may be in the torus or in the sphere. Assume the puncture is in the torus. The puncture is in the cd torus, so l_2 must be in the ab torus. Since the cut along l_2 leaves us with a single piece, l_2 must not separate the 1-torus. We claim that the non-separating curve on the one-torus missing a disc is also a non-separating curve on the T_2. When we proved 2.1 we showed that a cut along a non-separating curve was homeomorphic to b. There is a simple homeomorphism on T_2 which takes b to each of the loops which corresponds to a word consisting of one of the letters which generates the free group (ie. a, b, c, d) thus each of these loops is also non-separating. Consider a $\tilde{T}_2$, a 1-torus with a single boundary component. Assume we cut along one of the curves which corresponds to a generator of $\tilde{T}_2$, we will not separate $\tilde{T}_2$. As with a one-torus without boundary, a non-separating curve yields a sphere with two boundary components, however on $\tilde{T}_2$ one of the boundary components is comprised of two loops. Any non-separating curve on $\tilde{T}_2$ will yield a sphere with two boundary components, one of which arises from two loops, and we can define a homeomorphism between any non-separating loop on $\tilde{T}_2$ and any loop corresponding to a generator of the free group on two letters that is isomorphic to $\pi_1(\tilde{T}_2)$. When λ is cut, we are left with two 1-tori, each missing a disc, so the free groups corresponding to the 1-tori are $f(a,b)$, and $f(c,d)$. Assume that l_2 is made on the ab torus. We know that b is a non-separating curve on $\tilde{T}_2$, and it is a non-separating curve on T_2, so cutting along a curve homotopic to b will yield a sphere. The word corresponding to this cut is $\lambda\bar{b}$. Alternatively, l_2 may be cut on the cd torus, in which case we know that d is a non-separating curve on $\tilde{T}_2$ as well as on T_2, so we will obtain a punctured sphere by cutting along any loop freely homotopic to d. The word corresponding to this case is $d\bar{\lambda}$. Note that this last case is equivalent to the case when $l_2$lies on the ab-torus, and the puncture lies on the ab-torus rather than the cd-torus. Hence, we can describe this same l with the word $\bar{c}dcd\bar{b}$.

Now consider the case where the first cut is in the free homotopy class of a loop described by Δ. Since Δ bounds a punctured disc, when we add a face we obtain a sphere in region α, with $\chi = 2$. We may have one or two other surfaces, and again the requirement that the Euler characteristic of all of our remaining pieces must be 2 limits the possible topological types of the remaining surfaces. Since the puncture must be in region α, the remaining regions will not have a puncture. We may have the following combinations:

1. Region β is a sphere, and region γ is a 2-torus.
2. Region β is a torus and region γ is a sphere.
3. Region β and γ are connected as a torus.
4. Region β and region γ are both 1-tori.

When case 1 occurs, we know that a sphere in region β implies that l_1 is homotopic to l_2 so the any such curve is in the free homotopy class of $\Delta\Delta$. When case 2 occurs, region γ is a sphere without a puncture, so l_2 must be homotopic to a point, and we may ignore this case.

When case 3 occurs, l_2 must be a non-separating curve on the surface since regions β and γ are connected as a 1-torus. When we had cut along Δ as l_1 we were left with the sphere in region α and a 2-torus missing a disc. We claim that a non-separating curve on the 2-torus which is missing a disc is also a non-separating curve on T_2. When we cut along a non-separating curve on a 2-torus missing a disc, we increase the euler characteristic by 1 since we add a vertex. We must then add two faces to fill in the boundary components and we obtain a surface without boundary with Euler characteristic 0, ie. a torus. Similarly when we cut along a non-separating curve on T_2 we were left with a torus after adding in the appropriate faces. The topological type that results from the cutting and pasting is the same for a non-separating curve on T_2 and on the two-torus missing a disc, and if we could deform l_1 to a point, the boundary of the two surfaces would be equivalent. So, the non-separating curve on the 2-torus missing a disc is non-separating on T_2. Hence, without loss of generality, we can describe the loop which yields case 3 with the word Δb.

We note that case 4 is in fact a reflection of case 2 of the cuts where $l_1 = \lambda$ so there must be a homeomorphism between the loop which gives rise to this case and $\lambda\bar{\Delta}$.

Now consider the case where we cut along b first. Since b is non-separating, at least two of the three regions must be connected. The combinations of topological types for the regions that arise are restricted by the requirement that we must be left with one or two surfaces after our cutting and pasting process, and the Euler characteristic of all of our surfaces must be 2. The following combinations of surfaces may occur:

1. Region α and β are connected as a torus, and region γ is a sphere.
2. Region α and β are connected as a sphere and region γ is a torus.
3. Region α and γ are connected as a torus, and region β is a sphere.
4. Regions α, β, and γ are connected as a sphere.
5. Region α and γ are connected as a sphere and region β is a torus.

We have already addressed cases 1 and 2. Case 1 is a reflection of the third case for which Δ is the first cut, and case 2 is a reflection of the third case for which λ is cut first.

When case 3 occurs, the puncture may be in the sphere or in the torus. If the puncture is in the torus, we have a sphere in region β thus l_2 must be homotopic to l_1. The word which corresponds to this loop must be bb. If the puncture is in the sphere, the two loops would be homotopic on the unpunctured 2-torus, and the word that describes the loop is $b\bar{\Delta}b = bab\bar{a}\bar{a}\bar{d}cd\bar{c}b$.

When case 4 occurs l_2 must be a non-separating curve on the 1-torus missing two discs since we are left with one piece, a sphere after making the cuts. In order for two curves, l_1 and l_2 to not separate T_2, they most correspond to nonseparating curves of each of the two single tori which are the connected sum which form T_2. View T_2 as two one-tori, the ab-torus and the cd-torus. We know that on a one-torus, cutting a generator, say b doesn't separate. We have seen, though, that cutting along a second b does then separate the torus (see case 3), since it does not form a generating pair of the one-torus. Allowing the second cut to be a would not separate, however, we know that the intersection of these two loops would be non-transverse. Hence, this argument along with a similar one for the cd-torus, allows us to claim that a loop which creates case 4 is equivalent to two nonseparating loops from different one-tori. Thus we have described this curve as bd.

Finally, consider case 5. This case arises when the two loops correspond to two nonseparating curves from the same one-tori which would not be homotopic to each other on the unpunctured 2-torus. Without loss of generality, we can assume that they both lie on the ab-torus. Therefore, we can then let l_1 be b. First, let the puncture lie on the one-torus, or region β. If the intersection were trivial (ie the orientation of l_2 were reversed), we see that $b\bar{l_2}$ would be homotopic to λ and would therefore have the same name to describe it as λ. Hence, we can call l $bab\bar{a}$ On the other hand, if the puncture lies on the sphere of region α and γ, then we have $b\bar{l_2}$ being freely homotopic to the curve described by the word $d\bar{c}\bar{d}c$. Therefore, we can describe l with the word $b\bar{c}dc\bar{d}b$. $\diamond$

4 Classification of loops with two self-intersections on the two-torus

We now classify the free homotopy classes of loops on the once punctured two holed torus, T_2, with two self intersections. Consider any loop with two self intersections and look at a neighborhood that encloses both intersection points. The resulting figure will look like Figure 4a.

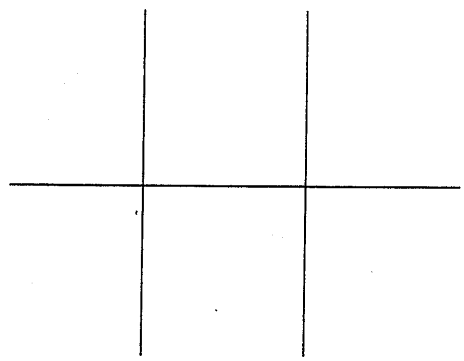


Figure 4a. A neighborhood around the two intersection points of a loop with two interections.

If we consider symmetry agruments, then the only configurations that we have to consider are displayed in Figure 4b. By a lemma stated earlier, when considering loops with two intersections, we only need to consider the compositions of three simple loops. So we collapse the configurations of Figure 4b along some axis. For configurations 1 and 2, no matter what axis we collapse along, we will get some rotation or reflection of the corresponding base point graphs 1 and 2, shown in Figure 5. For both configurations 3 and 4, no matter which axis you collapse along, a rotation or reflecion of base point graph 3, shown in Figure 5, will result. Therefore, when going through the analysis of the two intersector case, we need only consider the three base point graphs shown in Figure 5. We note that a and a' denote the initial and final segment of a single simple loop, as does b and b' and c and c'.

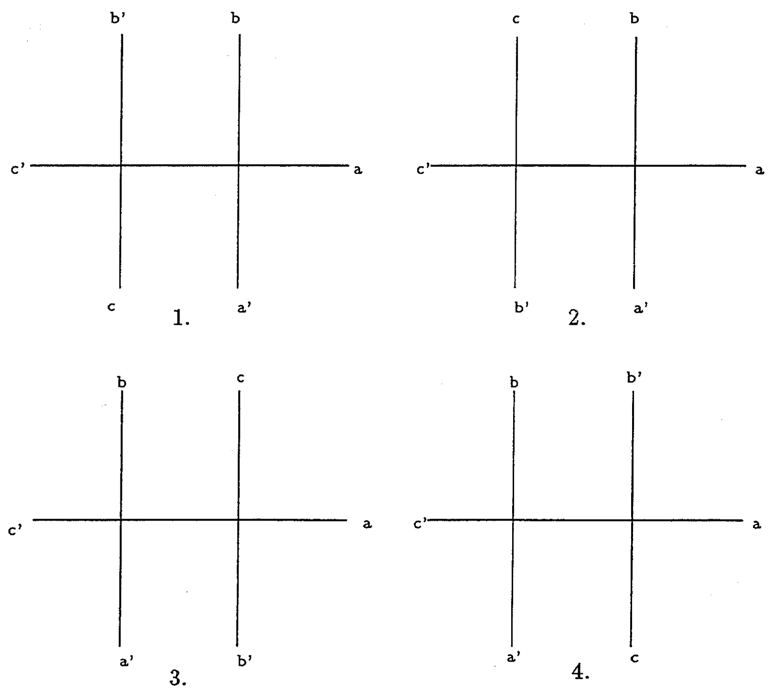


Figure 4b. The four possible configurations for loops with two intersecions.

When we consider free homotopy classes of loops on T_2 with two self intersections, we will need to consider the Euler characteristic. When you cut along the composition of three simple loops, you will produce two copies of each simple loop. This cutting will produce two additional vertices, no edges, and some number of boundary curves. In the end $\chi_{new}=\chi_{old}+2+$ #

of boundary curves, where χ_{old}=-2. Therefore, χ_{new}= # of boundary curves.

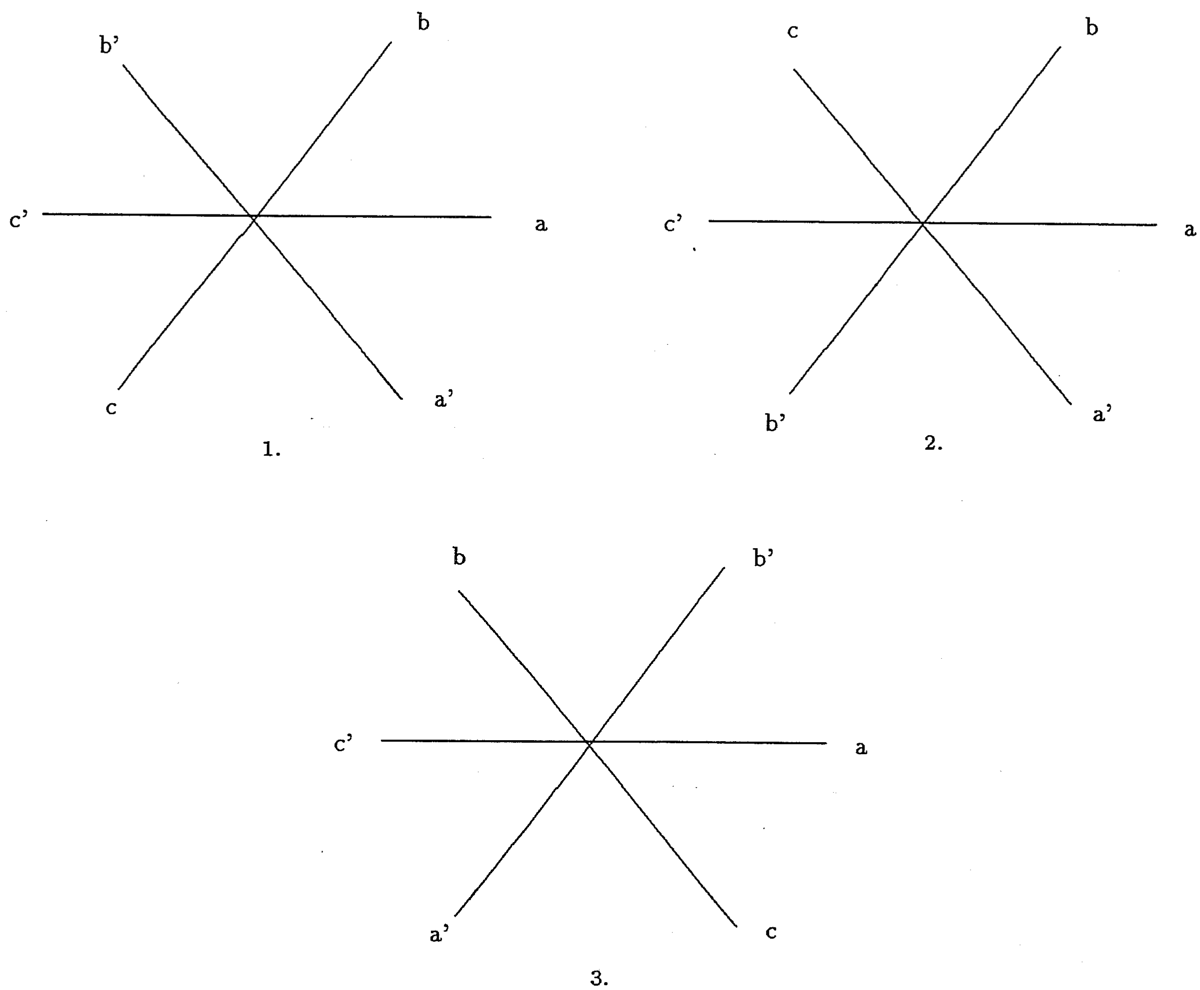


Figure 5. The corresponding base point graphs that need to be considered in the analysis of loops with two intersecions.

Consider basepoint graph 1 of Figure 5. After connecting the initial and final segments correctly and cutting along each of these simple loops, Figure 6a will result.

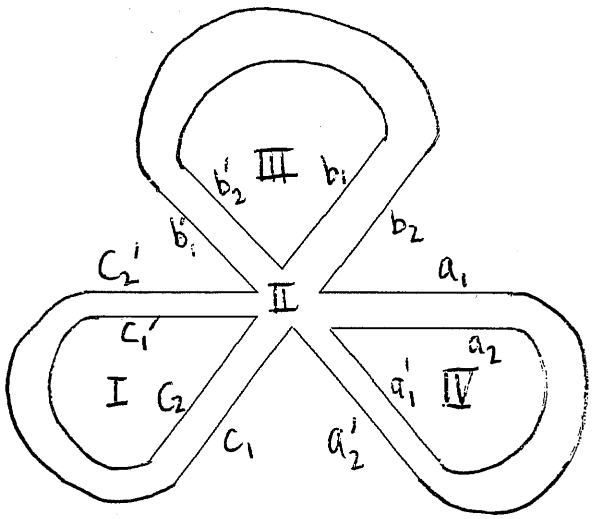


Figure 6a. The configuration corresponding to base point graph 1 after cutting along the simple loops.

After cutting along the simple loops a total of twelve loop boundaries will result. We define each boundary curve by starting at one loop's initial point, and follow the loop's paths through the two copies of the loops. noting which of the other segments we cross over in the order we cross over them, until we come back to the segment that we started with. We name each of the other boundary curves in this fashion until all the segments have been accounted for. We note that each segment will be used once. We have the following boundary curves for base point graph 1: $C_1 = c_1' \rightarrow c_2$, $C_2 = c_1 \rightarrow c_2' \rightarrow b_1' \rightarrow b_2 \rightarrow a_1 \rightarrow a_2'$, $C_3 = b_2' \rightarrow b_1$, and $C_4 = a_2 \rightarrow a_1'$. There are a total of four boundary curves, therefore, χ_{new}=4. We also note that four regions are defined by these boundary curves. We let region I be enclosed by C_1, region II be enclosed by C_2, region III be enclosed by C_3, and region IV be enclosed by C_4.

Consider basepoint graph 2 of Figure 5. After connecting the initial and final segments correctly and cutting along each of these simple loops, Figure 6b will result.

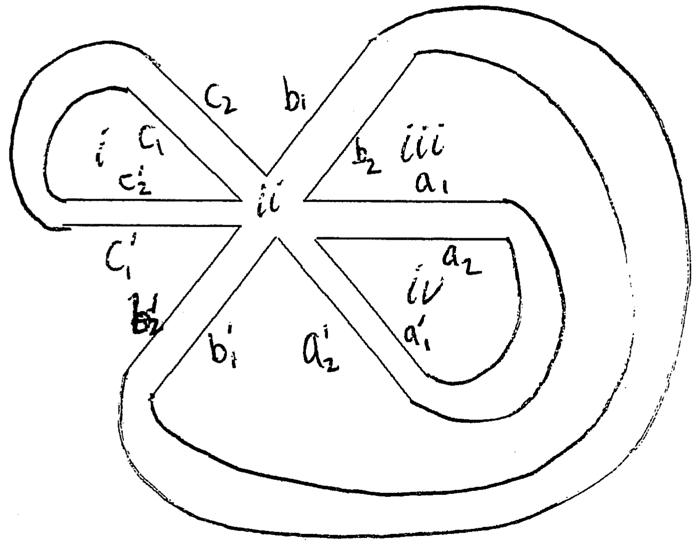


Figure 6b. The configuration corresponding to basepoint graph 2 after cutting along the simple loops.

We have the following boundary curves for base point graph 2: $C_1 = c_1 \rightarrow c_2'$, $C_2 = c_1' \rightarrow c_2 \rightarrow b_1 \rightarrow b_2'$, $C_3 = b_2 \rightarrow b_1' \rightarrow a_2' \rightarrow a_1$, and $C_4 = a_2 \rightarrow a_1'$. There are four boundary curves, therefore, χ_{new}=4. We also note that four regions are defined by these boundary curves. We let region i be enclosed by C_1, region ii be enclosed by C_2, region iii be enclosed by C_3, and region iv be enclosed by C_4.

Consider basepoint graph 3 of Figure 5. After connecting the initial and final segments correctly and cutting along each of these simple loops, Figure 6c will result.

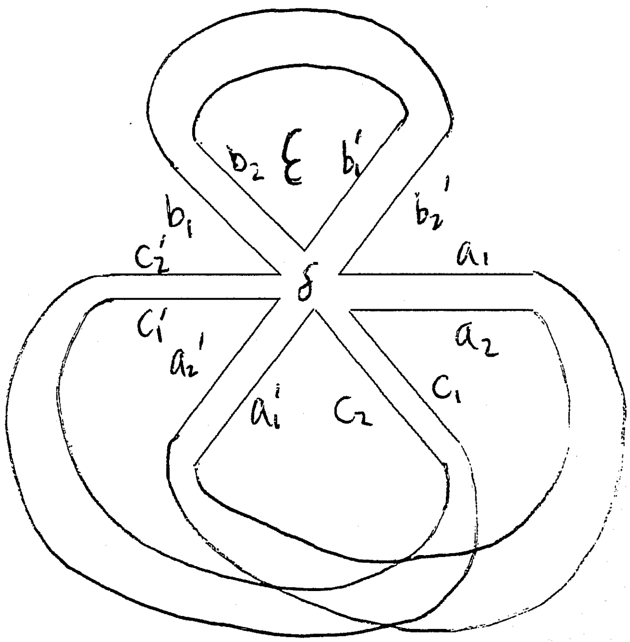


Figure 6c. The configuration corresponding to basepoint graph 3 after cutting along the simple loops.

We have the following boundary curves for base point graph 3: $C_1 = b_2 \rightarrow b_1'$ and $C_2 = b_1 \rightarrow b_2' \rightarrow a_1 \rightarrow a_2' \rightarrow c_1' \rightarrow c_2 \rightarrow a_1' \rightarrow a_2 \rightarrow c_1 \rightarrow c_2'$. There are a total of two boundary curves, therefore, χ_{new}=2. We also note that two regions are defined by these two boundary curves. We let region ϵ be enclosed by C_1 and region δ be enclosed by C_2.

In our analysis, we outline two approaches. In the first approach we consider the base point graphs of Figure 5. We notice that in all three cases we have an initial and final segment of a single simple loop right next to each other. We let that curve be given as one of our simple loops. We then analyze each base point graph and the regions that are defined by the boundary curves by considering every possible combination of surfaces that give us the desired Euler characteristic.

4.1 The First Approach

4.1.1 Let the given curve be a λ or a $\bar{c}dc\bar{d}$ curve

Let the given curve be either a λ curve or a $\bar{c}dc\bar{d}$ curve. We note that no other simple loop can enclose either of these curves, since this would create too many intersections. For the first case, we consider basepoint graph 1. For the second case we consider basepoint graph 2 and for case 3 we consider basepoint graph 3. In each of the cases, the initial and final segments of the other undetermined simple loops will be set, as defined by the configuration of each of the base point graphs considered.

Case 1. We now consider basepoint graph 1. We let the given curve start at c and end at c'. We let the simple loop that starts at b and ends at b' be l_2 and the simple loop that starts at a and ends at a' be l_3. The actual orientations of these loops will be considered later. Refer to Figure 3a and the explination of the boundary curves. Recall we have χ_{new}=4. We note that C_1 consists of a copy of either the λ curve or the $\bar{c}dc\bar{d}$ curve, C_2 consists of copies from all three simple loops, C_3 consists of a copy of l_2, and C_4 consists of a copy of l_3. We know from the one intersecting case that λ seperates off a 1-torus from T_2, which has a χ=0. So region I is a 1-torus. This leaves us with the following subcases:

1. region II be a sphere, region III be a sphere, region IV be a 1-torus.
2. region II be a sphere, region III be a 1-torus, region IV be a sphere.
3. region II be a 1-torus, region III be a sphere, region IV be a sphere.
4. regions II and III be a single sphere, region IV be a sphere.
5. regions II and IV be a single sphere, region III be a sphere.
6. regions III and IV be a single sphere, region II be a sphere.

*Subcase*1 Let region II be a sphere, region III be a sphere, and region IV be a 1-torus. We have yet to introduce which region the puncture is located. In this subcase, the puncture must be located in region III, otherwise l_2 would bound a disc and would be homotopic to a single point. This implies that the given loop is a $\bar{c}dc\bar{d}$ curve. Prior to adding a face to region III, we have a punctured disc with boundary curve C_3. So, l_2 must bound a punctured disc which we have shown is homotopic to Δ. We are left with regions II and IV. Since these are seperate surfaces, l_3 must be a seperating loop. Consider region IV. Prior to adding a face to region IV, we have a 1-torus with boundary curve C_4. So l_3 must have seperated off a 1-torus with boundary curve C_4, which consists of a single copy of l_3. We have shown that this type of loop is homotopic to λ. We note that with this combination of simple loops, region II will be a sphere. In order to have only two transverse intersections, a word that describes this loop is $\bar{c}dc\bar{d}\bar{\Delta}\lambda$.

*Subcase*2 Let region II be a sphere, region III be a 1-torus, and region IV be a sphere. This argument is the same as *Subcase*1. Since we have the same number of boundary curves, Euler characteristic, and orientability, there is a homeomorphism of T_2 that will take this loop to the loop described in *Subcase*1.

*Subcase*3 Let region II be a 1-torus, region III be a sphere, and region IV be a sphere. This case cannot happen. Consider regions III and IV. Prior to adding a face to both of these regions, we have two discs, one with boundary curve C_3 and the other consisting of C_4. Now we have only one puncture, so either l_2 or l_3 will bound a disc and will therefore be homotopic to a single point. This would produce a loop with only one transverse intersection.

*Subcase*4 Let regions II and III be a single sphere and region IV be a sphere. In this case, the puncture must be in region IV, otherwise l_3 would be homotopic to a single point. This implies that the given curve is a $\bar{c}dc\bar{d}$ curve. Following the same argument as in *Subcase*1, l_3 is homotopic to Δ. We are left with regions II and III. Since we have a single sphere, l_2 must be a non-seperating loop. Recall back to case 3 of the single intersecting case when the given cut was λ. It was shown that a non-seperating loop on a 1-torus missing a disc is also a non-seperating loop on T_2. It was also shown that the λ cut did not effect the non-seperating curve, that is, the non-seperating curve on T_2 is still non-seperating after a λ cut is made. We claim that after a $\bar{c}dc\bar{d}$ and Δ cut on T_2, a non-seperating cut will still be non-seperating. Consider T_2. After making a $\bar{c}dc\bar{d}$ cut you will be left with a 1-torus missing a disc. This 1-torus has a boundary component consisting of a copy of the $\bar{c}dc\bar{d}$ curve. After making a Δ cut you will be left with a 1-torus missing a disc. This 1-torus has a boundary component consisting of copies from both the $\bar{c}dc\bar{d}$ and Δ curves. So after both cuts, we will be left with a 1-torus missing a disc. Therefore, a non-seperating cut on the resulting 1-torus missing a disc will be non-seperating on T_2. It was shown that any non-seperating loop was homeomorphic to the curve b. In order to have only two transverse intersections, a word that describes this loop is $\bar{c}dcd\bar{\Delta}\bar{b}$.

*Subcase*5 Let regions II and IV be a single sphere and region III be a sphere. The argument for this subcase is the same as that done in *Subcase*4. Since we have the same number of boundary curves, Euler characteristic, and orientability, there is a homeomorphism of T_2 that will take this loop to the loop described in *Subcase*4.

*Subcase*6 Let regions III and IV be a single sphere and region II be a sphere. Recall that regions III and IV are both bounded by a single copy of one of the simple loops. Since the two regions are connected to form a single sphere, the two simple loops must be non-seperatng ones. We note that since regions III and IV are seperate from region II, l_2 and l_3 do not form a generating pair. In fact, if you consider l_2 and l_3 as seperate curves, one of the curves be freely homotopic to the other, assuming the puncture does not get in the way. Prior to adding the faces to regions III and IV, we must have had a cylinder with two boundary components. Up to homeomorphism, the only way for this combination of surfaces to arise is shown in Figure 7.

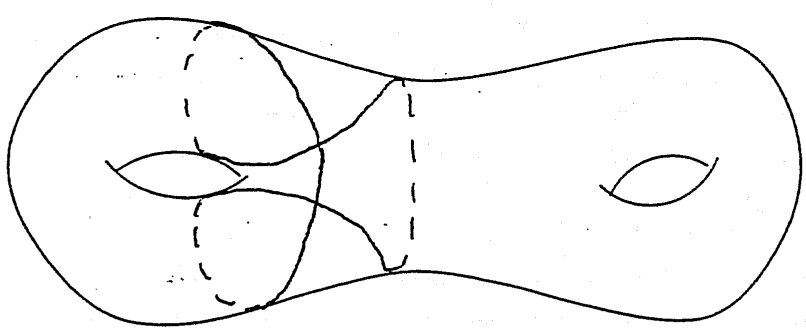


Figure 7.

The only question is where the puncture is located. It can be on region I, region II, or on regions III and IV. Consider the puncture on region I. In order for there to be two transverse intersections, a word that describes this loop is $b\bar{a}\bar{b}\bar{b}ab$. Consider the puncture on region II. In order for there to be two transverse intersections, a word that describes this loop is $\bar{c}dc\bar{d}\bar{b}ab\bar{a}$.

Consider the puncture on regions III and IV. In order for there to be two transverse intersections, a word that describes this loop is $\bar{c}dcd\bar{b}\bar{c}dcd\bar{b}$.

Case 2. We next consider basepoint graph 2. We let the given curve start at c and end at c'. We let the simple loop that starts at b and ends at b' be l_2 and the simple loop that starts at a and ends at a' be l_3. The actual orientations of these loops will be considered later. Refer to Figure 6b and the explination of the boundary curves. Recall we have χ_{new}=4. We note that C_1 consists of a copy of either the λ curve or the $\bar{c}dc\bar{d}$ curve, C_2 consists of copies from the λ curve or the $\bar{c}dc\bar{d}$ curve and l_2, C_3 consists of copies of l_2 and l_3, and C_4 consists of a copy of l_3. We know from the one intersecting case that λ seperates off a 1-torus from T_2, which has a χ=0. So region i is a 1-torus. This leaves us with the following subcases:

1. region ii be a sphere, region iii be a sphere, region iv be a 1-torus.

2. region ii be a sphere, region iii be a 1-torus, region iv be a sphere.

3. region ii be a 1-torus, region iii be a sphere, region iv be a sphere.

4. regions ii and iii be a single sphere, region iv be a sphere.

5. regions ii and iv be a single sphere, region iii be a sphere.

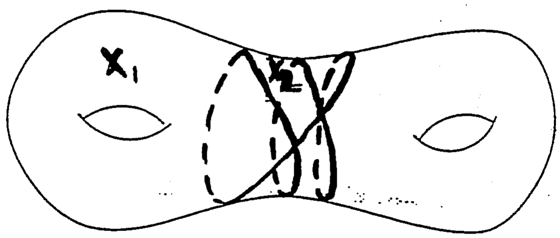
6. regions iii and iv be a single sphere, region ii be a sphere.

The analysis of these subcases are all similar to that done for Case 1. So what follows is a list of our results accompanied with a picture of the loops that produce each subcase.

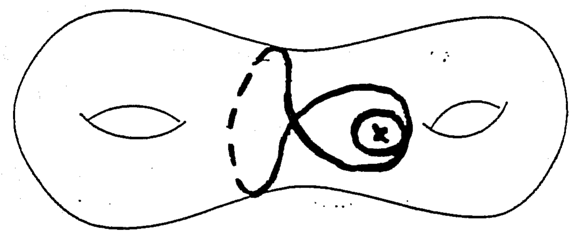
Case 3. We next consider basepoint graph 3. We let the given curve start at b and end at b'. We let the simple loop that starts at a and ends at a' be l_2 and the simple loop that starts at c and ends at c' be l_3. Refer to Figure 6c and the explanation of the boundary curves. Recall we have χ_{new}=2. We note that C_1 consists of a copy of λ or $\bar{c}dc\bar{d}$ and C_2 contains copies of all three simple loops. We know that a λ or $\bar{c}dc\bar{d}$ curve seperates off a 1-torus from the two holed torus, so region ϵ is a 1-torus. This implies that region δ is a shpere.

The analysis of this case is similar to that done for Case 1. So what follows is a list of our results accompanied with pictures of the loops that produced this case.

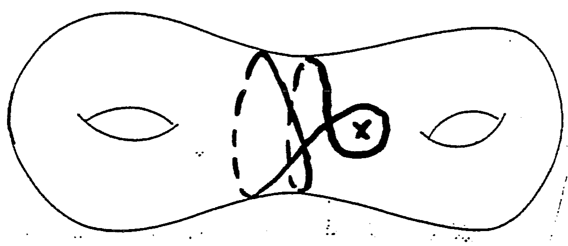
x_1-λ^3, x_2-$\bar{c}dc\bar{d}\lambda^2$



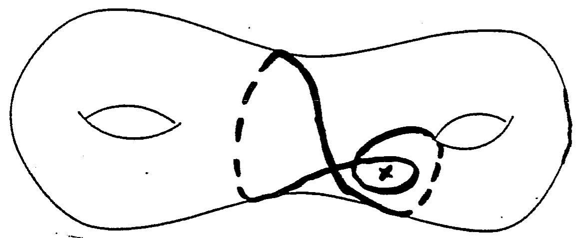
$\bar{c}dc\bar{d}\Delta^2$



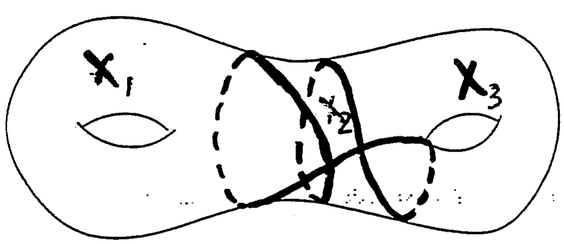
$\bar{c}dc\bar{d}\Delta\bar{c}dc\bar{d}$



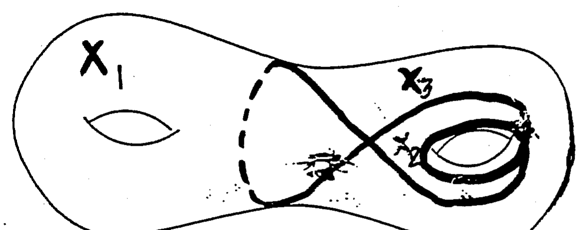
$\bar{c}dc\bar{d}\Delta\bar{b}$



x_1-$\lambda\bar{b}\lambda$, x_2-$\bar{c}dc\bar{d}\bar{b}\lambda$
x_3-$\bar{c}dc\bar{d}\bar{b}\bar{c}dc\bar{d}$

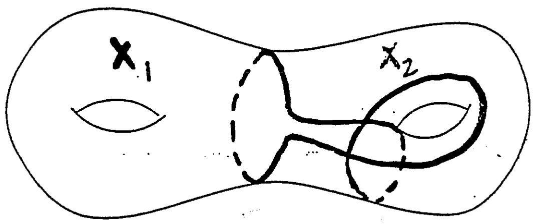


x_1-λa^2, x_2-$\bar{c}dc\bar{d}a\Delta a$
x_3-$\bar{c}dc\bar{d}a^2$



List 1. The list of results for Subsection 4.1.1, Case 2.

x_1-$\lambda\bar{a}b$, x_2-$\bar{c}dc\bar{d}\bar{a}b$



List 2. The list of results for Subsection 4.1.1, Case 3.

4.1.2 Let the given curve be a Δ curve

Let the given curve Δ. Refer to Figure 5. In Case 1 we consider basepoint graph 1. In Case 2 we consider basepoint graph 2 and in Case 3 we consider basepoint graph 3. In each of the cases, the initial and final segments of the other undetermined simple loops will be set, as defined by the configuration of each of the base point graphs considered.

Case 1. We consider base point graph 1. We let the given curve start at c and end at c'. We let the simple loop that starts at b and ends at b' be l_2 and the simple loop that starts at a and ends at a' be l_3. The actual orientations of these loops will be considered later. Refer to Figure 6a and the explination of the boundary curves. Recall we have χ_{new}=4. We note that C_1 consists of a copy of the Δ curve, C_2 consists of copies from all three simple loops, C_3 consists of a copy of l_2, and C_4 consists of a copy of l_3. We know from the one intersecting case that Δ seperates off a sphere from T_2, which has a χ=2. So region I is a sphere. In fact, region I will end up being a punctured sphere. This leaves us with the following subcases:

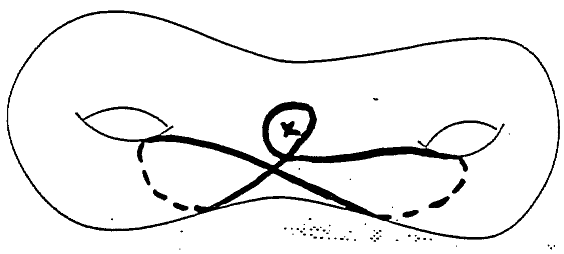
1. region II be a sphere, region III be a sphere, region IV be a 1-torus.

2. region II be a sphere, region III be a 1-torus, region IV be a sphere.

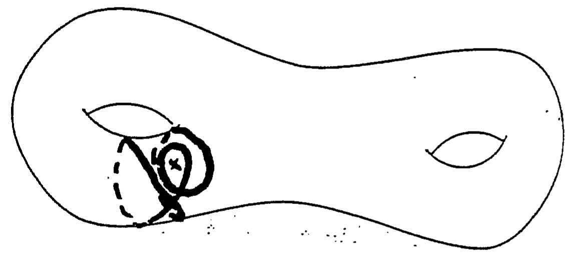
3. region II be a 1-torus, region III be a sphere, region IV be a sphere.

4. regions II and III be a single sphere, region IV be a sphere.

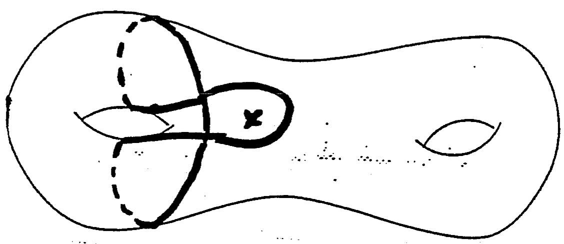
$\bar{d}\bar{\Delta}\bar{b}$



$\bar{d}\bar{\Delta}^2\bar{d}$



$\bar{d}\bar{c}ddc\bar{d}ba\bar{b}\bar{a}$



List 3. The list for Subsection 4.1.2, Case 1.

5. regions II and IV be a single sphere, region III be a sphere.
6. regions III and IV be a single sphere, region II be a sphere.
7. regions II, III, and IV be a single sphere.

The analysis of these subcases are all similar to that done for Case 1 when the given curve was either a λ or a $\bar{c}dc\bar{d}$ curve. So what follows is a list of our results accompanied with a picture of the loops that produce each subcase. We also note that any case that is a duplication of a case where a λ or a $\bar{c}dc\bar{d}$ curve was the given curve, is omitted.

Case 2. We next consider basepoint graph 2. We let the given curve start at c and end at c'. We let the simple loop that starts at b and ends at b' be l_2 and the simple loop that starts at a and ends at a' be l_3. The actual orientations of these loops will be considered later. Refer to Figure 6b and the explination of the boundary curves. Recall we have χ_{new}=4. We note that C_1 consists of a copy of the Δ curve, C_2 consists of copies from the Δ curve and l_2, C_3 consists of copies of l_2 and l_3, and C_4 consists of a copy of l_3. We have the following subcases:

1. region *ii* be a sphere, region *iii* be a sphere, region *iv* be a 1-torus.

2. region *ii* be a sphere, region *iii* be a 1-torus, region *iv* be a sphere.

3. region *ii* be a 1-torus, region *iii* be a sphere, region *iv* be a sphere.

4. regions *ii* and *iii* be a single sphere, region *iv* be a sphere.

5. regions *ii* and *iv* be a single sphere, region *iii* be a sphere.

6. regions *iii* and *iv* be a single sphere, region *ii* be a sphere.

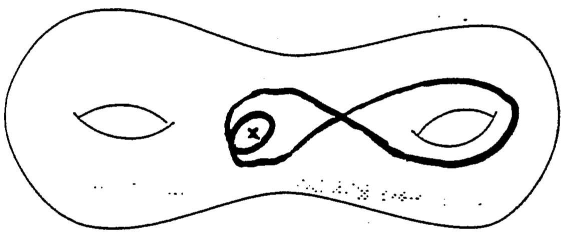
7. regions *ii*, *iii*, and *iv* be a single sphere.

The analysis of these subcases are all similar to that done for Case 1 when the given curve was either a λ or a $\bar{c}dc\bar{d}$ curve. So what follows is a list of our results accompanied with a picture of the loops that produce each subcase. We also note that any case that is a duplication of a case where a λ or a $\bar{c}dc\bar{d}$ curve was the given curve, is omitted.

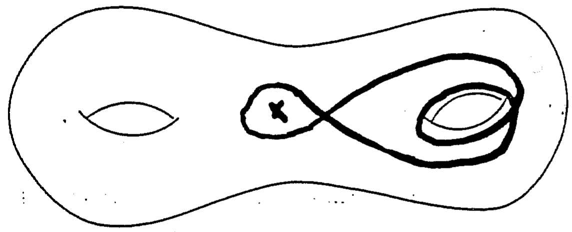
Case 3. We next consider basepoint graph 3. We let the given curve start at b and end at b'. We let the simple loop that starts at a and ends at a' be l_2 and the simple loop that starts at c and ends at c' be l_3. Refer to Figure 3c and the explination of the boundary curves. Recall we have χ_{new}=2. We note that C_1 consists of a copy of Δ and C_2 contains copies of all three simple loops. We know that a Δ curve seperates off a sphere from the two holed torus, so region ϵ is a sphere. This implies that region δ is a 1-torus.

The analysis of these subcases are all similar to that done for Case 1 when the given curve was either a λ or a $\bar{c}dc\bar{d}$ curve. So what follows is a list of our results accompanied with a picture of the loops that produce each subcase. We also note that any case that is a duplication of a case where a λ or a $\bar{c}dc\bar{d}$ curve was the given curve, is omitted.

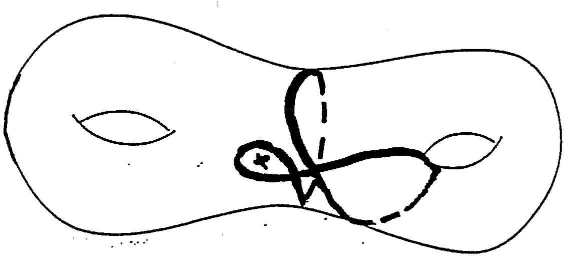
$a\bar{\Delta}^2$



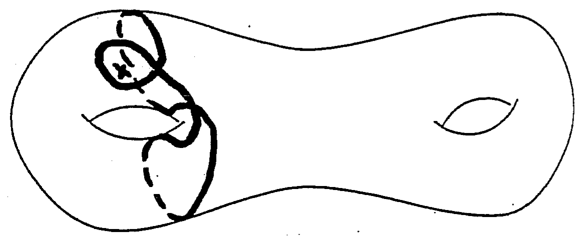
$a^2\bar{\Delta}$



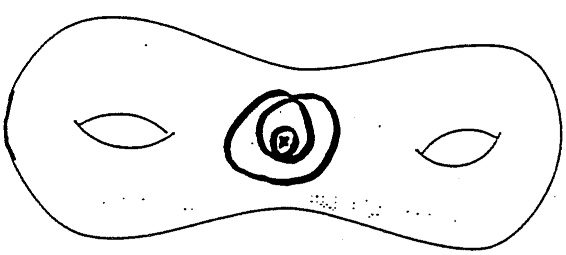
$\Delta\bar{\lambda}b$



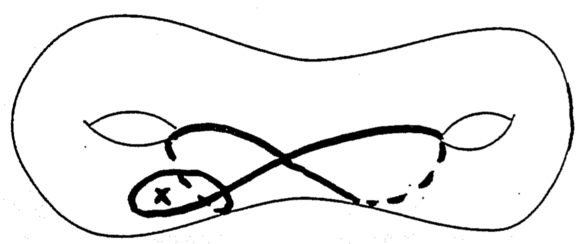
$\bar{d}^2\bar{\lambda}\Delta$



Δ^3

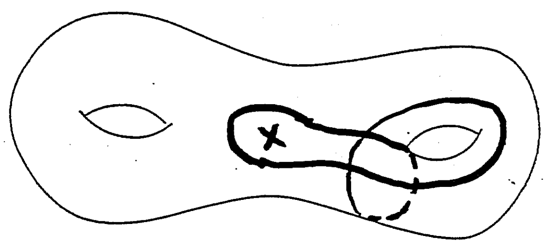


$\Delta d\Delta b$



List 4. The list for Subsection 4.1.2, Case 2.

$a\Delta\bar{b}$



List 5. The list for Subsection 4.1.2, Case 3.

4.1.3 Let the given curve be a non-seperating one.

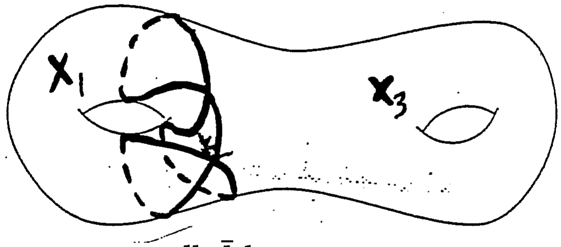
This section follows the exact same format as all the other cases. So what follows is a list of our results accompanied with a picture of the loops that produce each subcase. We also note that any case that is a duplication of a case where a λ or a $\bar{c}dc\bar{d}$ curve was the given curve or a Δ curve was the given curve is omitted.

4.2 Classification of loops with two self-intersections on the two-torus: approach 2

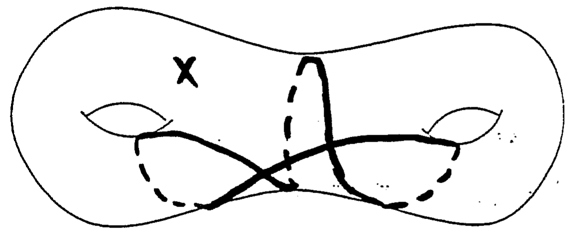
4.2.1 Technique

An alternative approach may be used to classify the loops with two self-intersections on T_2. This second approach also relies on the fact that there are only three basepoint graphs that we must consider. We note that each of the three basepoint graphs arises from a one-intersector case, ie. in each case we may remove one of the loops to obtain a loop with a single intersection. Since each of the possible basepoint graph configurations arises from a once self-intersecting loop, we will be able to identify all of the classes of loops with two-intersections by adding all possible simple loops to each of the classes of loops with single intersection that we have identified in the previous section. However, if this method is used alone we will find loops which are equivalent (under homeomorphisms) arise from multiple one-intersectors. In order to

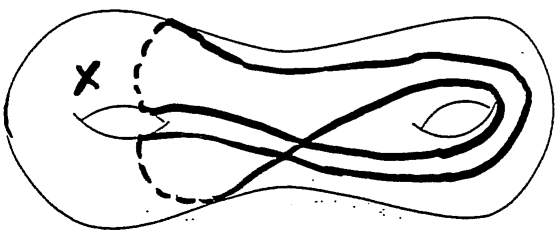
x_1-$dd\lambda d$, x_2-$dd\bar{c}ddc\bar{d}ba\bar{b}\bar{a}$
x_3-$dd\bar{c}dc$



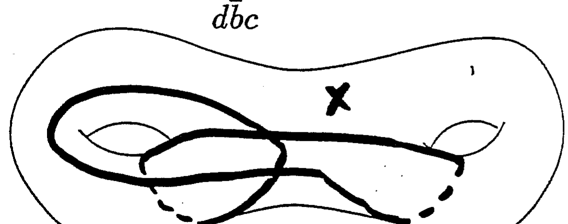
$d\bar{\lambda}b$



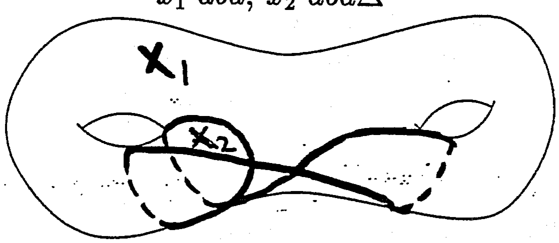
$db\bar{a}\bar{b}d\bar{a}$



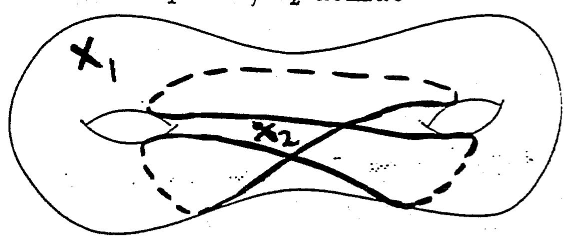
$d\bar{b}c$



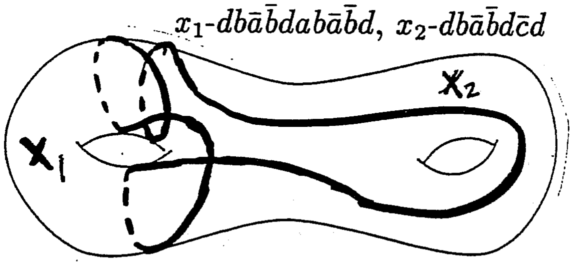
x_1-dbd, x_2-$dbd\Delta$



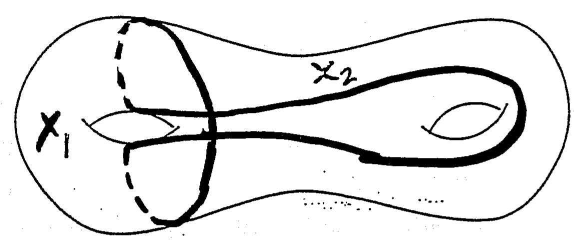
x_1-$db\bar{d}b$, x_2-$db\bar{\Delta}\bar{d}b$



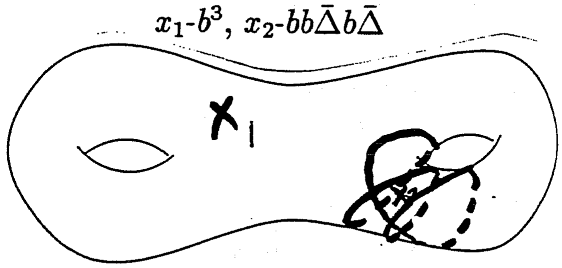
x_1-$db\bar{a}\bar{b}dab\bar{a}\bar{b}d$, x_2-$db\bar{a}\bar{b}d\bar{c}d$



x_1-$d\bar{a}\overline{d\lambda}$, x_2-$d\bar{a}\bar{c}\bar{d}c$



x_1-b^3, x_2-$bb\bar{\Delta}b\bar{\Delta}$



List 6. The list for Subsection 4.1.3.

eliminate cases which are equivalent we will look at the Euler characteristics of the resulting surfaces, using an approach similar to the one-intersector case. We rely on the fact that surfaces are equivalent up to homeomorphisms if they have the same number and type of boundary components and if they have the same Euler characteristic. We know that the Euler characteristic of all of the surfaces that we create by cutting along the loops must be 4 or 2, depending on whether the loop corresponds to basepoint graph 1 and 2 or basepoint graph 3, respectively. Considering all combinations of surfaces in the regions that arise after cutting along the loops for each basepoint graph insures that all possible classes of loops will be considered (up to the placement of the puncture).

Certain generalizations limit the possibilities for the combinations of surfaces in each of the regions for a given basepoint graph. For the first two basepoint graphs we may have four, three, or two surfaces after cutting along the given loop, and in the third basepoint graph we may have one or two surfaces. In the first case, we may not have one surface since that would imply that the Euler characteristic of the single surface must be 4, which is impossible. Also, the regions are still defined such that α and γ are replaced by any region with a single loop as boundary, and β corresponds to any region with a single boundary comprised of two loops. An analysis of the possible combinations of types and placement of surfaces for the regions of a given basepoint graph yields that there are 44 topologically distinct loops. We will demonstrate the basic aproach with a few sample cases, some of which are equivalent to those which were discussed in 4.1. We will close this section with a table listing all of the cases that we have found through this method and the method outlined in 4.1.

4.2.2 A demonstration of the approach

We will now explain our approach in detail for the two-intersector cases which arise from the once-intersecting loops with the name $\lambda\bar{\Delta}$. In our analysis of the one-intersectors, we determined the topological types of the surfaces that remained after we cut along the simple loops and in the case of $\lambda\bar{\Delta}$ we are left with a punctured sphere, a one-holed torus with one boundary component from a single loop, and a one-holed torus with one boundary component from two loops. In order to determine the combinations of simple loops we may add to the one-intersector, we need only consider simple loops which lie entirely on one of the three surfaces which remained after our procedure of cutting. If the simple loop we add lies on more than one surface the loop must be homotopic to a loop on a single surface. If the loop we consider adding is not homotopic to a loop which lies solely on one surface, the point at which the loop crosses between the surfaces will increase the intersection number, so we will be left with a loop with more than two self-intersection. Now, we need only consider the topologically distinct loops on each of the three surfaces that arise from the cuts of our one-intersector, $\lambda\bar{\Delta}$.

We will first consider the case where the loop is added on the one-holed torus with a single boundary from one loop. Using an Euler characteristic argument, we can demonstrate that there are only two topologically distinct simple loops on the one-holed torus with a missing disc: a separating loop and a non-separating loop. The separating loop produces a one-holed torus with a single boundary from one loop and a sphere with single boundary from two loops. Because of where the puncture is the boundary loops of the sphere must be homotopic. The name for this combination of three simple loops is $\lambda\bar{\Delta}\lambda$. This loop corresponds to basepoint graph 2 where region *i* is a punctured sphere with one boundary from a single loop, region *ii* is a 1-torus with one boundary from two loops, region *iii* is a sphere with one boundary from two loops, and region *iv* is a 1-torus with a boundary from a single loop. There is a homeomorphism of T_2 that takes this loop to $\bar{c}dc\bar{d}\Delta\bar{c}dc\bar{d}$, which was found in the first approach. If the added loop is non-separating, it will correspond to a non-separating loop on T_2. We have already demonstrated this correspondence with non-separators on T_2 in the discussion of the once-intersecting loops, and since the non-separator lies on the ab torus, we may assume the simple loop to be b. The name for this loop will be $\lambda\bar{b}\bar{\Delta}$. This loop corresponds to basepoint graph 2 where region *i* is a punctured sphere with one boundary from a single loop, region *ii* is a 1-torus with one boundary from two loops, region *iii* and *iv* form a single sphere with two boundaries, one from a single loop and the other from two loops. After comparing this loop to those found in the first approach, we see that this loop is in the free homotopy class of $\Delta\bar{\lambda}b$.

Next we will consider the case where the loop added is on the punctured disc. There is only one possibility for this loop: it must separate off a punctured disc from a sphere with a single boundary which is comprised of two loops. Thus the loop we add must be homotopic to the loop that bounds our punctured disc, that is the Δ loop. The name for the composition of these two homotopic loops and λ is $\lambda\bar{\Delta}\bar{\Delta}$. This loop corresponds to basepoint graph 2 where region i is a punctured sphere with one boundary from a single loop, region ii is a sphere with one boundary from two loops, region iii is a 1-torus with one boundary from two loops, and region iv is a 1-torus with a boundary from a single loop. Note that there is a homeomorphism of T_2 that takes this loop to $\bar{c}dc\bar{d}\Delta^2$, which was found in the first approach.

The final surface on which we may add a loop is the one-holed torus with a single boundary from two loops. For this torus there are four topologically distinct loops that may be added: one of which is non-separating, and the other three are separating. As in the torus with a single boundary which arises from one loop, the non-separating loop corresponds to a loop which is non-separating on T_2. Since this non-separating loop will be added on the cd torus, we may assume the loop to be d. The name of the loop will be $\lambda\bar{\Delta}\bar{d}$. The loop corresponds to basepoint graph 2 where region i is a punctured sphere with one boundary from a single loop, region ii and iii form a single sphere with two boundaries both from two loops, and region iv is a 1-torus with a boundary from a single loop. There is a homeomorphism of T_2 that takes this loop to $\bar{c}dc\bar{d}\Delta\bar{b}$, which was found in the first approach. There are three possible configurations for separating loops; however, two of the possibilities have already been considered, namely the two loops which separate a sphere with a single boundary which consists of two loops. In these two cases, the loop which is added must be homotopic to one of the two loops which created the boundary of the torus since the puncture lies on a different surface. The case in which the added loop is homotopic to λ had been considered when the loop was added to the ab torus, and the case where a loop is added on the punctured sphere corresponds to the case on the cd torus where the added loop is homotopic to Δ. The final separating case that must be considered, then, is the loop which separates off a one-holed torus with one boundary from a single loop and a sphere with a single boundary from three loops. This loop corresponds to basepoint graph 1 where region I is a 1-torus with one boundary from a single loop, region II is a sphere with one boundary from 3 loops, region III is a punctured sphere with one boundary from a single loop, and region IV is a 1-torus with one boundary from a single loop. The name for this loop is $\lambda\bar{c}dc\bar{d}\bar{\Delta}$. This loop is the same as $\bar{c}dc\bar{d}\bar{\Delta}\lambda$ that was found in the first approach.

A similar analysis may be followed for each of the remaining once-intersecting loops which are identified in 3.1. This analysis yields the 44 cases which appear in the following table. As an example of how to read the table, consider number 5. A punctured 1-torus–1bd,1 loop, is read a punctured one hole torus with one boundary consisting of one copy of a simple loop. A sphere–1bd,1 loop..1bd,1 loop, is read a sphere that has two boundaries each consisting of one copy of a simple loop.

Loops With Two Intersecions on T			
	Given Curve	*Word*	*Components*
1	λ	$\bar{c}dcd\Delta\lambda$	1-torus–1bd,1 loop; sphere–1bd,3 loops punctured sphere–1bd,1 loop; 1-torus–1bd,1 loop
2	$\vdots$	$\bar{c}dcd\bar{\Delta}\bar{b}$	1-torus–1bd,1 loop; punctured sphere–1bd,1 loop sphere–1bd,1 loop..1bd,3 loops
3		$d\lambda d\lambda$	1-torus–1bd,1 loop; sphere–1bd,3 loops punctured sphere–1bd,1 loop..1bd,1 loop
4		$d\bar{c}dc\lambda$	1-torus–1bd,1 loop; punctured sphere–1bd,3 loops sphere–1bd,1 loop..1bd,1 loop
5		$d\bar{c}ddcd$	punctured 1-torus–1bd,1 loop; sphere–1bd,3 loops sphere–1bd,1 loop..1bd,1 loop
6		$\lambda\bar{a}b$	punctured 1-torus–1bd,1 loop; sphere–1bd,5 loops
7		$\bar{c}dcd\bar{a}b$	1-torus–1bd,1 loop; punctured sphere–1bd,5 loops
8		λ^3	punctured 1-torus–1bd,1 loop; sphere–1bd,2 loops sphere–1bd,2 loops; 1-torus–1bd,1 loop
9		$\bar{c}dcd\lambda^2$	1-torus–1bd,1 loop; punctured sphere–1bd,2 loops sphere–1bd,2 loops; 1-torus–1bd,1 loop
10		$\bar{c}dcd\Delta^2$	1-torus–1bd,1 loop; punctured sphere–1bd,1 loop sphere–1bd,2 loops; 1-torus–1bd,2 loops
11		$\bar{c}dcd\Delta\bar{c}dcd$	1-torus–1bd,1 loop; sphere–1bd,2 loops punctured sphere–1bd,1 loop; 1-torus–1bd,2 loops
12		$\bar{c}dcd\Delta b$	1-torus–1bd,1 loop; punctured sphere–1bd,1 loop sphere–1bd,2 loops..1bd,2 loops
13		$\lambda b\lambda$	punctured 1-torus–1bd,1 loop sphere–1bd,2 loops; sphere–1bd,1 loop..1bd,2 loops
14		$\bar{c}dcdb\lambda$	1-torus–1bd,1 loop; punctured sphere–1bd,2 loops sphere–1bd,1 loop..1bd,2 loops

Loops With Two Intersecions on T (cont.)			
	Given Curve	*Word*	*Components*
15	λ	$\bar{c}dcdb\bar{c}dc\bar{d}$	1-torus–1bd,1 loop; sphere–1bd,2 loops punctured sphere–1bd,1 loop..1bd,2 loops
16	$\vdots$	λa^2	punctured 1-torus–1bd,1 loop sphere–1bd,2 loops; sphere–1bd,1 loop..1bd,2 loops
17		$\bar{c}dcda\bar{c}dcdbab$	1-torus–1bd,1 loop;punctured sphere–1bd,2 loops sphere–1bd,1 loop..1bd,2 loops
18		$\bar{c}dc\bar{d}a^2$	1-torus–1bd,1 loop; sphere–1bd,2 loops punctured sphere–1bd,1 loop..1bd,2 loops
19	Δ	$d\bar{\Delta}b$	punctured sphere–1bd,1 loop sphere–1bd,1 loop..1bd,1 loop..1bd,3 loops
20	$\vdots$	$d\Delta^2d$	punctured sphere–1bd,1 loop sphere–1bd,3 loops; 1-torus–1bd,1 loop..1bd,1 loop
21		$d\bar{c}ddc\bar{d}bab\bar{a}$	punctured sphere–1bd,1 loop sphere–1bd,1 loop..1bd,1 loop; 1-torus–1bd,3 loops
22		$a\bar{\Delta}b$	punctured sphere–1bd,1 loop 1-torus–1bd,5 loops
23		$a\bar{\Delta}^2$	punctured sphere–1bd,1 loop; sphere–1bd,2 loops 1-torus–1bd,1 loop..1bd,2 loops
24		$a^2\bar{\Delta}$	punctured sphere–1bd,1 loop; sphere–1bd,2 loops 1-torus–1bd,1 loop..1bd,2 loops
25		$\Delta\lambda b$	punctured sphere–1bd,1 loop; 1-torus–1bd,2 loops sphere–1bd,1 loop..1bd,2 loops
26		$\bar{d}^2\lambda\Delta$	punctured sphere–1bd,1 loop; 1-torus–1bd,2 loops sphere–1bd,1 loop..1bd,2 loops
27		Δ^3	punctured sphere–1bd,1 loop; sphere–1bd,2 loops sphere–1bd,2 loops; two holed torus–1bd,1 loop

Loops With Two Intersecions on T (cont.)			
	Given Curve	*Word*	*Components*
28	Δ	$\Delta d \Delta b$	punctured sphere–1bd,1 loop sphere–1bd,1 loop..1bd,2 loops..1bd,2 loops
29	*nonsep.*	$dd\lambda d$	1-torus–1bd,2 loops; sphere–1bd,2 loops punctured sphere–1bd,1 loop..1bd,1 loop
30	⋮	$dd\bar{c}ddc\bar{d}ba\bar{b}\bar{a}$	1-torus–1bd,2 loops; punctured sphere–1bd,2 loops sphere–1bd,1 loop..1bd,1 loop
31		$dd\bar{c}dc$	punctured 1-torus–1bd,2 loops; sphere–1bd,2 loops sphere–1bd,1 loop..1bd,1 loop
32		$d\lambda b$	punctured sphere–1bd,1 loop..1bd,2 loops sphere–1bd,1 loop..1bd,2 loops
33		$d\bar{a}\lambda d\bar{a}$	punctured sphere–1bd,1 loop..1bd,2 loops sphere–1bd,1 loop..1bd,2 loops
34		$d\bar{b}c$	punctured sphere–1bd,1 loop..1bd,5 loops
35		$d\bar{b}d$	sphere–1bd,2 loops punctured sphere–1bd,1 loop..1bd,1 loop..1bd,2 loops
36		$d\bar{b}d\Delta$	punctured sphere–1bd,2 loops sphere–1bd,1 loop..1bd,1 loop..1bd,2 loops
37		$d\bar{b}db$	punctured sphere–1bd,1 loop..1bd,1 loop..1bd,1 loop sphere–1bd,3 loops
38		$db\Delta db$	sphere–1bd,1 loop..1bd,1 loop..1bd,1 loop punctured sphere–1bd,3 loops
39		$db\bar{a}bdab\bar{a}bd$	punctured sphere–1bd,1 loop..1bd,1 loop sphere–1bd,2 loops..1bd,2 loops
40		$db\bar{a}bd\bar{c}d$	sphere–1bd,1 loop..1bd,1 loop punctured sphere–1bd,2 loops..1bd,2 loops
41		$d\bar{a}d\lambda$	punctured sphere–1bd,1 loop..1bd,1 loop sphere–1bd,1 loop..1bd,3 loops

Loops With Two Intersecions on T (cont.)			
	Given Curve	*Word*	*Components*
42	*nonsep.*	$d\bar{a}\bar{c}dc$	sphere–1bd,1 loop..1bd,1 loop punctured sphere–1bd,1 loop..1bd,3 loops
43	$\vdots$	b^3	sphere–1bd,2 loops; sphere–1bd,2 loops punctured 1-torus–1bd,1 loop..1bd,1 loop
44		$bb\Delta b\Delta$	sphere–1bd,2 loops; punctured sphere–1bd,2 loops 1-torus–1bd,1 loop..1bd,1 loop

5 Generalizing the once-intersecting loop on T_n

Recall that T_n is the n-holed, once punctured torus whose fundamental group is isomorphic to the free group on $2n$ letters, $F(a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$. In this section we will limit ourselves to the cases where $n > 2$, since the cases where $n \leq 2$ has been dealt with either in previous works or in the previous section. Further, note that for convenience in dealing with T_n, the orientation of the generators on T_n differ from T_2 in the previous section (see figure 2 versus figure 3). Therefore, some of the words describing the loops are also changed in comparison to the naming scheme for the two-intersectors.

5.1 simple loops

Simple loops, nonseparating loops and separating loops are described in the introduction. The same cutting and gluing technique is used for T_n as for T_2. as in previous sections. Using this technique on T_n, we can make the following theorem:

Theorem 5.1 *On the once punctured, n-holed torus, T_n, there exists a homeomorphism that takes any simple closed loop, l, to one of the following:*

1. *a nonseparating loop, b_1,*
2. *a loop enclosing a disc,*
3. *a loop, Δ, enclosing a punctured disc, which can be described by the word $a_1b_1\bar{a}_1\bar{b}_1a_2b_2\bar{a}_2\bar{b}_2 \ldots a_nb_n\bar{a}_n\bar{b}_n$, or*

4. a loop, λ_i, which separates an i-holed torus from T_n, where $0 < i < n$. This loop can be described by the word $a_1 b_1 \bar{a}_1 \bar{b}_1 \ldots a_i b_i \bar{a}_i \bar{b}_i$.

Proof The Euler characteristic of an n-holed torus is given as $2-2n$. After cutting and separating along l, we must add two discs to remove the boundary components from the surface(s). By doing so, we have created one or two or surfaces without boundary and can therefore utilize the classification theorem for orientable surfaces to verify homeomorphic surfaces without boundary. Furthermore, if two loops create homeomorphic surfaces, then there is a homeomorphism which maps one loop onto the other. We will use this fact throughout the proof.

Recall that when cutting one loop, l, we can create at most two surfaces, since each new surface must contain at least one of the two boundary components. Hence we have a total $\chi = 4 - 2n$ for one or two surfaces. Given these restrictions, only the following possiblities can occur:

1	a $(n-1)$-torus
2	a sphere and a n-torus,
3	a one-torus and a $(n-1)$-torus,
4	a two-torus and a $(n-2)$-torus,
$\vdots$	$\vdots$
*	a $\lfloor n/2 \rfloor$ and a $n - \lfloor (n/2) \rfloor$.

Each of these cases creates either a surface with $\chi = 4-2n$ or two surfaces whose combined Euler characteristic is $4-2n$.

Consider the first case. Since there is only one surface left after cutting l, then l must be a nonseparating loop. We claim that a nonseparating curve on the n-torus missing a disc is also a nonseparating curve on the n-torus. The analogous claim on T_2 is proved in section 2.3 3 and the generalization to tori of higher genus is trivial. Because a generator of T_n is a nonseparating loop on T_n, there is a homeomorphism which takes l to any of the $2n$ generators. Without loss of generality, assume this generator is b_1.

Next, assume that upon cutting l, we create case 2. We have a separating loop since we have divided T_n into two surfaces. First, assume that the puncture lies on the n-torus. On the sphere, there is a loop, say l_0, corresponding to a boundary component of l. Since the puncture is not on the sphere, then l_0 is contractible to a a single point, and therefore l is homotopic to the identity, or a loop bounding a disc. Now assume that the puncture lies on the sphere. The loop l_0 is no longer contractible to a single point, but it is homotopic to a loop enclosing a punctured disc. We will name such a curve Δ and describe it with the word $a_1b_1\bar{a}_1\bar{b}_1a_2b_2\bar{a}_2\bar{b}_2\dots a_nb_n\bar{a}_n\bar{b}_n$.

Consider the third thru last cases. In each of these, a separating loop divides T_n into an m-holed torus and an $(n-m)$-torus, for $m < \lfloor n/2 \rfloor$. Such a loop will be called λ_m. However, the puncture can lie on any one of these surfaces. Hence, letting the puncture lie on the a_nb_n-torus, λ_i will be a curve which creates an i-holed torus and a punctured $(n-i)$-holed torus, where $0 < i < n$, and the puncture lies on the a_nb_n-torus. In terms of the generators stated given, we will describe λ_i with the word $a_1b_1\bar{a}_1\bar{b}_1\dots a_ib_i\bar{a}_i\bar{b}_i$. Therefore, any loop which separates a T_i from a punctured $T_{[(n-i)]}$ will be equivalent to λ_i. (Note that if $i = n$ we have a punctured sphere and a n-torus which is Δ and described by the word $a_1, b_1, a_2, b_2, \dots, a_n, b_n$). Furthermore, if the puncture were to lie on T_i, which does not contain the a_nb_n-torus, then this loop is described with the word $b_na_n\bar{b}_n\bar{a}_n\dots b_{(i+1)}a_{(i+1)}\bar{b}_{(i+1)}\bar{a}_{(i+1)}$, which for convenience we will define as λ_i'.

In order to see the orientations of the above simple loops, see figure 6:

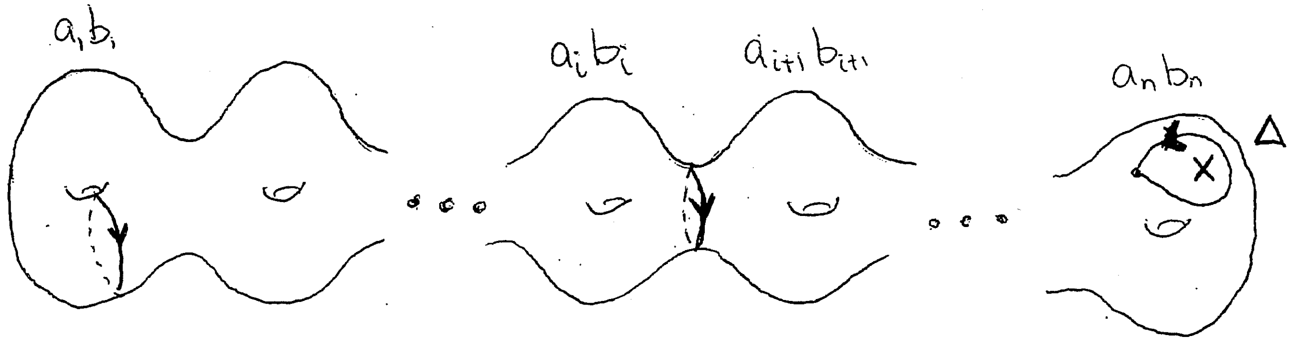


Figure 6

Note that we disregard the identity since it is contractible to a single point.

5.2 Once-Intersecting loops on T_n

Recall that on T_2, a closed loop, l with a single transverse self-intersection could be considered to be the composition of two simple loops, l_1 and l_2. This same claim can be made on T_n with the same proof [[14]] applied to the resulting n-holed torus. Therefore, we can view a loop l as two simple loops l_1 and l_2 sharing a common basepoint.

Also, recall that when cutting a loop with a single transverse intersection, we create three regions, α, which is bounded by l_1, β, bounded by l_1 and l_2, and γ, bounded by l_2. Again we may make use of claim 4.2.1 regarding these three regions on T_n with the same proof as on T_2.

With this claim established, we can now state the following theorem classifying once self-intersecting loops on T_n.

Theorem 5.2 *On T_n, for any loop l with a single transverse self-intersection which is not freely homotopic to a simple loop or a single point, there is a homeomorphism mapping l onto an equivalence class of one of the following:*

1. $\lambda_i\lambda_k$, *for* $0 < i, k < n$,
2. $\lambda_i\lambda_k'$, *for* $0 < i < n$ *and* $i \leq k \leq (n-i)$,
3. $\lambda_i\Delta$, *for* $0 < i < n$,
4. $\lambda_i b_j$, *for* $0 < i < n$ *and* $0 < j \leq n$,
5. $\Delta\Delta$,
6. Δb_1,
7. $b_1 b_1$,
8. $b_1\bar{\Delta}b_1$,
9. $b_1 b_2$,
10. $b_i\bar{\lambda}_i b_i$ *for* $0 < i \leq n$,
11. $b_{(i+1)}\bar{\lambda}_i b_{(i+1)}$, *for* $0 < i \leq n$.

Proof Recall from section 2.6 that when cutting two simple loops with a common basepoint we increase the Euler characteristic by 4 (the same argument used for T_2 applies to T_n), since we must attach discs to remove the three boundary components, and we have added a vertex. Therefore, by cutting the n-torus along l, we have altered χ so that it equals $6 - 2n$. Also, cutting along l creates three new boundary components, so there can be one, two or three new surfaces resulting from the cut. Hence, by cutting l we have one, two, or three surfaces with a sum $\chi = 6 - 2n$.

We let l, be the composition of the two simple loops l_1 and l_2. We will first consider the three possiblities for l_1: it is equivalent to λ_i, Δ, or a nonseparating curve, b_1 (Note that we will ignore any loop or combination of loops which is freely homotopic to a simple loop or the identity). Then we can analyze the three differenct possibilities for the regions α, β and γ given $\chi = 6 - 2n$ and the corresponding l_2.

First, assume l_1 is a λ_i for some $0 < i < n$. Cutting λ_i separates T_n into an i-torus, or T_i, and an $(n-i)$-torus, or $T_{(n-i)}$. Hence, l will either create a total of two or three surfaces after l_2 is cut. We can assume that $T_{(n-i)}$ includes the $a_n b_n$-torus. If this is not the case when we first cut λ_i, then there exists a homeomorphism taking T_n to T_n which would make this assumption valid.

First, let region α be T_i and l_2 lie in $T_{(n-i)}$. Recall that a loop intersecting λ_i at only one point must lie entirely to one side of λ_i. So, if l_2 lies on $T_{(n-i)}$, then it will cut that torus into one or two surfaces with a total $\chi = 2+2i-2n$. This is a straight result from the total Euler characteristic of T_n after cutting l, $6 - 2n$, minus the χ of T_i, $2 - 2i$.

For one or two regions with a total $\chi = 2 + 2i - 2n$, we have the following possibilities for the regions β and γ without boundary (recall that the puncture lies in the $(n-i)$-torus) hence it lies in either β or γ:

	β	γ
1	sphere	$(n-i)$-torus
2	$(n-i)$-torus	sphere
3	1-torus	$(n-i-1)$-torus
4	2-torus	$(n-i-2)$-torus
$\vdots$	$\vdots$	$\vdots$
*	$(n-i-1)$-torus	1-torus

or

**	both regions form a single $(n-i-1)$-torus.

First, assume case 1 occurs. If the puncture does not lie in the region β, then by the claim made before the theorem, l_2 must be homotopic to l_1. Hence, if the puncture lies in γ on the $(n-i)$-torus, then we can describe l as $\lambda_i\lambda_i$. On the other hand, if the puncture lies between l_1 and l_2, in region β, then the two loops are no longer homotopic. This case is described by the word $\lambda_i\lambda_i'$.

Next, consider case 2. First, assume the puncture is in $T_{(n-i)}$. The boundary component created by l_2 which lies on the sphere of region γ is then contractible to a single point. This case can be ignored since l is therefore homotopic to a simple loop. Now, assume the puncture lies on the sphere of region γ. The boundary component of l_2 on the sphere is therefore no longer freely homotopic to the identity. Instead, it can be contracted to a loop around the puncture, which we have labeled as Δ. Hence, we have l as being described by the word $\lambda_i\Delta$.

Consider cases 3 thru *. Each of these cases can be generalized to the following situation: region α is a i-torus, region β is a $(k-i)$-torus, and region γ is a $(n-k)$-torus, where k is some integer where $i < k < n$. The second simple loop, l_2 is therefore equivalent to λ_k since it separates a $T_{(k-i)}$ from $T_{(n-i)}$. Note that it is no longer true that $\lambda_{(k-i)}$ is equivalent to $\lambda_{(n-(k-i))}$ due to the boundary components of the surfaces. If the puncture lies in the $(n-k)$-torus, which we can assume contains the a_nb_n-torus, then l can be described by the word $\lambda_i\lambda_k$, where $0 < i < k < n$. If β is the punctured region, though, l would be identified as $\lambda_i\lambda_k'$.

Lastly, let l_2 create the case **. Since this cut does not separate $T_{(n-i)}$

into two pieces, it must be a nonseparating cut. We have shown that on T_n, a nonseparating cut is equivalent to any of the $2n$ generators. Similarly, on $T_{(n-i)}$, which is the connected sum of each of the a_jb_j-tori for $i < j \leq n$ any nonseparating curve will be equivalent to b_j for some $i < j \leq n$. In order to name this particular l, we will recall that the puncture lies on the original $T_{(n-i)}$ and can then describe this l by the word $\lambda_i b_j$, for some $i < j \leq n$.

Next, let l_2 lie on T_i. Therefore, the region α is now $T_{(n-i)}$ and contains the puncture. We have the following possiblities for β and γ given χ and the limit of one or two surfaces resulting from the cutting l_2 on T_i:

	β	γ
1	sphere	i-torus
2	i-torus	sphere
3	1-torus	$(i-1)$-torus, $i > 1$
4	2-torus	$(i-2)$-torus, $i > 2$
$\vdots$	$\vdots$	$\vdots$
*	$(i-1)$-torus, $i > 1$	1-torus

or

**	both regions form a single $(i-1)$-torus, if $i > 1$
***	both regions form a single sphere if $i = 1$.

Recall that the neighborhood around the basepoint is symmetric such that α, β, γ is equivalent to γ, β, α. With this fact in mind, it is obvious that case one is equivalent to the previous case one, where l_2 was on $T_{(n-i)}$. We have already identified this case in the previous section.

Next, consider case 2. Since the puncture is not in the γ region since it is in the α region, we have previously shown that l_2 will be contractible to a single point, so we can ignore this case.

In cases 3 thru *, we can again make a generalization: region α is a $(n-i)$-torus, region β is a $(i-k)$-torus, and region γ is a k-torus, for some positive k less than i. Hence, we have l_2 equivalent to λ_k since it separates a $T_{(i-k)}$ from T_i. Now, assume that the puncture lies on the $(n-i)$-torus. Then, we can describe l by the word $\lambda_i\lambda_k$, for $0 < k < i < n$. (Note that if the puncture were in the β region, then l would be labeled as $\lambda_i'\lambda_k$. However, there is a homeomorphism taking this to $\lambda_i\lambda'_k$ for a k less than i, and therefore this is the same case as in the previous section, cases 3 thru *, where β contains the puncture). Therefore, this case combined with cases 1 and 2 from the previous section form together for the loop described by $\lambda_i\lambda_k$ for $0 < i, k < n$. However, in order to remove duplicates from these lists (for example, on T_9, $\lambda_3\lambda'_4$ is equivalent to $\lambda_4\lambda'_3$, which both satisfy the above restriction on i and k), we must further limit the values of i and k. Therefore, we have distinct loops which are described with the word $\lambda_i\lambda'_k$ such that $0 < i < n$ and $i \leq k \leq (n-i)$.

Now, consider case **, when $i > 1$. Since both regions β and γ are one piece after l_2 is cut, then l_2 must be a nonseparating curve. We can describe this nonseparating loop as b_j, where $1 < j \leq i$, since l_2 lies on T_i which is generated by the free group $F(a_1, b_1, a_2, b_2, \ldots, a_i, b_i)$. Recall, though, that it cannot be the case, here that i is one. This is considered in the next case.

Considering case ***, we see that λ_1 was l_1, since $i = 1$. The second cut, which lies on the i-torus and is nonseparating, must then be equivalent to b_1. The loop l can be described as $\lambda_1 b_1$ since the puncture lies on the $(n-i) = (n-1)$ holed torus. Hence, together with case ** from the previous section and the previous case in this section, we have the curve $\lambda_i b_j$ for $0 < i < n$ and $0 < j \leq n$.

Now, assume that l_1 is Δ, a loop enclosing a punctured disc. Therefore, we have separated a punctured sphere from T_n. Let region α be this punctured sphere. After cutting l_2, the regions β and γ will form either one or two surfaces, and we will then have a total of either two or three surfaces. When the boundary components are removed, we will again have a total $\chi = 6 - 2n$ amongst these two or three pieces without boundary. Hence, we have the following possiblities:

	β	γ
1	sphere	n-torus
2	n-torus	sphere
3	1-torus	$(n-1)$-torus
4	2-torus	$(n-2)$-torus
$\vdots$	$\vdots$	$\vdots$
*	$(n-1)$-torus	1-torus

or

**	both regions form a single $(n-1)$-torus.

In case one, β is a sphere without a puncture. We have established that this implies l_2 is freely homotopic to l_1. Hence we can describe l as $\Delta\Delta$. In this case, l_2 lies either on the sphere or the n-torus which l_1 created. Furthermore, note that this is the only case where l_2 can lie on the sphere created by l_2. Any loop on the interior of Δ must be enclosed by Δ. If it weren't, then the loop would have to intersect Δ at a second point giving too many intersections. Recall that the only loop which can be enclosed by Δ is one that is homotopic to Δ. Therefore, the first case is the only case where l_2 may lie in the region α. Hence, from now on, assume l_2 lies on T_n.

Consider case 2. Since the puncture lies on α, it cannot lie on γ as well. Therefore, l_2 is contractible to a single point, and this particular l is homotopic to a simple loop and can be disregarded.

Now, assume cases 3 thru * occur. Again, we can make a generalization: region α is a punctured sphere, region β is a i-torus, and region γ is a $(n-i)$-torus where $0 < i < n$. Therefore, l_2 must be equivalent to λ_i since it separates off a T_i, and we have a case which is the same as the second case from the first section of this proof.

Finally, consider case **. Due to the fact that β and γ are one surface, l_2 must be a nonseparating curve. Since l_2 lies on T_n, it will be equivalent to any of the $2n$ generators. Without loss of generality, we can call this curve Δb_1.

Now we can consider the final case, where l_1 is a nonseparating loop. Without loss of generality, we will assume this cut is b_j, for $0 < j \leq n$.

After cutting along l_1, we will then have a single surface, $T_{(n-1)}$. Hence after cutting l_2, we will have regions α, β and γ combining in various ways to form one or two new surfaces. Using the fact that $\chi = 6 - 2n$ amongst the one or two surfaces without boundary, we have the following:

1	α and γ are $(n-1)$-torus, β is a sphere
2	α and β are $(n-1)$-torus, γ is a sphere
3	α is $(n-i)$-torus, β and γ are (i-1)-torus, if $i > 1$
4	α is $(n-1)$-torus, β and γ are a sphere, if $i = 1$
5	α is (i)-torus, β and γ are $(n-i-1)$-torus
6	α, β and γ are $(n-2)$-torus
7	α and γ are a sphere, and β is $(n-1)$-torus
8	α and γ are a punctured $(i-1)$-torus, and β is $(n-i)$-torus, $0 < i < (n-1)$
9	α and γ are a $(n-1)$-torus and β is a sphere
10	α and γ are a $(n-i)$-torus and sphere, and β is a punctured $(i-1)$-torus where $0 < i < (n-1)$

In case one, we know that if the puncture is not in the β region, then l_1 is homotopic to l_2. Without loss of generality, we will assume that l_1 is described with the word b_1. Hence, we can describe l by $b_1 b_1$. If the puncture is on the sphere of region β, though, then the two loops are no longer homotopic. It must be the case, though, that l_2 is equivalent to a generator of the $a_1 b_1$-torus, since it must lie on the same torus as l_1. We will describe this case with the word $b_1 \bar{\Delta} b_1 = b_1 b_n a_n \bar{b}_n \bar{a}_n \ldots b_1 a_1 \bar{b}_1 \bar{a}_1 b_1$.

Looking at case 2, we see that we have already considered this case when l_1 was Δ, and we have named this loop as Δb_1.

Considering cases 3 and 4, we have also seen these previously. They correspond to the 3rd thru * cases and the ** case of the λ_i case when l_2 was on T_i, respectively.

Case 5 we have also seen earlier. It is the same case as the the ** case of the second part of the λ_i argument, where l_2 lies on $T_{(n-i)}$.

If case 6 arises, we will have all three regions as one piece. Hence, both l_1 and l_2 must be nonseparating curves. However, in viewing T_n as the connected sum of n one-holed tori, it cannot be the case that both l_1 and l_2 correspond to generators of the same one-holed torus, since two generators of the same one-tori will separate the surface. Hence, we can describe l by the word $b_1 b_2$.

In cases 7 and 8, we again have two nonseparating loops. Each of these loops correspond to generators of the same one-holed torus in the connected sum. However, it is not the case that the two curves would be homotopic to each other on the unpunctured n-holed torus (as in case 1). We show an example of this case in figure 7.

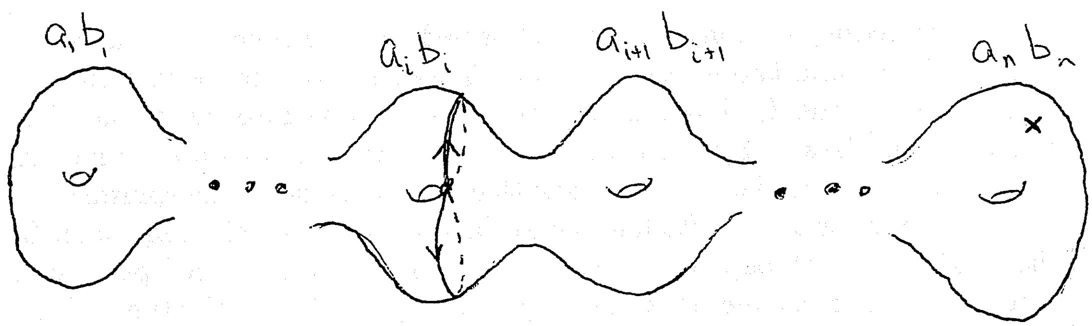
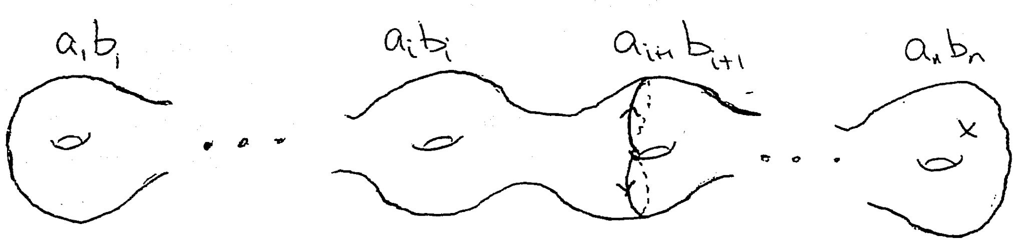


Figure 7

Letting l_1 be b_i, we will name l as $b_i \bar{\lambda}_i b_i = b_i b_i a_i \bar{b}_i \bar{a}_i \ldots b_1 a_1 \bar{b}_1 \bar{a}_1 b_i$, for $0 < i \leq n$. Note that if $i = n$, we have case 7, where the puncture must lie in the α and γ region or else this curve is homotopic to a simple loop. For any other i, we have case 8.

Cases 9 and 10 are similar to the previous case. Again, they are two non-separating curves from the same single torus which would not be homotopic on the non-puncutred n-holed torus. However, they intersect at the other side of the hole, as follows in the diagram below:



In this case, we let l_1 be $b_{(i+1)}$. This particular l is described with the word $b_{(i+1)}\bar{\lambda}_{(i)}b_{(i+1)}$, for $0 < i < n$. Note further that if $i = 1$ we have case 9. The remainding i's yeild case 10.

$\diamond$.

6 Distinctness of Loops via Whitehead's Algorithm

In the preceeding sections, we have classified the free homotopy classes of loops with one and two-intersections on T_2, and generalized loops with one-intersection on the T_n; however, we must now demonstrate the distinctness of each of the classes. By distinctness, we mean that the loops are distinct up to homeomorphisms, ie. we must show that there is no homeomorphism of T_2 or T_n which maps a loop which is in one class to a loop in a different class. In the classification of loops with two-intersections on the one-holed punctured torus a few different methods are proposed to prove the distinctness of the equivalence classes, and the method which is used involves a study of the corresponding loops on an unpunctured torus. [5] Our initial approach to proving the distinctness of loops on T_n for $n \geq 2$ will make use of Whitehead's algorithm, a method suggested for the one-holed torus. Through Whitehead's algorithm we will examine the distinctness of the classes of words in the free group which correspond to loops on T_n. This will provide a partial proof of the distinctness of the classes of loops up to homeomerphisms. We use the fact that homeomorphisms of the n-holed torus induce automorphisms of the free group of $2n$ generators. Thus, by showing that two words are not equivalent, we will have shown that the corresponding loops are distinct up to homeomorphisms of the torus. However, we know that not all automorphisms of the free group are realizable as homeomorphisms of the n-holed torus if $n \geq 2$. Hence, equivalence under automorphisms need not imply the existence of any corresponding homeomorphisms between the surfaces. In fact, Stallings has identified an automorphism of the free group on three generators which is not realizable by a homeomorphism, and he discusses the correspondence between homeomorphisms of a surface and automorphisms of the free group for more general cases [13]. For classes of loops which are equivalent under automorphisms we will need further means to prove the distinctness of the classes. We determine the distinctness of words within the free group through an algorithm given by Whitehead [16]. Whitehead provides an algorithm which reduces the question of whether two words are equivalent to the question of whether words are 'Whitehead equivalent' under a smaller set of automorphisms, the Whitehead automorphisms which are defined as follows. [9]

The algorithm uses determines the equivalence of words under the Whitehead automorphisms, and thus all automorphisms through the following steps.

1. From the initial list of words generated in the preceeding section, determine the new list of 'minimal' words. A word is said to be minimal if the length of the word is less than or equal to the length of all Whitehead automorphic images of the word.

2. We separate the minimal words by word length. Minimal words of different lengths are not Whitehead equivalent, and thus such words are not equivalent. If only one minimal word has a given length, it is distinct from all other words, and we are done. If there is more than one minimal word of a given length we count the number of times each generator or its inverse appears in the word. The set of these numbers is invariant for words which are equivalent, according to Whitehead, though the generator to which a given number corresponds may change. Thus, words which have different sets of these numbers are also distinct.

3. Words which are equivalent must have identical sets of occurrence numbers of generators; however, words with identical sets of occurrence numbers need not be equivalent. For the final step in Whitehead's algorithm we generate families of words which are Whitehead equivalent to a word of a given length, ie. we perform all length preserving Whitehead automorphic images on a word of given length. If two words have the same length but are not in the same set of Whitehead automorphisms then they are not equivalent.

We implement the first step of Whitehead's algorithm through a computer program in C which is attached at the end of this paper. The number of occurences of generators and their inverses can easily be calculated by hand, and the step three of Whitehead's algorithm may be easily programmed, though time constraints prevented us from completing step three.

Included in the comments of the program are suggestions for modifications which would program step three. The program can be used for free groups with n generators, though in its current form it runs for free groups with 4 generators. We ran the program for our list of one-intersectors on T_2, and for our two-intersectors on T_2. In the following table we present the results of the program and our implementation of the first two steps of Witehead's algorithm. We find that we have now shown 8 out of the 12 one-intersectors to be distinct, and 12 out of the 44 two-intersectors to be distinct.

Original Word	Minimal Word	Equivalent Word
abABabAB	unchanged	
CdcDabAB *	unchanged	bbabABdCDc
CdcDCdcDbaBA	unchanged	
dbaBA **	a	bd
ddCDc	unchanged	
CdcDbaBACdcDbaBA	unchanged	
CdcDbaBAb	unchanged	
bb	unchanged	
bbabABdCDc *	babADCDc	CdcDabAB
bd **	d	dbaBA
dabABd	unchanged	
dCdc	unchanged	

Our implementation of the first two steps of Whitehead's algorithm has proved the distinctness of the classes of loops which are not starred. We must still prove the distinctness of the two classes with single stars and the two classes with double stars. It is possible that distinctness of the classes with single stars may be proved by applying the final step of Whitehead's algorithm. The two classes which are identified with double stars are Whitehead equivalent to simple loops. It is clear that an application of the third step of Whitehead's algorithm will be useless for these two classes, so an alternate method of proof must be determined. However, we have found that a partial implementation of Whitehead's algorithm has reduced our proof of the distinctness of the 12 once-intersecting loops to a prove of the distinctness between two pairs of one-intersectors. For the loops with two-intersections Whitehead's algorithm again restricts the numbers of cases among which we must prove distinctness, however Whitehead's algorithm leaves many sets of classes which are equivalent under the first two steps of the algorithm so that the distinctness of classes within a given class remains to be proved. The computer program reveals that the classes in the following table are distinct. (Note that we use the letter X to denote *barx*)

CdcDabABdCDcB	DabABdabAB
abABabABabAB	CdcDCdcDbaBACdcDbaBA
CdcDCdcDbaBACdcD	abABBabAB
aabABdCDcabABdCDc	CdcDbaBAbaBAb
DDbaBACdcDbaBA	CdcDbaBACdcDbaBACdcDbaBA
dbDb	bbb

Within each of the following tables, the distinctness of the classes of loops must be proved. The following eight classes are equivalent to minimal words of length one, which correspond to simple loops.

dACDc	dbABdCd
dbd	dBc
dbaBAb	CdcDBCdcD
CdcDCdcDbaBAB	CdcDAB

The following classes are minimal words with length five.

abABAb	ddCdc

The classes of loops in the following table are all equivalent to minimal words of length six. Implementing step two of Whitehead's algorithm allows us to determine that the two words in the top row are distinct from the two words in the bottom row.

DCddcD	abABaa
CdcDaa	dAabABdA

The following three classes with minimal length seven must be shown to be distinct from each other.

dADbaBA	dbABdabABd	ddabABd

The following classes have minimal length eight.

DCdcabAB	dbabABdCDcDb

The following classes have minimal length nine.

bbabABdCDcbabABdCDc	dbdCdcDbaBA	aCdcDbaBAB
CdcDBabAB	ddCddcDbaBA	

The classes in the following table all have minimal length ten, though the classes in the top row are distinct from the classes in the bottom row.

CdcDbaBAdCdcDbaBAb	DabABdCDcB
DCddcDbaBA	aaabABdCDc

The classes in the following table have minimal length twelve.

CdcDaCdcDbaB	CdcDabABabAB

The following classes both have minimal length sixteen.

dCdcDbaBACdcDbaBAd	CdcDabABdCDcabAB

For some of the sets of classes which appear to be equivalent, the third step of Whitehead's algorithm may work. However, to prove the distinctness of some of the classes of loops we will require further means of proof. Certainly we will require an alternate method to prove distinctness for the set of eight loops which are Whitehead equivalent to simple loops.

7 Conclusion

The techniques used in this paper to develop a classification of the once and twice intersecting loops on a two-holed once-punctured torus and the once intersecting loops on the n-holed once-punctured torus may be generalized to classify loops with higher intersection numbers. However it is evident that new approaches are required to complete the proof of the distinctness of the classes which we obtained. The attempt at proving the distinctness of the classes of loops through Whitehead's algorithm raises a number of interesting questions. It would be interesting to determine when Whitehead automorphisms and automorphisms of the free group correspond to homeomorphisms of T_n, and to determine whether the partial results of Stallings [13] would help prove distinctness among the remaining classes. It might be interesting to examine the connection between the Whitehead equivalence classes of minimal words of a given length and the corresponding geodesics.

Additionally, there are a number of interesting number theoretic questions that might be interesting. It would be interesting to determine the Markoff values associated with the classes of geodesics that we have found [11]. Also it might be interesting to determine whether there is any connection among the geodesics which are Whitehead equivalent to loops with lower self-intersection numbers and their Markoff numbers.

References

[1] Daniel E. Cohen. *Combinatorial Group Theory: a topological approach.* London Math. Soc., 1989.

[2] D. Crisp. *The Markoff Spectrum and Geodesics on the Punctured Torus.* PhD thesis, Univ. Adelaide, 1993.

[3] D. Crisp and W. Moran. The markoff spectrum and geodesics with one self-intersection on the punctured torus. *Flinders Univ. pre-print*, 1995.

[4] Fred H. Croom. *Principles of Topology.* Saunders College Publishing, 1989.

[5] David Crisp et al. Closed curves and geodesics with two self-intersections on the punctured torus. Research started during the Summer 1994 NSF REU Program at Oregon State University., 1996.

[6] R.V. Gamkrelidze. *Encyclopaedia of Mathematical Sciences*, volume 58. Springer-Verlag, 1990.

[7] P.J. Higgins and R.C. Lyndon. Equivalence of elements under automorphisms of a free group. *J. London Math. Soc.*, (Second series, 8):254–258, 1974.

[8] William S. Massey. *Algebraic Topology: An Introduction.* Springer-Verlag, 1967.

[9] J. McCool. A presentation for the automorphism group of a free group of finite rank. *J. London Math. Soc.*, 8:259–266, 1974.

[10] E.S. Rapaport. On free groups and their automorphisms. *Act. Math.*, 99:139–163, 1958.

[11] C. Series. The geometry of markoff numbers. *Math. Intell.*, 7(3):20–29, 1985.

[12] Caroline Series. The markoff spectrum in the hecke group g5. *Proc. London Math. Soc.*, pages 151–181, 1988.

[13] John R. Stallings. Topologically unrealizable automorphisms of free groups. *Proc. Amer. Math. Soc.*, 84(1):21–24, 1982.

[14] John Stillwell. *Classical Topology and Combinatorial Group Theory: second edition.* Springer-Verlag, 1993.

[15] J.H.C. Whitehead. On certain sets of elements in a free group. *Proc. London Math. Soc.*, 41:48–56, 1936.

[16] J.H.C. Whitehead. On equivalent sets of elements in a free group. *Ann. of Math.*, 37:782–800, 1936.

```
/*My program to use Whitehead's algorithm to determine whether words representin

#include <stdio.h>
#include <string.h>
#include <stdlib.h>
#include <ctype.h>

struct key {char fixed; char subset[8];};
struct key automorph[504];


/*read in the values from a function, keep this array of structures as a global
int getauto(FILE *in)
{
  int end, numbercheck, power, fixnum;
  char word[10];
  char letter;
  end=fscanf(in, "%s", &letter);
  fixnum=0;
  numbercheck=0;
  for (power=0; power<504; power=power+1)
    {
      automorph[power].fixed=letter;
      end=fscanf(in, "%s", automorph[power].subset);
      if (end==EOF)
        {return EOF;}
      numbercheck=numbercheck+1;
       if (power%5==0)
        fprintf(stderr, "\n");
      if (automorph[power].subset[0]!='Z')
        {
          continue;
        }
      if (automorph[power].subset[0]=='Z')
        {
          fscanf (in, "%s", &letter);
          while (letter=='Z')
            {
              fscanf (in, "%s", &letter);
            }
          power=power-1;
        }
    }
  return numbercheck;
}

/*I still need a function to get the permutations*/

/*this function gets a word from the file of all the words.*/
char *getword( FILE *in, char* word)
{
  char *wordptr;
  int end, i;
  i=0;
  wordptr=word;
  for (i=0; i<50; i++)
    (*(wordptr+i)='\0';)
  end=fscanf (in, "%s", wordptr);
  fprintf (stderr, "\n \nThe gotten word is %s\n ", wordptr);
  if (end==EOF)
    wordptr="Q";
  return wordptr;
}
```

```
}



/*this function will perform an 'elementary reduction' on the word.  Ie. a reduc

char *elementary (char *wordptr)
{
  char temp1, temp2;
  char tempword[100];
  char *holdword;
  int check=1;
  int i;
  holdword=wordptr;
  while (check!=0)
    {
      check=0;
      i=0;
      for (temp1=*holdword; temp1!='\0'; temp1=*holdword)
        {
          holdword++;
          if (*holdword=='\0')
            {
              tempword[i]=temp1;
              i=i+1;
              tempword[i]='\0';
              i=i+1;
              break;
            }
          temp2=*holdword;
          if (temp1==temp2)
            {
              tempword[i]=temp1;
              i=i+1;
              continue;
            }
          if ((temp1!=temp2+32) &&(temp1!=temp2-32))
            {
              tempword[i]=temp1;
              i=i+1;
            }
          if ((temp1==temp2 - 32)||(temp1==temp2+32))
            {
              holdword=holdword+1;
              check=check+1;
            }
        }
      tempword[i]='\0';
      strcpy (holdword, tempword);
    }
  return holdword;
}
/*IN order to implement the final step of WHitehead's algorithm, it might be a g

char *minimal(char *wordptr)
{
  int check, aut, i, hold, action;
  char temp1;
  char tempword[100];
  char holdword[100];
  char *redword;
  check=1;
  hold=0;
  fprintf(stderr, "I am now entering minimal with word %s \n", wordptr);
  strcpy(holdword, wordptr);
```

```
char *cyclic(char *wordptr)
{
  char *tempword;
  char tempfirst, templast;
  int count, i, check;
  check=count=1;
  tempfirst=wordptr[0];
  tempword=wordptr;
  fprintf (stderr, "I am now in cyclic with word %s \n", wordptr);
  while (check!=0)
    {
      check=0;
      for (templast=wordptr[count]; templast!='\0'; templast=wordptr[count])
        {count=count+1;}
      count=count-1;
      /*Deal with the case where the word is a single letter*/
      if (count==0)
        {return wordptr;}
      templast=wordptr[count];
      if (templast==tempfirst+32||templast==tempfirst-32)
        {
          wordptr[count]='\0';
          tempword=&wordptr[1];
          check=check+1;
        }
      count=1;
      strcpy (wordptr, tempword);
      tempfirst=wordptr[0];
      tempword=wordptr;
    }
  return wordptr;
}



/*this function determines which of the type 2 (ie. non-permutation) whitehead a

int morph(char xletter, int aut)
{
  int i, value;
  char current;
  i=value=0;
  current=automorph[aut].fixed;
  if (xletter==current)
    return 1;
  current=automorph[aut].subset[0];
  while (current != '\0')
    {
      if (tolower(xletter)==xletter)
        {
          if (current==xletter)
            value=value+2;
          if (current==toupper(xletter))
            value=value+26;
        }
      if (toupper(xletter)==xletter)
        {
          if (current==xletter)
            value=value+3;
          if (current==tolower(xletter))
            value=value+27;
        }
      i=i+1;
      current=automorph[aut].subset[i];
    }
  return value;
```



```
  while (check!=0)
    {
      check=0;
      for (aut=0; aut<504; aut=aut+1)
        {
          i=0;
          hold=0;
        /*this for loop performs the actual automorphism, and afterwards it does
          for (temp1=holdword[0]; temp1!='\0'; temp1=holdword[hold])
            {
              /*if word is one letter, no more reduction needed*/
              if (holdword[1]=='\0')
                {
                  strcpy(wordptr, holdword);
                  return wordptr;
                }
              action=morph(temp1, aut);
              if (action==0)
                {
                  tempword[i]=temp1;
                  i=i+1;
                }
              if (action==1)
                {
                  tempword[i]=automorph[aut].fixed;
                  i=i+1;
                }
              if (action==2||action==3)
                {
                  tempword[i]=temp1;
                  i=i+1;
                  tempword[i]=automorph[aut].fixed;
                  i=i+1;
                }
              if (action==26||action==27)
                {
                  if (tolower(automorph[aut].fixed)==automorph[aut].fixed)
                    {
                      tempword[i]=(toupper(automorph[aut].fixed));
                      i=i+1;
                    }
                  if (toupper(automorph[aut].fixed)==automorph[aut].fixed)
                    {
                      tempword[i]=(tolower(automorph[aut].fixed));
                      i=i+1;
                    }
                      tempword[i]=temp1;
                      i=i+1;
                }
              if (action==28||action==30)
                {
                  if (tolower(automorph[aut].fixed)==automorph[aut].fixed)
                    {
                      tempword[i]=(toupper(automorph[aut].fixed));
                    }
                  if(toupper(automorph[aut].fixed)==automorph[aut].fixed)
                    {
                      tempword[i]=(tolower(automorph[aut].fixed));
                    }
                  i=i+1;
                  tempword[i]=temp1;
                  i=i+1;
                  tempword[i]=automorph[aut].fixed;
                  i=i+1;
                }
              hold=hold+1;
```

```
            }

          tempword[i]='\0';
          redword=elementary(tempword);
          if (strlen(redword) < strlen(holdword))
            {
              strcpy(holdword, redword);
              fprintf (stderr, "Comparison made, real %s\n", holdword);
              fprintf (stderr, "Automorphism was (%c, %s).\n", automorph[aut].fi
              check=check+1;
            }
        }
    }
strcpy(wordptr, holdword);
return wordptr;
}

void main()
{
  char *word, *reduced, *cyclical, *firstelem, *final;
  char empty[100];
  char rightreduced[100], holdred[100];
  FILE *in;
  FILE *automor;
  FILE *out;
  FILE *cycle;
  int numwords, end;
  automor=fopen("allauto", "r");
  in=fopen ("loops1.txt", "r");
  out=fopen ("finloop2", "w");
  cycle=fopen("cycle", "w");
  end=getauto(automor);
  numwords=0;
  fprintf (stderr, "Now I have all automorphisms\n");
/*this loop reads in a word, does an elementary reduction of it, and then saves
  for (word=getword(in, empty); *word!='Q'; word=getword(in, empty))
    {
      firstelem=elementary (word);
      fprintf (stderr, "The elementary word is %s", firstelem);
      cyclical=cyclic(firstelem);
      reduced=minimal (cyclical);
      final=cyclic(reduced);
      fprintf (stderr, "\ncyclic final is %s!!\n", reduced, final);
      fprintf (out, "%s", final);
      fputc ('\n', out);
      numwords=numwords+1;
    }


  fclose (out);
  fclose(automor);
  fclose (in);
  fclose(cycle);
  printf ("I think I am done, you should have %d words", numwords);
}
```