The Knight's Tour Problem on Boards with Holes

Holly R. Wiitala

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Abstract

This paper looks at the knight's tour problem on ringboards of width two. We find that no closed knight's path exists for any square ringboard, and that an open knight's path exists on the 4m+1 case. We also look at the possibilty of NP-completeness for the knight's tour problem with holes.

1 Introduction

The knight's tour problem asks: using only legal knight moves is it possible to place a knight on a nxm chessboard and visit every square *exactly* once? The problem is solved for both rectangular [1, 5] and square chessboards [2].

The knight's tour problem is really a special case of the Hamiltonian path problem in graph theory, which is known to be NP-complete[3]. Interest lately has moved to mapping the knight's tours onto boards with holes [4]. On a board with holes, does the problem become Np-complete? The conjecture is that the problem does indeed become Np-complete. The paper offers a discussion about the conjecture at the end. The paper also examines specific kinds of boards with holes, namely the ringboard of width two.

2 Preliminaries

2.1 Review of Necessary Terms

Recall that a **legal knight's move** requires two vertical moves followed by a horizontal move, or two horizontal moves followed by a single vertical move. A knight's tour is called *closed* if the last move is a legal knight's move away from the starting square. The term closed knights tour will be used interchangeably with **Hamiltonian cycle.** An *open knights tour* occurs when the last move is not a knights move away from the starting sqaure. The term open knights path will be used interchaneably with **Hamiltonian path.** It is often useful to look at the knight's tour problem in the context of a **graph.** A graph consists a vertex set and an edge set. The **knight's graph** is constructed by letting vertices represent each square of the chessboard. An edge is drawn between two vertices if a legal knight move is possible between the squares represented by the aforementioned vertices. Both the traditional chessboard method and knight's graph method will be used in this paper. The squares and columns will be named using matrix notation, starting by labeling the upper left hand corner (1,1) and ending by naming the bottom right hand corner with (n, m), where n denotes the number of rows, and m the number of columns.

2.2 Definitions

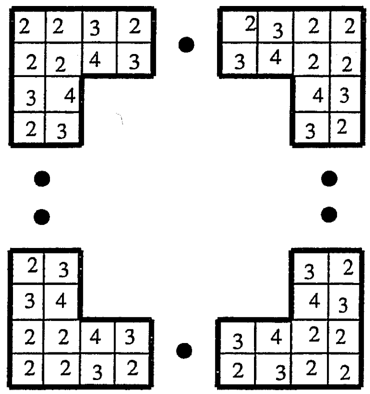
- **Ringboard:** an nxm chessboard with the middle missing. For our pruposes we will only be considering ringboards of width 2.
- **2-cycle:** a closed cycle which includes only edges adjacent to at least one vertex degree 2.
- **degree structure of a ringboard:** shows the number of moves possible from each square.
- **degree structure of a cycle:** refers to the sequence of degrees from the vertices included in a 2-cycle.

2.3 Degree Structure of the Ringboard

vExamine the middle piece of one of the sides of the width 2 ringboard. It should be clear that there are only two choices for a knight to move from each square. In the following diagrams the numbers in each square represent the number of choices available from that square.

2	2	2	2
2	2	2	2

The knight must move to the opposite row that it is because there are only two rows (or columns), and a legal knights move requires a "width" of two squares. So the only question as to the degree structure of the ringboard occurs on the corners. We know that a legal knights move requires a "length" of three squares, so once the number of squares missing in the middle of the ringboard becomes four or greater, the degree structure of the corners stays the same, and the appropriate number of squares of degree 2 needs to be added between the constant corner blocks.



To preserve this structure on nxm ringboard the smallest side must be greater than or equal to eight.

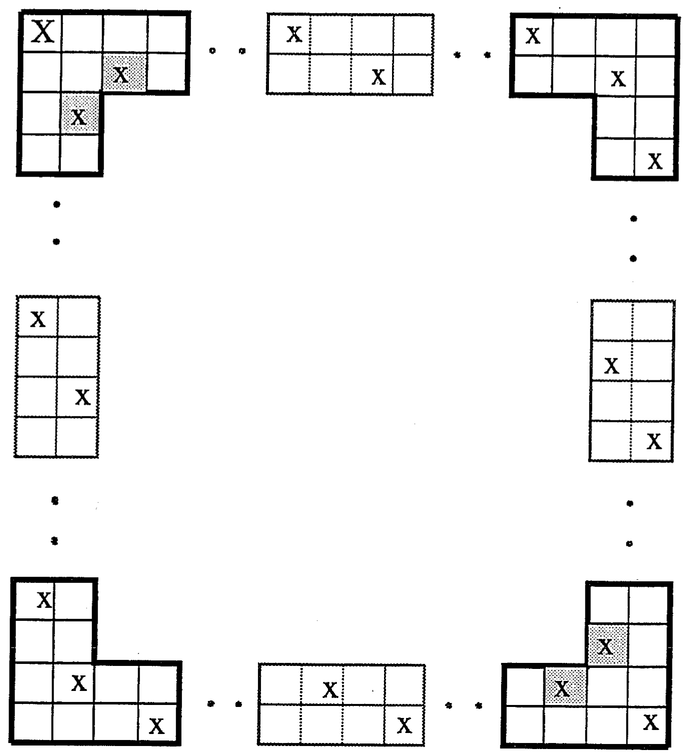
3 Closed Tours on nxn ringboards

Claim 1 *There exist no closed tours on any nxn ringboard.*

Theorem 1 *There exists a 2-cycle on every nxn ringboard.*

Proof. Our proof uses four cases– $4m$, $4m+1$, $4m+2$, and $4m+3$. In each of the cases we will construct a path, then prove this path is in fact a 2-cycle. First

Case 1, $4m$, $m \geq 2$**:** First we break the board into the appropriate blocks. The corners will always be drawn the same way, with the number of repeating blocks changing depending on m. Note that for the 8x8 case there will be no repeating blocks but only the corner blocks.



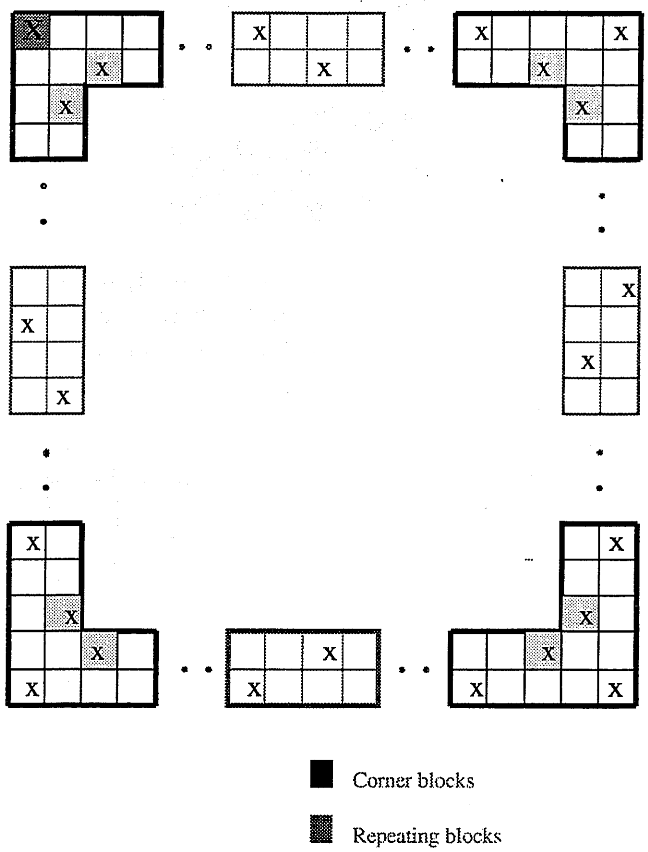
When we chck we see this indeed represents a ringboard of size $4m$x$4m$ since each block has four squares in it, and there will be m number of blocks. Duplicate the X's as shown,and then connect the X's that are a legal knight's move apart.

Next we show that this construction is in fact a cycle. Start at the big X in square $(1,1)$. Follow the edge to the x in square $(2,3)$ and continue to follow the edges already drawn. It is clear that each x is a legal knight's move away from exactly two x's. Because of this, we have a path, and we end at square $(3,2)$, a knight' move away from where we began. Therefore we have a cycle.

Next we need to show that this cycle is a 2-cycle. Using what we know about degree structure of ringboards we find four squares included in our cycle not of degree 2 (these are the shaded squares). However each is adjacent to a square of degree 2, and so we see that we do have a 2-cycle on all $4m$x$4m$ ringboards.

If m=1, we dod not have a ringboard, and so we are done.

Case 2,$4m+1$, $m \geq 2$: First we break the board into the appropriate blocks. The corners will always be drawn the same way, with the number of repeating blocks changing depending on m. Note that for the 9x9 case there will be no repeating blocks but only the corner blocks.

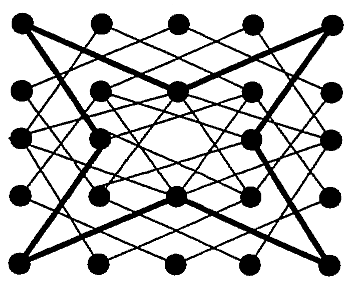


Upon inspection we see this indeed represents a $4m+1$x$4m+1$ ringboard. Duplicate the X's as shown, and then connect the X's that are a legal knight's move apart.

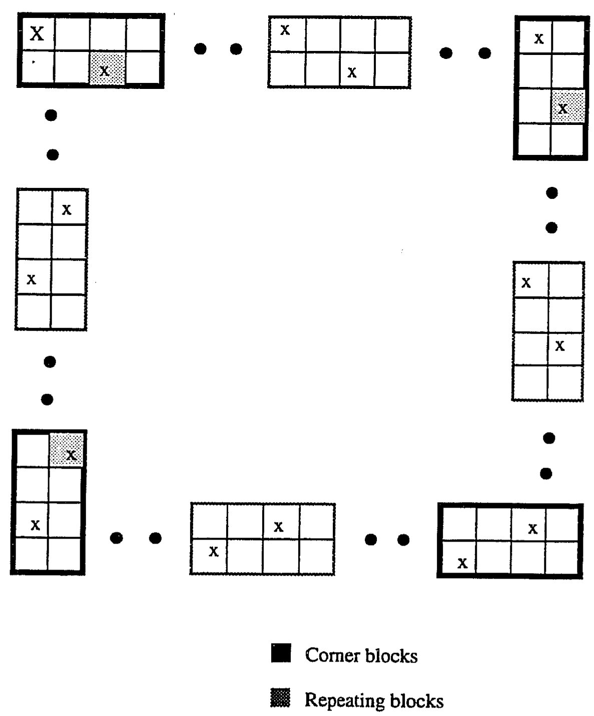
Next we show this is in fact a cycle. Start at the big X in square $(1,1)$. Follow the edge to the x in square $(2,3)$ and continue to follow the edges already drawn. It is clear that each x is a legal knight's move away from exactly two x's. Because of this, we have a path, and we end at square $(3,2)$, a knight's move away from where we began. Therefore we have a cycle.

Next we need to show that this cycle is a 2-cycle. By inspection, we find eight squares included in our cycle not of degree 2 (these are the shaded squares). However each is adjacent to a square of degree 2, and so we see that we do have a 2-cycle on all $4m+1 \mathrm{x} 4m+1$ ringboards.

If m=1, the following 2-cycle results for the 5x5 ringboard.



Case 3: First we break the board into the appropriate blocks. The corners will always be drawn the same way, with the number of repeating blocks changing depending on m. Note that for the 6x6 case there will be no repeating blocks but only the corner blocks.

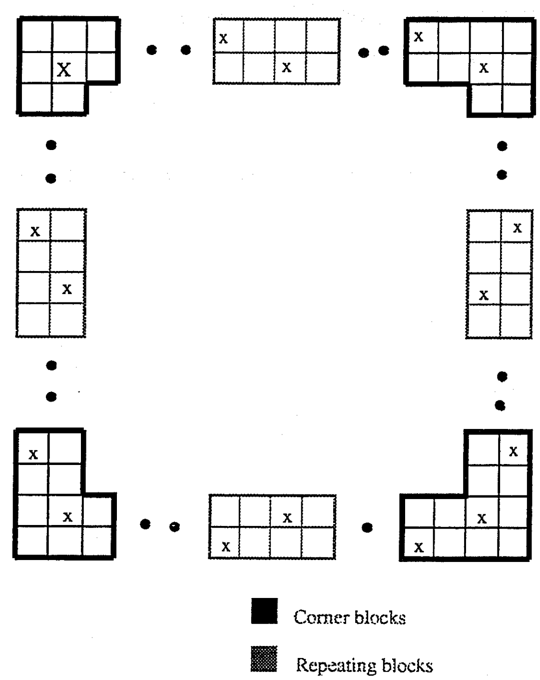


Upon inspection we see this indeed represents a square $4m+2$ ringboard. Duplicate the X's as shown,and then connect the X's that are a legal knight's move apart.

Next we show this is in fact a cycle. Start at the big X in square $(1,1)$. Follow the edge to the x in square $(2,3)$ and continue to follow the edges allready drawn. It is clear that each x is a legal knight's move away from exactly two x's. Because of this, we have a path, and we end at square $(3,2)$, a knight' move away from where we began. Therefore we have a cycle.

Next we need to show that this cycle is a 2-cycle. By inspection, we find six squares included in our cycle not of degree 2. Three of these are the shaded squares, the other three occur at squares $(2,4m-1),(3,2)$ and $(4m+2,3)$. However each is adjacent to a square of degree 2, and so we see that we do have a 2-cycle on all square $4m+2$ ringboards.

Case 4: First we break the board into the appropriate blocks. The corners will always be drawn the same way, with the number of repeating blocks changing depending on m. Note that for the 7x7 case there will be no repeating blocks but only the corner blocks.



We see this indeed represents a $4m+3$x$4m+3$ ringboard. Duplicate the X's as shown,and then connect the X's that are a legal knight's move apart.

Next we show this is in fact a cycle. Start at the big X in square $(2,2)$. Follow the edge to the x in square $(1,4)$ and continue to follow the edges allready drawn. It is clear that each x is a legal knight's move away from exactly two x's. Because of this, we have a path, and we end at square $(4, 1)$, a knight's move away from where we began. Therefore we have a cycle.

Next we need to show that this cycle is a 2-cycle. By inspection, we find no squares with degree other than 2, and so we are done.

And so we have shown that there exists a 2-cycle on every nxn ringboard. Now we are ready to prove Claim 1.

Theorem 2 *There are no closed knight's tours on any nxn ringboard of width 2.*

Proof. Assume a closed tour, H, exists. We know H is a Hamiltonian cycle. In a Hamiltonian cycle, each vertex is visited only once, so we must enter and exit the vertex exactly once. This implies that each vertex must contribute two edges to the cycle. So we know that both edges from any vertex of degree 2 must be included in H since this is the only possible way to enter and exit the vertex. But if there exists a 2-cycle then we know that there cannot be a Hmailtonian cycle, since not all of the verices are included. From Theorem 1 we know that every nxn ringboard has a 2-cycle, and so we conclude that no closed knight's tours exist on any square ringboard of width 2.

4 Properties of 2-cycles

This section will summarize some of the information which came out of the above proof and will be useful in proving the open tour case.

- $4m$ By symmetry, we find a total of two 2-cycles. Two corners are included in each 2-cycle, and the degreestructure contains four vertices of degree different than 2.

- $4m + 1$ By inspection we find two 2-cycles, one of which includes all corners. The degree structure includes a total of eight vertices that have degree differnt than 2.

- $4m + 2$ By symmetry we find four 2-cycles, one corner is included in each. There are six vertices included in the 2-cycle which do not have degree 2.

- $4m+3$: Upon inspection, we find one 2-cycle only. The 2-cycle contains only vertices of degree 2.

5 Open Tours on the Ringboard

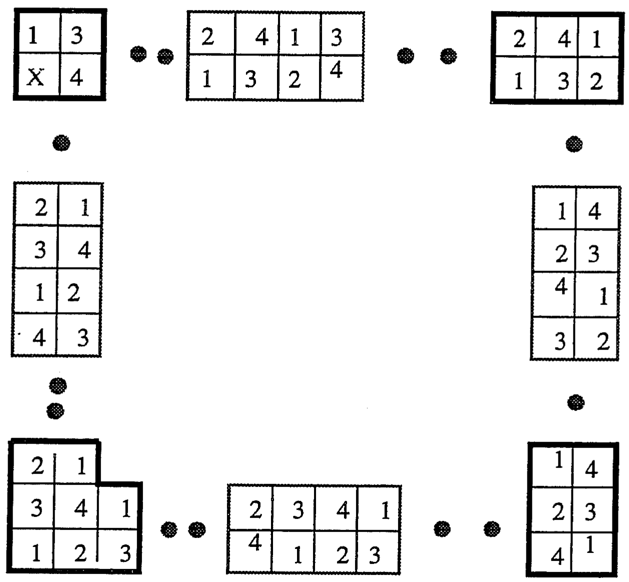
Theorem 3 *On ringboards of width 2, an open knight's path exists only in the* $4m+1$ *x* $4m+1$ *case.*

Intuitively, this makes sense, since the $4m+1$ case is the only that has a 2-cycle which includes all outside corners. Therefore, in some sense it is the only one that has a chance of including all vertices in a path.

Proof. We will prove that no such tour exists on the $4m$, $4m+2$, and $4m+3$ cases, and then contruct a path for the $4m+1$ case.

- $4m$ **case:** We know we must either start or end at a vertex of degree 2. Otherwise the only way to reach the vertices of degree 2 is closed in the aforementioned 2-cycle, and therefore unreachable in a tour of the entire graph.

- $4m+1$ **case:** Our argument will be very similar to the arguments used to show a 2-cycle exists. First seperate the ringboard into the appropriate blocks, shown below. The blocks in bold are the corner blocks and the others are repeating blocks. Note that if $m = 1$ then we have no repeating blocks, only the corner blocks. 1, 2, 3, and 4 denote which trip around the ringboard we are on. The X denotes the end of the path.



Duplicate the numbers in the appropriate manner. Begin at square $(1,1)$, and then draw edges between all the 1's,then all the 2's, all the 3's, and then the 4's, with the only exceptions in the lefthand corner. Here the last "1", square $(3,2)$, with the first "2" in square $(1,3)$, etc. After this is done, you will notice that each square is connected to exactly 2 other squares– with the exception of square $(1,1)$ and $(2,1)$, which are connected to only one other square. Hence we have a path that includes every square. The final square $(2,1)$ is not a legal knight's move away from the starting square, $(1,1)$, and so we have an open knight's tour on any $4m+1$ ringboard.

- $4m+2$: Assume that a Hmailtonian path exists. We know that every vertex must somehow be included in the path. We designate a vertex as a starting point that has degree 2 but is also adjacent to a vertex of higher degree. This allows us to travel all vertices in the 2-cycle, as we know this is the only way to include them while still allowing us to exit to another 2-cycle. So we exit to the next 2-cycle. We must enter via a vertex that is not degree 2 also, say u, or else this is not possible. However in order to be in the 2-cycle u must have at least 2 edges (one entering, one leaving) that are each adjacent to a vertex of degree 2. In other words, u must be directly adjacent to 2 vertices of degree 2, w and v. Without loss of generality, choose w. The path will come to a dead end at v, having no place to go but back to u, which has already been visited once, and leaving two 2-cycles out of the path. We have a contradiction. Hence no open knight's path can exist on the $4m + 2$ ringboard.

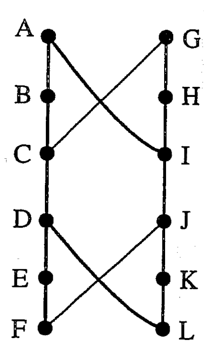
- $4m + 3$: Assume an open path exists. Such a path must start or end at a vertex of degree 2. So then we must follow the 2-cycle to be sure to include all the vertices of degree 2 in our path. Because all vertices in a 4m+3 2-cycle have degree 2, we end at a vertex of degree 2, which implies we have reached a dead end. The only way to leave the dead end vertex is by the "unused" edge, but then we are back to the start without including all the vertices, and we have a contradiction. There is no way to reach the rest of the graph from any of the vertices in the 2-cycle. Therefore no open knight's path exists on the $4m + 3$ ringboard.

6 NP-Completeness and Boards with Holes

We know that the vertex cover problem reduces to the Hamiltonian path problem[3]. So the question is naturally, does the Hmailtonian path problem reduce to the knight's tour problem with holes? Although we were not able to finish a proof of this, a discussion of what was accomplished will follow.

Conjecture 1 *On a board with holes, the knight's tour problem becomes NP-complete.*

In the book by Garey and Johnson, the vertex cover given to prove the Hamiltonian path problem NP-complete is as follows. We'll name the graph VC.

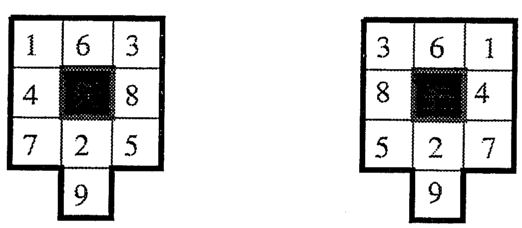


So the idea is to replace the vertices with subboards that have knights tours, replace the edges by connecting the boards with legal knight's moves. When we are done we will have a chessboard, P, with holes. We do this to construct a Hamiltonian path through P that keeps with the specific properties of VC.

6.1 Crossovers

The first problem came with constructing a way to crossover from subboard to subboard. It should be readily apparent that there is no way to connect some subboard to crossover in this manner. In the graph, vertex A connects to vertex I at the same time that vertex G connects to vertex C, we needed to connect subboards this way. So, we decided to create a board as a crossover that would allow us to enter in one corner and exit at the opposite lower corner. We looked at a variation of a $3x3$ board with a hole in it. Starting in the upper left hand corner, we see that there are two choices for the next square. However after that first choice, each square is determined, and we see that from the upper corner there is only one path that works.

By symmetry, we find there are two paths– one from each upper corner.



This showed great promise as a crossover, until we recently discovered that two more paths were possible on this board:

7	2	5
4		8
1	6	3
	9	

5	2	7
8		4
3	6	1
	9	

This causes problems later on , and will be discussed in the subsection titled "Problem".

Next we needed to consider boards that we knew had specific kinds of tours on them. A lemma from Cull's paper [1] shows us that there are specific paths on 5xm board. The big X denotes the starting square and the little x's possible ending squares.

X					
				x	
				x	

X				
			x	
			x	

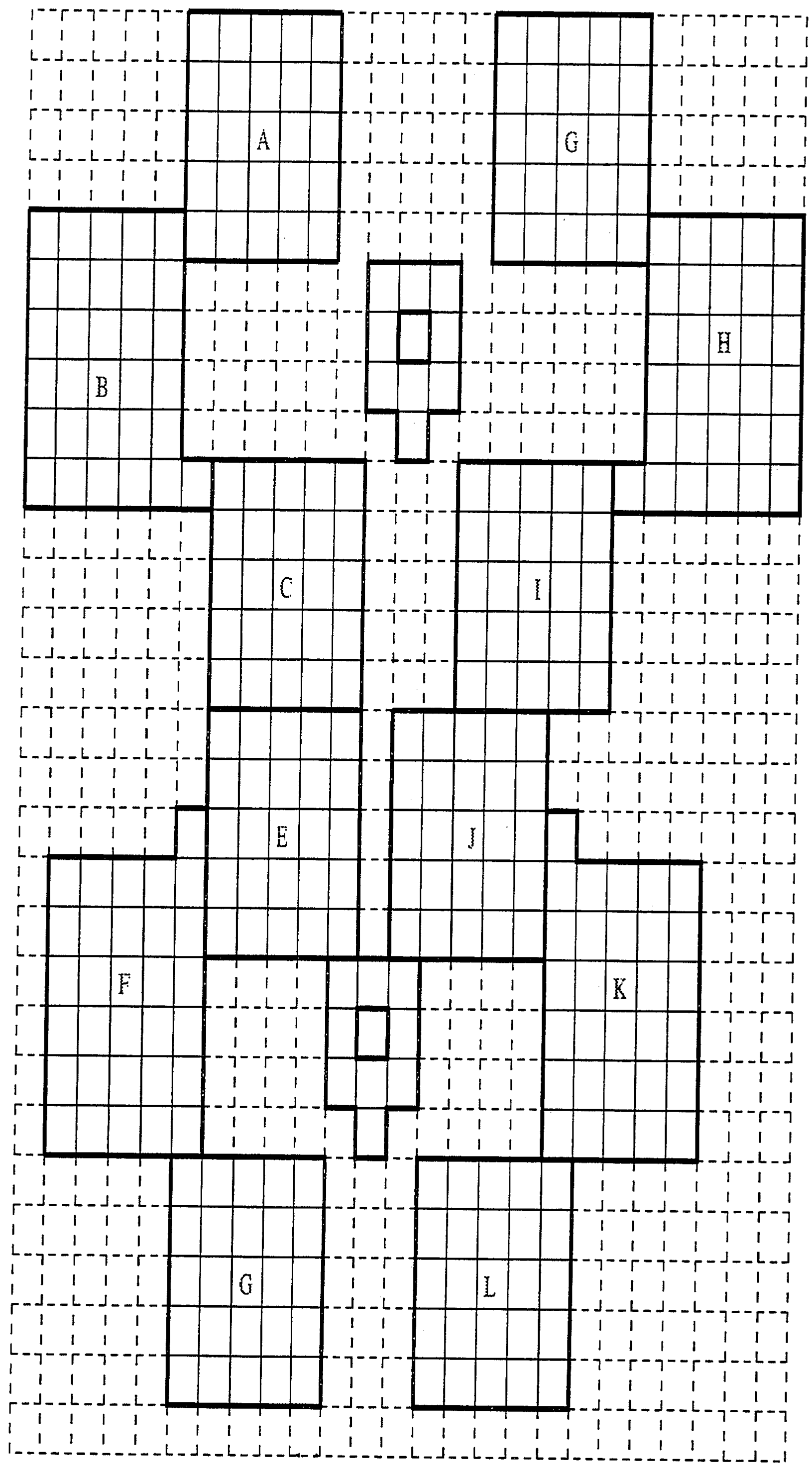
Also note that other open tours are possible from the same starting square on the 5x5 board:

X	12	17	22	3
18	x	2	11	16
13	8	23	4	21
24	19	6	15	10
7	14	9	20	5

X	14	9	20	3
24	19	2	15	10
13	8	23	4	21
18	x	6	11	16
7	12	17	22	5

6.2 The Board with Holes

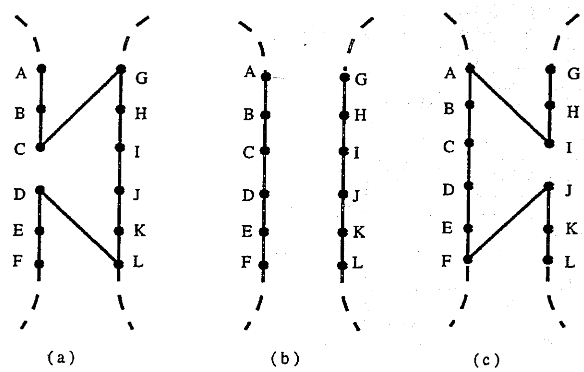
Now we are ready to begin trying to duplicate the vertex cover shown before. After much trial and error, we found a way to connect the boards that preserved the correct connections. The board follows. The x's denote possible starting or ending points to illustrate that the various boards are in fact connected by knight's moves. The names of the subboards correspond with the names of the vertices in VC. Dashed lines represent the missing squares.



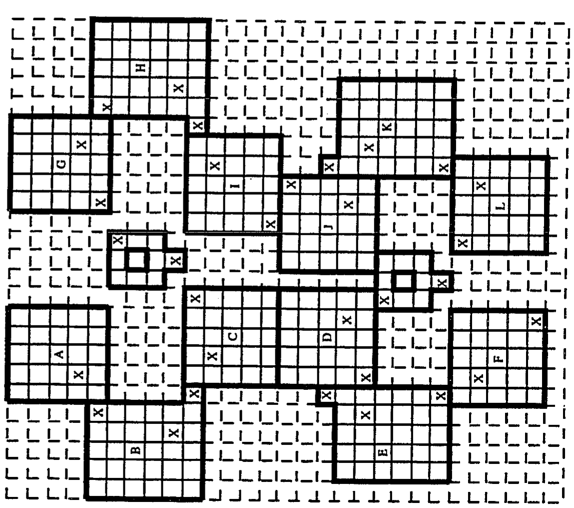
A
G
B
H
C
I
E
J
F
K
G
L

6.3 Necessary Properties of this Board

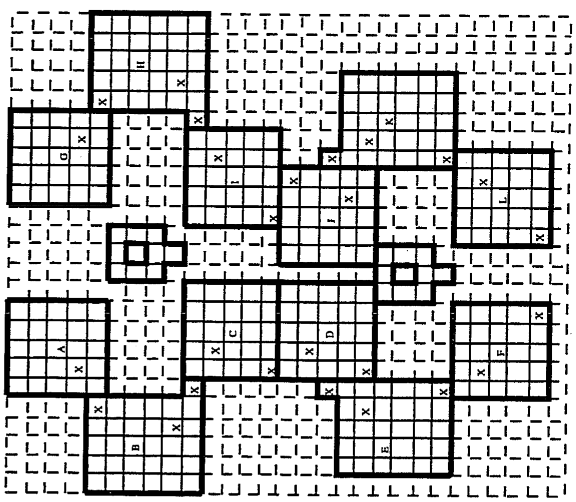
The vertex cover has these three basic properties[3]:



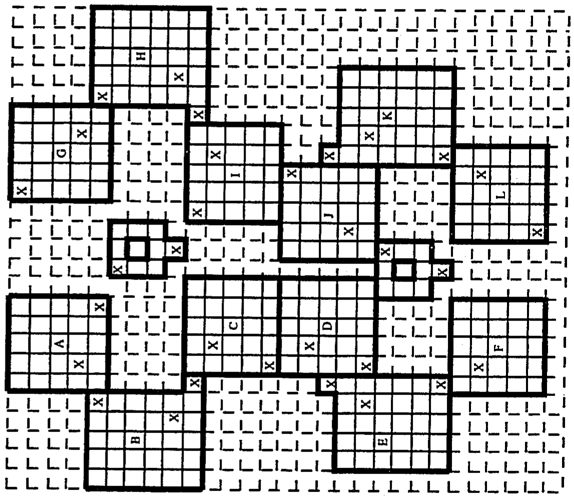
It is easy to show that our chessboard, P, has these properties also. Paths are shown again by x's:



(A)



(B)



(C)

Now we must show that these are the *only* ways to traverse this area of the board with a Hamiltonian path. Most cases are easy to rule out. I will go through the ruled out cases, and leave the problematic cases for the next section.

- **Case 1:** One will notice that a path exists from subboard A to subboard C, but no edge from vertex A to vertex$C$. If this path is taken, a Hamiltonian path is then not possible thorugh the rest of the chessboard. This is easy to see because either the board B is missed, or the path reaches a dead end in B and it is not posssible to reach the rest of the graph.

  Similarly for the cases involving the paths from subboards G to H, D to F, and J to L.

- **Case 2:** It is not possible to travel from A to G, since no knight's path exists on the crossover that begins in the upper left hand corner and ends in the upper right, so we need not consider this case at all.

- **Case 3:** If we combine subboards C and D, there is a knight's path that starts in $(2,2)$ and ends in the lower right hand square, $(10,5)$ [1]. It is then possible to make a knights move from board D to J. Again this is a connection that is not in VC. But, again, a Hamiltonian path is not possible with this connection and we disregard the case.

- **Case 4:** We know that there exists a knight's path on subboard C that starts in square $(2,2)$ and ends in the bottom right hand corner, $(5,5)$. From there a legal knight's move exists to the top left hand square of subboard J. The only possible knight's path would have to start at subboard A, go to subboard B, continue to C, hop to J, then K, L, F, E, D. From D the path must continue on to I, however this is not possible, so again, no knight's path is possible and and we can disregard the case.

With the exception of the next example, there are no other possible ways to traverse the board.

6.4 Problem

The problem arises with the fact that it is possible to traverse the crossover in a knight's path beginning with any of the corners. The lower corners of the crossovers are unfortunately reachable to several boards by a single knight's move, and still exit from the usual place. We see this causes two more possible ways to traverse P on our board, one of which causes a problem, one of which doesn't.

- **Case 1** We see it is now possible to travel from C to I by way of the crossover. However no matter how we traverse the rest of the board it is readily apparent that no Hamiltonian path exists if we consider this connection. So this is not a problem.
- **Case 2** It is also possible to reach L from F by entering the crossover from the lower left corner with a knight's move, and exiting out the bottom square, a knight's move away from board L. This is a problem because we now have a path which starts at subboard A, ends with subboard G and covers every single subboard in P– in other words– a Hamiltonian path propertyVC doesn't have. We therefore are not able to reduce the Hamiltonian path problem to the knight's tour problem with holes, and cannot prove that isNP-complete.

7 Conclusion

The obvious next question asks where to go from here. Perhaps there is a way to embed several of these P's into the graph so that it is not possible to have a Hamiltonian path through the entire graph using the fourth and problematic property of P. There may be a way to reposition the subboards so this cannot happen although both seem unlikely. One may want to look at a different crossover, or start over and reconstruct a chessboard with holes much like P, possibly using the same types of boards as crossovers for the subboards. In any event, the conjecture seems to be intuitively obvious. It seems the more holes poked in a chessboard, the more difficult the problem becomes. One can easily see that even with the ringboard the problem has become much more difficult than on the more traditional boards. However we were not able to complete the proof of NP-completeness this summer.

Other possibly interesting questions would investigate ringboards of different widths or investigating other graohs which share having the 2-cycle property.

References

[1] P. Cull and J. De Curtains. Knight's tour revisited. *Fibonacci Quarterly*, 16:276–285, 1978.

[2] A. Conrad et al. Solution of the knight's hamiltonian path problem on chessboard. *Discrete Applied Mathematics*, 50:125–134, 1994.

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