Classification of Loops with Self-intersections on the Twice Punctured Torus

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Abstract

In this paper we examine the conjugacy classes of once and twice transverse self intersecting loops on the twice punctured torus. We obtain a complete list of all classes in a systematic method which utilizes Euler characteristics. We then use Whitehead's algorithm and topological arguments to assure distinctness.

1 Introduction

Much work has been done recently with classifying loops on tori [1], [5], [6], [7]. These papers have looked at loops with between 1 and 3 self intersections on the one and two holed tori with one puncture. They have also examined loops on the n holed torus with one puncture. However, none of them have explored loops on the twice punctured torus. That is what we will examine in this paper.

The once punctured torus is shown in Figure 1.

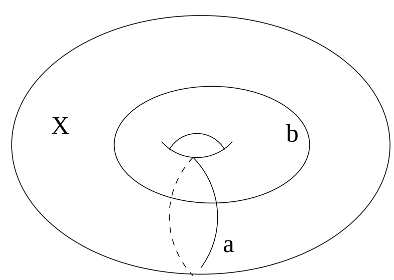

Figure 1: Once punctured torus

It can be formed by Figure 2 where a and b are called generators.

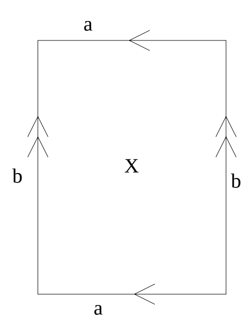

Figure 2: Once punctured torus cut along a and b and flattened to form a rectangle

We wrap the rectangle around into a cylinder so that the b's are together as shown in Figure 3.

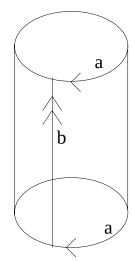

Figure 3: Once punctured torus cut along a to form a cylinder

Then we wrap the cylinder around so that the a's meet as in Figure 1. Any closed curve, called a loop, on this torus can be represented by a "word" which is a combination of the letters a, $\bar{a}$, b, and $\bar{b}$.

The twice punctured torus is similar, but requires a third generator to distinguish between loops surrounding one puncture from loops surrounding both punctures. Thus we take Figure 4 and create our twice punctured torus shown in Figure 5.

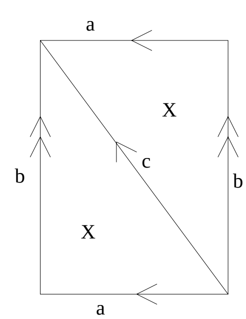

Figure 4: Twice punctured torus cut along a and b and flattened to form a rectangle

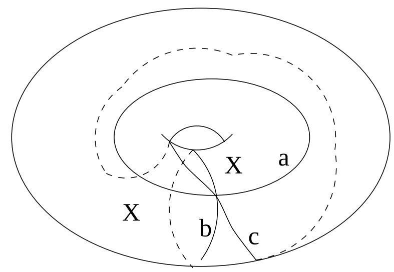

Figure 5: Twice punctured torus

We can also cut along a and lay the torus out like a disc to obtain Figure 6. This often aids in visualization.

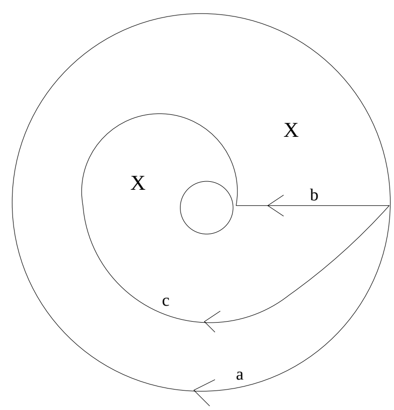

Figure 6: Twice punctured torus cut along b and flattened to form a disk

(Note: Due to the symmetry seen in Figure 4 it is unnecessary to distinguish between the two punctures.)

Past papers such as [1], [4], [5], [6], [7] have classified loops on once punctured 1 - n holed tori. These classifications have certain applications to number theory. In this paper we wish to classify once and twice non-trivial self intersecting loops on the twice punctured torus. The applications of these classifications are not readily apparent at this time. Never-the-less, they are classifications worth examining for possible future applications as well as for mathematical completeness.

2 Simple Loops

Before exploring the classification of loops with one and two non-trivial self intersections on the twice punctured torus, we will first look at the classification of simple (non-self intersecting) loops on this torus.

The cunjugacy class of a simple loop l on the twice punctured torus with set generators a, b, and c as in Figures 4, 5, and 6 is either:

(a) l bounds a disk and is equivalent to the identity

(b) l bounds a once punctured disk and is equivalent to the word abc

(c) l bounds a twice punctured disk and is equivalent to the word $ab\bar{a}\bar{b}$

(d) l is a non separating curve and is equivalent to a

(Note: $\bar{a}$ is the inverse of a, and similarly for $\bar{b}$ and $\bar{c}$. Further, equivalent here and throughout the paper means that there is a homeomorphism or permutation which takes one loop to the other.)

This classification of simple loops is helpful because of the following lemma:

Lemma 1: Every loop with k self intersections is the composition of k+1 simple loops.[4]

3 The classification of loops with a single non-trivial self intersection on the twice punctured torus

After applying Lemma 1 our goal becomes to classify both the compositions of 2 simple loops and 3 simple loops on the twice punctured torus. However, if we were to start composing 2 and 3 simple loops on the torus, we would surely miss one or more possibilities. Thus we must use a systematic method to insure that we obtain all such possible loops. The method we shall use involves Euler characteristics. As the classification of once self intersecting loops is less complicated than twice self intersecting loops, we will start with it and later apply many of the same basic principles to the more complicated twice intersecting loop classification.

As stated earlier, we cannot begin by simply listing all the combinations of compositions of 2 simple loops on our torus for fear of missing some of the possibilities. Instead, let us assume that we already have an entire list of all possible single self intersecting loops on the torus. Now it only remains for us to discover what is on that list.

Obviously, on the torus in the neighborhood of the single intersection point we have something of the form shown in Figure 7.

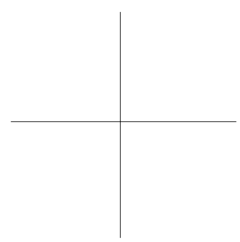

Figure 7: The torus in the neighborhood of the single intersection point

If we allow a and a' to denote initial and final segments of a simple loop (and similarly for b and b'), we obtain the three possible configurations depicted in Figure 8.

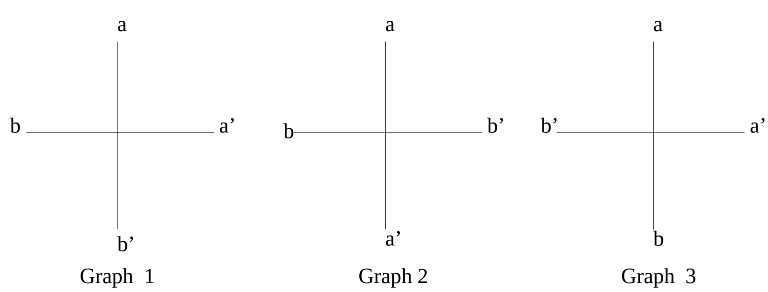

Figure 8: The three possible configurations for loop l with a single self intersection point

Graphs 2 and 3 can be deformed in such a manner that the intersection is actually trivial. Thus, if graphs 2 and 3 were placed on our torus, the result may be a simple loop rather than a loop with exactly one transverse self intersection. As we are souly interested in loops with a single transverse intersection, we will only consider graph number 1 which we will call a base point graph. Our base point graph can be completed in the manner depicted in Figure 9.

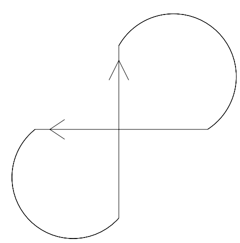

Figure 9: The possible base point graph configuration

Thus, our base point graph is somehow situated on the twice punctured torus. However, we still do not have a list of all such possible situations. In order to obtain such a list, we must cut the torus along our base point graph. This will result in three curves or boundaries which are depicted in Figure 10.

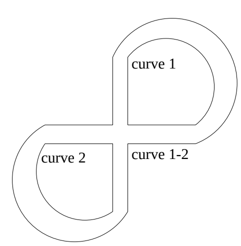

Figure 10: The curves which result after cutting along the base point graph

We now have curve 1, curve 2, and curve 1-2 (the curve encompassing curve 1 and curve 2). Originally, the twice punctured torus had an Euler characteristic of 0. By cutting along the base point graph, we have added an extra vertex. Thus, the Euler characteristic goes up by 1. Furthermore, the torus has been separated into three regions with the three curves as boundary components. However, we do not want to have any boundary components, so it is necessary to "cap off" the three regions by attaching a disk to each boundary component. However, each disk that we attach adds 1 more to the Euler characteristic. Thus, the Euler characteristic is now 0 + 1(for the 1 new vertices) + 3(for the 3 disks we attached). This creates a total Euler characteristic of 4. So, where we once had a twice punctured torus, we now have some number of surfaces with a total Euler characteristic of 4.

Since the Euler characteristic must total 4, we know that we must have at least 2 spheres (the Euler characteristic of a sphere is 2). Thus we have 2 spheres and some number of tori. However, we only have three capped off regions, so we can only have a total of three surfaces. Thus, after cutting along the base point graph and capping off the regions, we have either two spheres or two spheres and a torus.

On these surfaces are our three curves and two punctures from the original torus. Each surface contains at least one curve (otherwise it would not have been separated into a new surface when we cut along our base point graph). Furthermore, curve 1 cannot be alone on a sphere. If it were, then it could be deformed into a single point in which case our original base point graph would not have had a transverse intersection. Similarly, curve 2 cannot be alone on a sphere. Curve 1-2 does not share this problem as it contains the transverse intersection.

Keeping this in mind, it is now a simple matter to list all of the ways in which the three curves and two punctures can be placed on our two or three surfaces. Table 1 is a complete list of possibilities where 1 denotes curve 1, 2 denotes curve 2, 1-2 denotes curve 1-2, and x denotes a puncture. The curve or puncture is listed under the surface containing it, and each row depicts a single possible arrangement. (Note that curve 1 and curve 2 are symmetric and thus need not be differentiated.)

Table 1			
	Sphere	*Sphere*	*Torus*
1	1, 1-2, X	2, X	—
2	1-2, 1	2, X, X	—
3	1-2, X	1, 2, X	—
4	1-2, X, X	1, 2	—
5	1-2	1, 2, X, X	—
6	1-2	1, X	2, X
7	1-2	1, X, X	2
8	1-2, X	1, X	2
9	1, X	2, X	1-2

The manner in which this table was formulated, assures us that we have a complete list of all possible configurations. It now remains for us to attach the curves so as to reform our uncut base point graph on the torus. By doing so we will obtain a complete list of possible pictures of tori containing single self intersecting loops. From these pictures we can discover the corresponding words as well. By gluing back together each of our cut base point graphs in Table 1 we obtain the corresponding words and pictures shown in Figure 11. (Note: the tori have been displayed in the manner of Figure 6 with dashed lines representing the loops.)

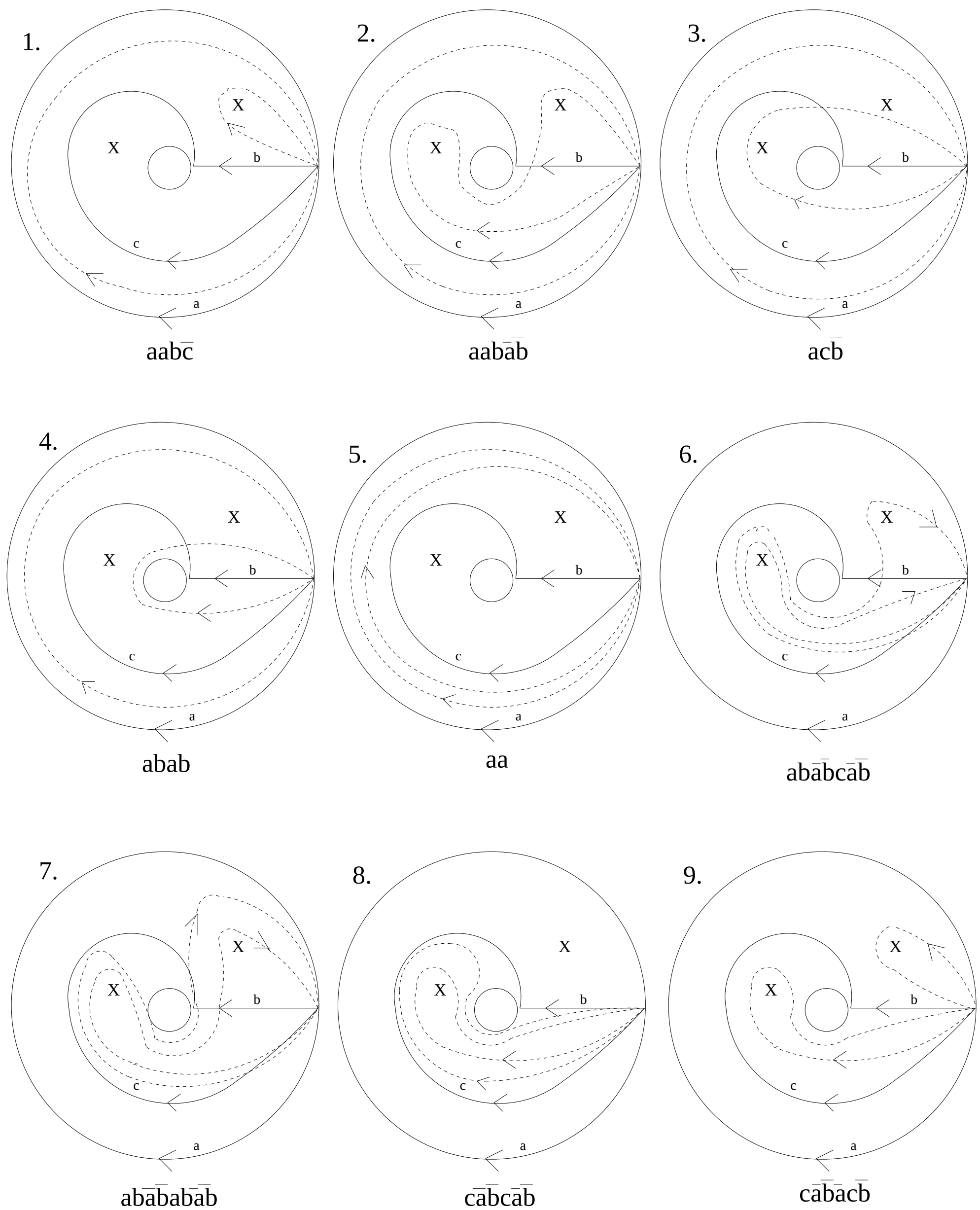

Figure 11: The pieces from Table 1 glued back into tori along with the word representations

Now that we have a complete list of possible loops with one non-trivial self intersection on the twice punctured torus, we must check for duplications. We will start by using Whitehead's algorithm. We can use a program created by Michael Lau to determine the minimal length of each word. If two words have different minimal length, then they are necessarily distinct. If they do not, then a different argument must be used to determine distinctness.[2], [3]

Lau's program gave the following results shown in Table 2. It displays the original and minimal length words corresponding to single self intersecting loops on the twice punctured torus.

Table 2

	Original Word	*Minimal Word*	*Length of Minimal Word*
1	$aab\bar{c}$	c	1
2	$aaba\bar{b}$	$aaba\bar{b}$	5
3	$ac\bar{b}$	c	1
4	$abab$	bb	2
5	aa	aa	2
6	$aba\bar{b}ca\bar{b}$	c	1
7	$aba\bar{b}aba\bar{b}$	$aba\bar{b}aba\bar{b}$	8
8	$cabcab$	c	1
9	$ca\bar{b}ac\bar{b}$	$ca\bar{b}ac\bar{b}$	6

We see that words 1, 3, 6, and 8 share a minimal length of 1 and words 4 and 5 share a minimal length of 2. According to Whitehead's theorem, the words of different minimal length are necessarily distinct. So now the only loops we need to check for distinctness are loops 1, 3, 6 and 8; and loops 4 and 5. (Note that we do know that loop 1 is distinct from loop 4 and etc. because they are of different minimal length.) We will test these for distinctness by replacing one of the punctures, thereby transforming our torus into a once punctured torus. If the resulting loops are distinct on the once punctured torus, then they are necessarily distinct on the twice punctured torus. (Note that due to possible transformations of the torus, it is necessary to verify that the loops formed by replacing one of the punctures are distinct regardless of which puncture is replaced.) After following this method for all relevant loops, we find that loops 1, 3, 6, and 8 are all distinct, but loop 4 is equivalent to loop 5.

The results are that this topological argument works for all but loops 4 and 5. These loops still need to be distinguished from each other. Some other type of argument must be found to show that this pair of loops is or is not distinct. Thus, after discarding all of the known duplications and assuming that the loops mentioned above are not distinct (by assuming that they are equivalent we are looking at a worst-case scenario) we find the conjugacy class, depicted in Figure 12, of single self intersecting loops on the twice punctured torus with set generators a, b, and c. (Note: some of the pictures represent more than one case. For example, the two X_1's represent one possible position for the punctures and the two X_2's represent a different possible location for the punctures and therefore, a different class.)

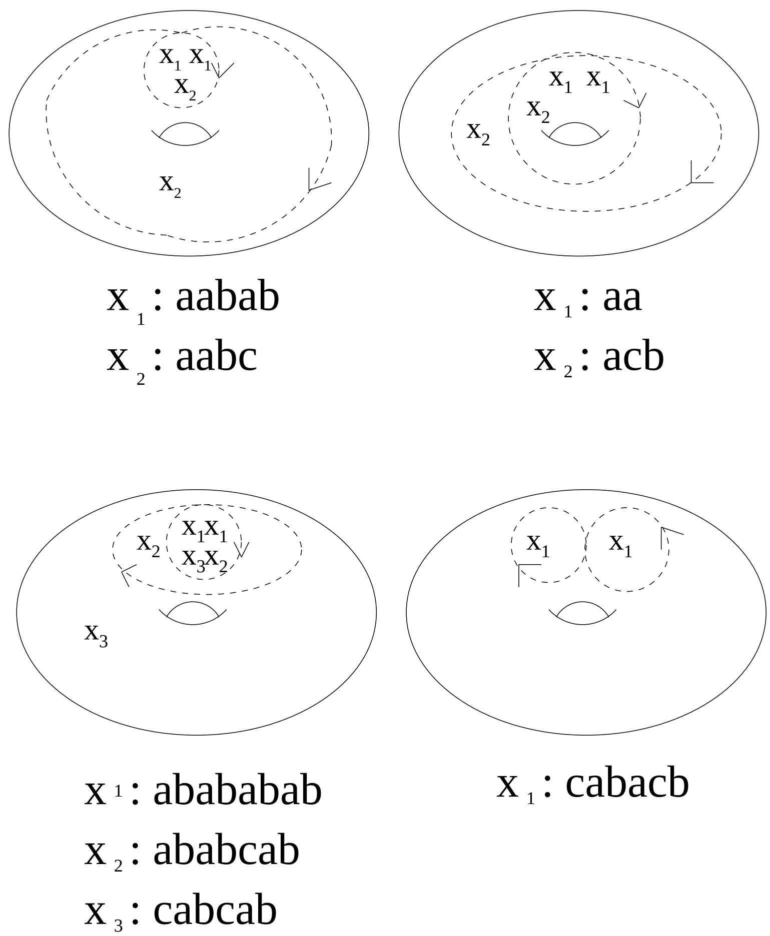

Figure 12: The conjugacy classes for loops with one transverse self intersection on the twice punctured torus

4 The classification of loops with two non-trivial self intersections on the twice punctured torus

Although the two self intersection case is slightly more complicated than the one; the method of classifying loops with two self intersections on the twice punctured torus is similar to that used for loops with one self intersection. As in the one self intersection case, we will start by assuming that we have an entire list of all possible twice self intersecting loops on the torus. Now we must discover what is on that list. In the neighborhood of the two intersection points, we have something on our torus which takes the form shown in Figure 13.

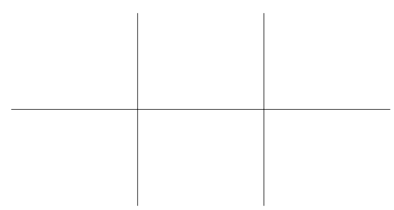

Figure 13: The torus in the neighborhood of the two intersection points

In 1996 Gould, Steiner, and Steinhoff listed all of the possible ways to attach the six segments shown in Figure 13. They discarded all duplications due to symmetry and all cases not resulting in exactly two transverse intersections, and they found that only three possibilities remained. These possibilities are shown in Figure 14.[7]

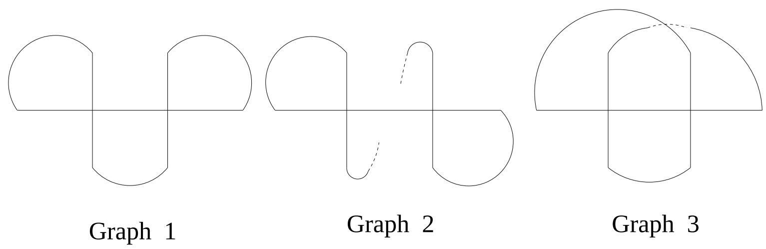

Figure 14: The three possible configurations for the loop l with two self intersection points

Thus our classification list consists of twice punctured tori each of which have either graph 1, graph 2, or graph 3 wrapped around it in some manner. Again, we must cut the torus along the graph in order to obtain a complete list of all the possible ways in which this can be done. Let us first cut apart the tori containing graph 1. When we cut along graph 1 the four curves depicted in Figure 15 will result.

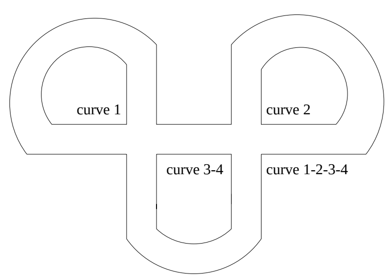

Figure 15: The four curves which result when we cut along Graph 1

We now have curve 1, curve 2, curve 3-4, and curve 1-2-3-4 (the curve encompassing curve 1, curve 2, and curve 3-4). Although we now have four curves as opposed to three, the following argument is very similar to the one used in the one self intersection case. Originally, the twice punctured torus had an Euler characteristic of 0. By cutting along graph 1, we have added two more vertices. Thus, the Euler characteristic goes up by 2. Furthermore, the torus has been separated into four regions with the four curves as boundary components. However, we do not want to have any boundary components, so it is necessary to "cap off" the four regions by attaching a disk to each boundary component. However, each disk that we attach adds 1 more to the Euler characteristic. Thus, the Euler characteristic is now 0 + 2(for the 2 new vertices) + 4(for the 4 disks we attached). This creates a total Euler characteristic of 6. So, where we once had a twice punctured torus, we now have some number of surfaces with a total Euler characteristic of 6.

Since the Euler characteristic must total 6, we know that we must have at least 3 spheres (the Euler characteristic of a sphere is 2). Thus we have 3 spheres and some number of tori. However, we only have four capped off regions; so we can only have a total of four surfaces. Thus, after cutting along graph 1 and capping off the regions, we have either three spheres or three spheres and a torus. On these three or four surfaces lie curve 1, curve 2, curve 3-4, curve 1-2-3-4, and the two punctures from the original twice punctured torus. As in the one self intersection case, each surface contains at least one curve and curve 1 cannot be alone on a sphere. Curve 1 and curve 2 are fundamentally the same as they are symmetric, thus curve 2 cannot be alone on a sphere and curves 1 and 2 need not be differentiated. For similar reasons to those discussed in the one intersection case, curve 3-4 cannot be alone on a sphere either but curve 1-2-3-4 does not share this restriction.

With these restrictions in mind, it is now a simple matter to list all of the possible ways in which the four curves and 2 punctures can be placed on our three or four surfaces. Table 3 is a complete list of these possibilities and it follows the same conventions used in Table 1.

Table 3					
	Sphere	*Sphere*	*Sphere*	*Torus*	*Corresponding picture from figures 18 and 19*
1	1, X	2, X	3-4, 1-2-3-4	—	14
2	1, X, X	2, 3-4	1-2-3-4	—	5
3	1, X	3-4, X	1-2-3-4, X	—	27
4	1, 2	3-4, X	1-2-3-4, X	—	2
5	1, X	2, 3-4	1-2-3-4, X	—	6
6	1, 2, X	3-4, X	1-2-3-4	—	3
7	1, X	3-4, X	1-2-3-4	2	28
8	1, X	2, X	1-2-3-4	3-4	8
9	1, 2	3-4, X, X	1-2-3-4	—	4
10	1, X	2, 3-4, X	1-2-3-4	—	7

Now we must go through a similar process for graph 2. When we cut along graph 2 the four curves depicted in Figure 16 will result.

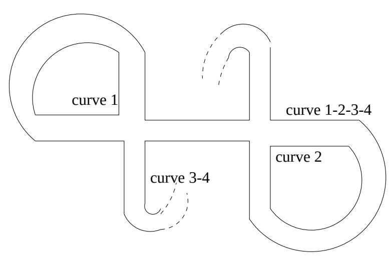

Figure 16: The four curves which result when we cut along Graph 2

The curves, regions, and surfaces thus formed are the same as those formed by cutting along graph 1. The only difference is that in this case, curve 3-4 encompasses one of the intersection points, and thus it can be alone on a sphere since it cannot be deformed to a single point. With this in mind, it is again a simple matter to list all of the ways in which the four curves and 2 punctures can be placed on our three or four surfaces. Table 4 lists these possibilities in the same manner as Table 3.

Table 4					
	Sphere	*Sphere*	*Sphere*	*Torus*	*Corresponding picture from figures 18 and 19*
1	1, 2, X, X	3-4	1-2-3-4	—	21
2	1, 2, X	3-4, X	1-2-3-4	—	25
3	1, 2	3-4, X, X	1-2-3-4	—	23
4	1, X	2, 3-4, X	1-2-3-4	—	13
5	1, X, X	2, 3-4	1-2-3-4	—	11
6	1, 2, X	3-4	1-2-3-4, X	—	24
7	1, 2	3-4, X	1-2-3-4, X	—	26
8	1, X	2, 3-4	1-2-3-4, X	—	12
9	1, X, X	3-4	1-2-3-4, 2	—	10
10	1, X	3-4, X	1-2-3-4, 2	—	9
11	1, 2	3-4	1-2-3-4, X, X	—	22
12	1, X	3-4	1-2-3-4, 2, X	—	29
13	1, X, X	3-4	1-2-3-4	2	15
14	1, X	3-4, X	1-2-3-4	2	18
15	1, X	2, X	1-2-3-4	3-4	30
16	1, X	3-4	1-2-3-4	2, X	17
17	1, X	3-4	1-2-3-4, X	2	16
18	1, X	2, X	3-4	1-2-3-4	1
19	1, X	2, X	1-2-3-4, 3-4	—	31

Again, the process for graph 3 is similar, however; cutting along graph three results in only 2 curves rather than four. These are shown in Figure 17.

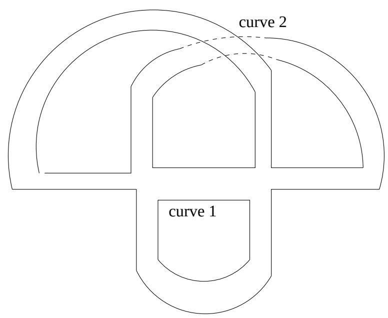

Figure 17: The two curves which result when we cut along Graph 3

Again, this adds two more vertices and thus adds 2 to the Euler Characteristic. It also separates the torus into 2 regions which, when capped off, add another 2 to the Euler characteristic. Thus we have a total Euler characteristic of 4. This means that we must now have at least 2 spheres. However, as we only have 2 capped off regions, we can only have a maximum of 2 surfaces. Thus we know that cutting along graph 3 and capping off the regions results in 2 spheres upon which are the 2 curves and 2 punctures. Again, curve 1 cannot be alone on a sphere as it would then be possible to deform it into a single point and thus lose one of the intersection points. We can now make another table, Table 5, similar to Tables 3 and 4 which shows the possible ways of placing the 2 curves and 2 punctures on the 2 spheres.

Table 5

	Sphere	*Sphere*	*Torus*	*Corresponding picture from figures 18 and 19*
1	1, X, X	2	—	20
2	1, X	2, X	—	19

The manner in which Tables 3, 4, and 5 were created assures us that we have a complete list of all possible configurations of twice self intersecting curves on the twice punctured torus. However, in order to get the list in the desired form, it is necessary to once again attach the curves to reform our uncut graphs on the torus. We thus obtain a complete list of possible pictures of tori containing twice self intersecting loops. From these pictures we can obtain the corresponding words. These pictures and words are shown in Figures 18 and 19. (Note: the tori have been displayed in the manner of Figure 6 with dashed lines representing the loops.)

Now that we have a complete list of possible loops with two transverse self intersections on the twice punctured torus, we must check for duplications. We will do this in exactly the same manner in which we did the one self intersection case. First we use Lau's program to find the minimal words which correspond to each loop. Lau's program offers the minimum words found in Table 6.

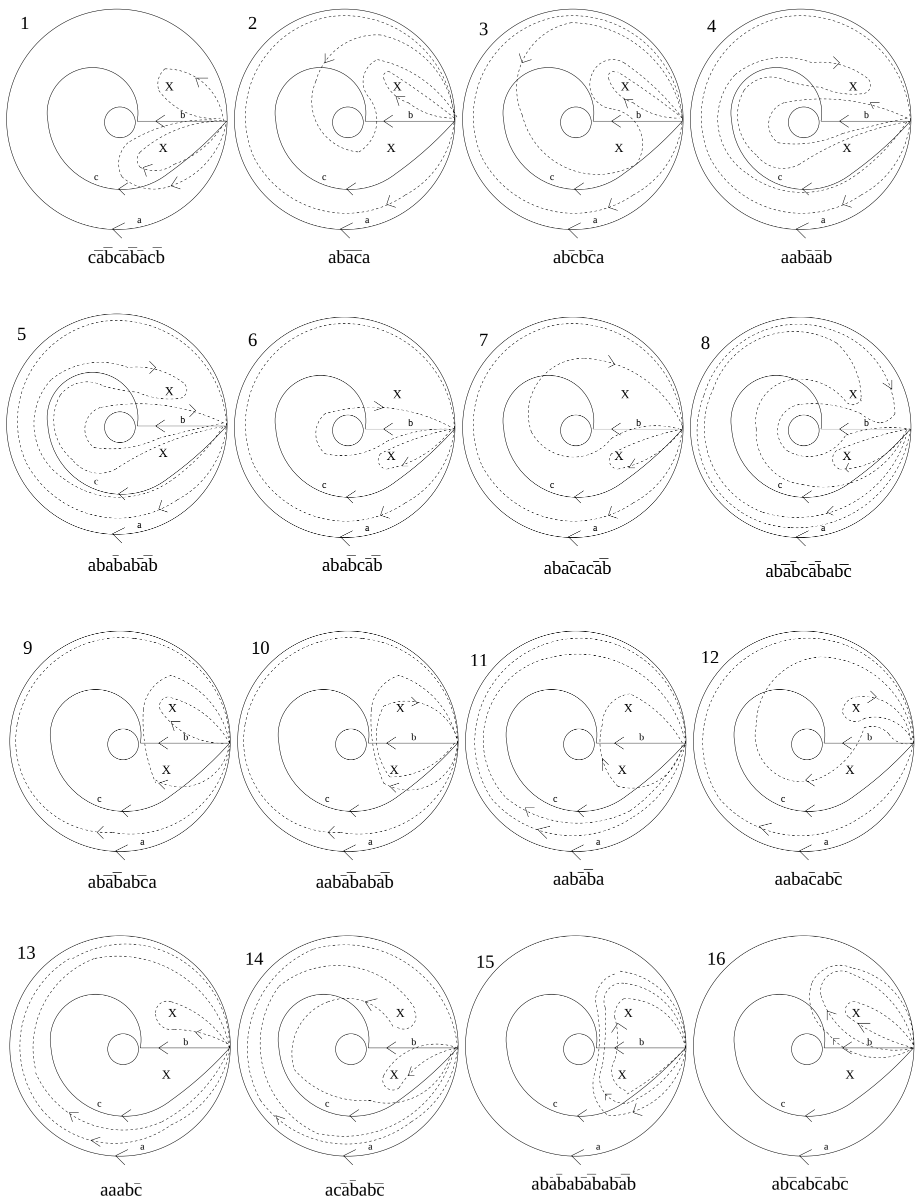

Figure 18: The pieces from Table 1 glued back into tori along with the word representations

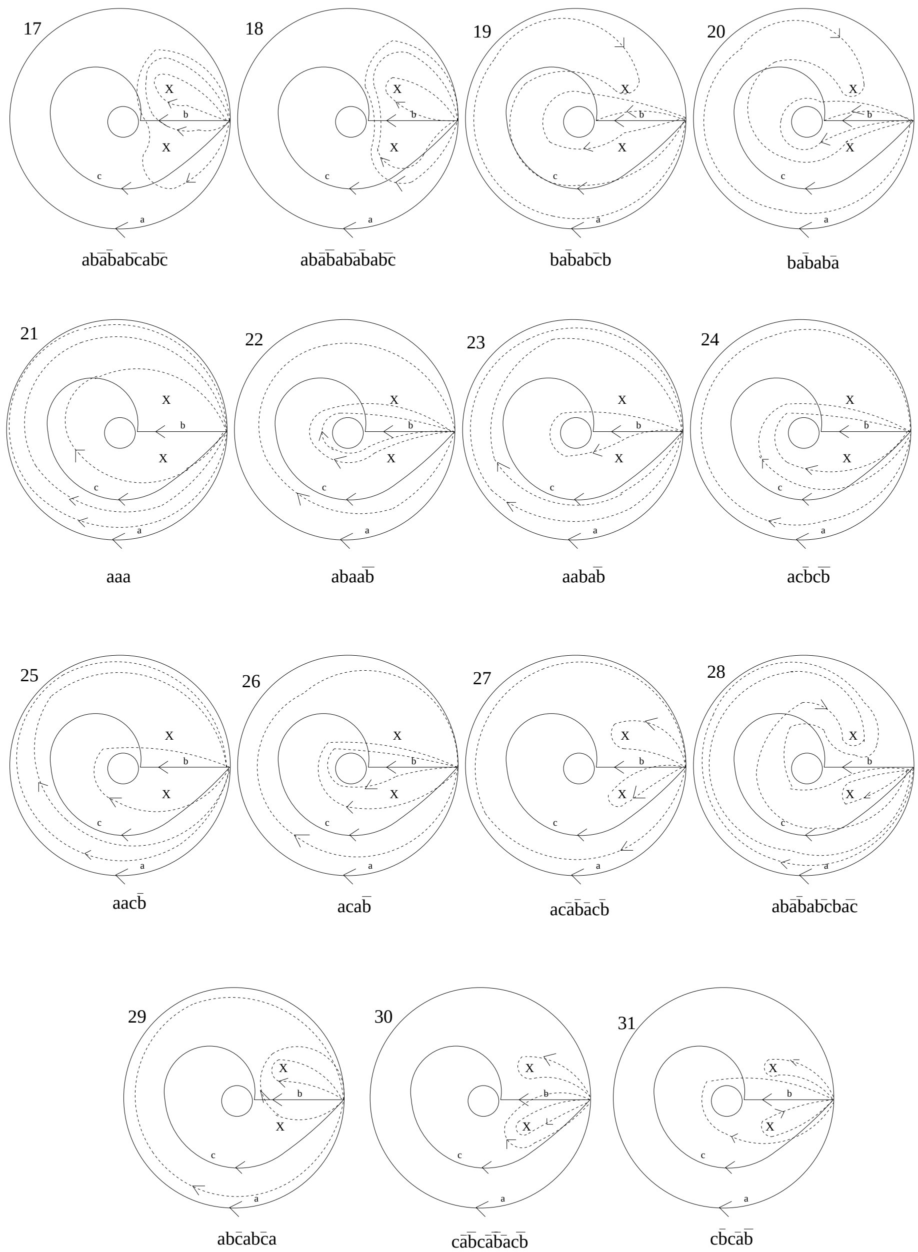

Figure 19: The pieces from Table 1 glued back into tori along with the word representations (continued)

	Table 6		
	Original Word	*Minimal Word*	*Length of Minimal Word*
1	$cabcabac\bar{b}$	$cc\bar{a}cba\bar{b}$	7
2	$ab\bar{c}b\bar{c}a$	$a\bar{c}ca$	4
3	$ab\bar{a}ca$	$\bar{c}$	1
4	$aa\bar{b}aab$	$aa\bar{b}aab$	6
5	$aba\bar{b}abab$	$aba\bar{b}abab$	8
6	$aba\bar{b}cab$	c	1
7	$aba\bar{c}aca\bar{b}$	$a\bar{c}ac$	4
8	$aba\bar{b}cababa b\bar{c}$	$aba\bar{b}cabab\bar{c}$	10
9	$ababab\bar{c}a$	$\bar{c}$	1
10	$aababa\bar{b}ab$	$aababa\bar{b}ab$	9
11	$aababa$	$aababa$	6
12	$aaba\bar{c}ab\bar{c}$	$aba\bar{c}b\bar{c}$	6
13	$aaab\bar{c}$	a	1
14	$acabab\bar{c}$	$acabab\bar{c}$	7
15	$ababababab ab$	$abababababab$	12
16	$ab\bar{c}ab\bar{c}ab\bar{c}$	$c\bar{c}c$	3
17	$ababab\bar{c}ab\bar{c}$	$baba\bar{c}c$	6
18	$ababababab\bar{c}$	$\bar{c}$	1
19	$babab\bar{c}b$	c	1
20	$ba\bar{b}ab\bar{a}$	$ba\bar{b}ab$	5
21	aaa	aaa	3
22	$abaab$	$abaab$	5
23	$aaba\bar{b}$	$aaba\bar{b}$	5
24	$ac\bar{b}c\bar{b}$	a	1
25	$aac\bar{b}$	c	1
26	$aca\bar{b}$	c	1
27	$aca\bar{b}acb$	$aca\bar{b}acb$	7
28	$ababab\bar{c}ba\bar{c}$	$ababab\bar{c}ba\bar{c}$	10
29	$ab\bar{c}ab\bar{c}a$	a	1
30	$cabcabac\bar{b}$	$cc\bar{a}cba\bar{b}$	7
31	$cbcab$	$\bar{a}$	1

Next we use the topological method of replacing punctures to determine if loops with the same minimal length are distinct. The results are that this topological argument works for all but the following loops. Loops 24 and 25, loops 1 and 30, loops 23 and 22, loops 6 and 13, loops 8 and 28, loops 14 and 27, and loops 11 and 12 still need to be distinguished from each other. Some other type of argument must be found to show that these 7 pairs of loops are or are not distinct. Thus, after discarding all of the known duplications and assuming that the loops mentioned above are not distinct (by assuming that each pair is equivalent we are looking at a worst-case scenario) we find the conjugacy class depicted in Figure 20, of twice self intersecting loops on the twice punctured torus with set generators a, b, and c. (Note: some of the pictures represent more than one case. For example, the two X_1's represent one possible position for the punctures and the two X_2's represent a different possible location for the punctures and therefore, a different class.)

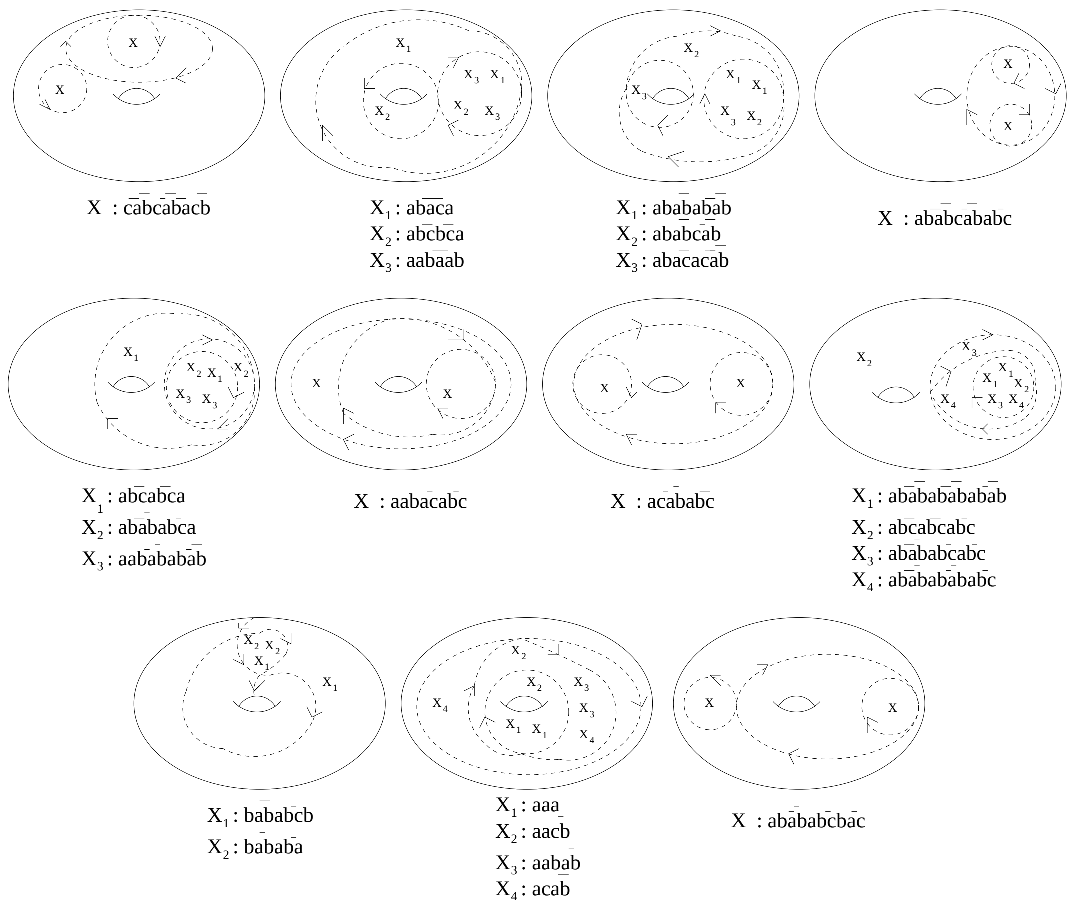

Figure 20: The conjugacy class for loops with two self intersections on the twice punctured torus

5 Conclusion

Thus we have basically succeeded in classifying loops with one and two transverse self intersections on the twice punctured torus. However, we still must go back and determine whether or not the loops left in question are or are not distinct. It also remains to be determined what, if any, are the applications of this classification. We could continue the project by classifying loops with three self intersections on the twice punctured torus as well as classifying loops on twice punctured tori of higher genus. It would be interesting to note the relationship, if any, of these classifications with those of the once punctured torus.

6 References

[1] Mande Butler et al. *GeodesicswithThreeIntersectionsonthePuncturedTorus.* Research started during the Summer 1995 NSF REU Program at Oregon State University, 1995.

[2] J.H.C. Whitehead. On certain sets of elements in a free group. *Proc.LondonMath.Soc*41; 48-56, 1936.

[3] J.H.C. Whitehead. On equivalent sets of elements in a free group. *Ann.ofMath.*. 782-800, 1936.

[4] D. Crisp. *TheMarkoffSpectrumandGeodesicsonthePuncturedTorus.* Univ. Adelaide. 1993.

[5] David Crisp et al. *ClosedCurvesandGeodesicswithTwoSelf−IntersectionsonthePuncture* Research started during the Summer 1994 NSF REU Program at Oregon State University. 1996.

[6] Susan Garner Garille, *AMathematicaProgramforClassifyingGeodesicswithkSelfInterse* Research started during the Summer 1995 NSF REU Program at Oregon State University. 1995.

[7] Ryan Gould et al, *ClassificationofLoopswithSelf−intersectionsontheOncePuncturedTor* Research started during the Summer 1996 NSF REU Program at Oregon State University. 1996.