Classification of Loops with Two Self-intersections on the Once Punctured Torus with Genus N

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Abstract

In this paper I classify the homotopy classes of loops with two self-intersections on the once punctured torus of genus n. I rely on a topological argument to systematically classify all such loops. A discussion of the distinctess of the homotopy classes as determined by the application of Whitehead automorphisms follows the classification.

1 Introduction

In this paper, I classify all possible twice self-intersecting curves on the surface of the once puntured torus of genus n. Aside from this, my goal in this paper is to establish an exhaustive method for solving such classification problems that can be easily applied to higher numbers of self-intersections and to curves on different types of surfaces. I will begin by giving background information, which will provide a concrete conception of the surface on which I am working as well as the types of curves that can be found on this surface. I will continue by establishing my method of analysis, which is based on the Euler Characteristic and which provides a way of ensuring thoroughness in my classifications. I will then give a classification of the curves. I will conclude with a discussion of the application of Whitehead's Theorem and its corresponding algorithm for the purpose of establishing distinctness of homotopy classes.

2 Background

For much of this background material, I am indebted to [3].

2.1 T_n

Throughout this paper, I will be working with a puctured torus of genus n, denoted T_n. The fundamental group of T_n, $\pi_1(T_n)$, is the free group on $2n$ letters, $F(a_1, b_1, a_2, b_2...a_n, b_n)$. There is a bijection between free homotopy classes of closed curves on T_n and conjugacy classes of elements of $F(a_1, b_1, a_2, b_2...a_n, b_n)$, a fact which will be useful in determining the distinctness of the curves via Whitehead's algorithm. Each curve on the surface of T_n can be represented algebraically by a word consisting of members of this free group.

T_n can be interpreted as the union of n one-holed tori, with each one-holed torus generated by a pair of letters in the free group, (a_i, b_i). More specifically, T_n is the quotient space of the $4n$-gon formed by associating identically labeled sides of the $4n$-gon as shown in Figure 1. The generators on T_n are oriented as shown in Figure 2. The "X" denotes the puncture. By convention, the puncture is considered to be on the nth torus, although my analysis includes all possible locations for the puncture in relation to the curves.

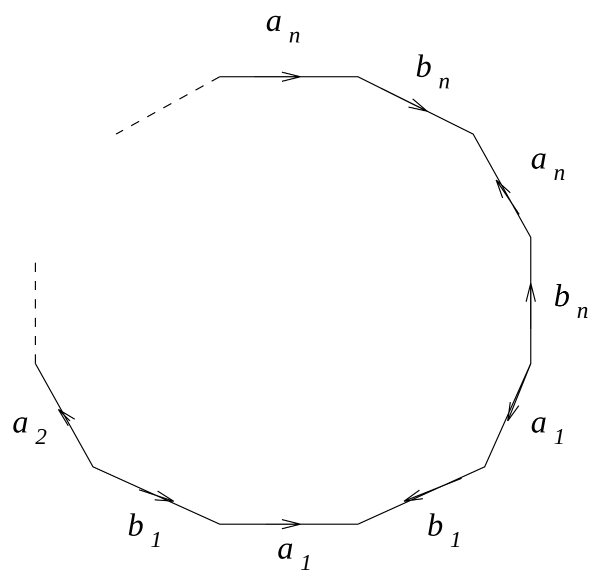


Figure 1

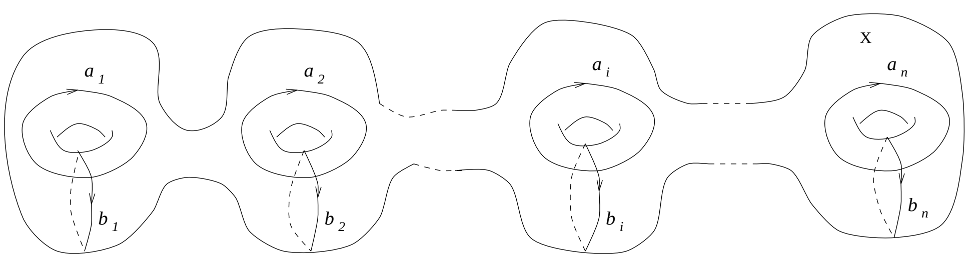


Figure 2

2.2 Loops on T_n

Definition 1 *A* **closed loop** *l on the surface of T_n is defined to be the image of a map $f : [0, 1] \rightarrow T_n$ such that $f(0) = f(1)$. l is said to be simple if and only if $f(i) = f(j) \Rightarrow i = 0$ and $j = 1$. Otherwise, l has a self intersection when $f(i) = f(j)$ where $0 < i, j < 1$.*

Throughout this paper, when I speak of a loop with a certain number of self intersections, I assume that the intersections are transverse and that the loop is not homotopic to a simple loop, or to any loop with a smaller number of self intersections.

2.3 Cutting and Gluing

My argument in this paper will be based on the systematic classification of curves according to the surfaces formed when T_n is "cut open" along the curves. If two loops create homeomorphic surfaces when they are cut along, then there is a homeomorphism which maps one loop onto the other [3].

First we need a precise definition of what it means to "cut" along a simple loop, l. The fact that T_n is a manifold determines the existence of an open interval around each point on l such that removing the interior produces two identical copies of l. Each of these copies is a **boundary component** of a region, i.e. cutting along l has created two new boundary components, each one a copy of l. The loop l is said to be non-separating if both boundary components created when T_n is cut along l lie on the same region, and separating if two separate regions are formed, each with one boundary component.

The regions formed by this process can be made into surfaces by "gluing" an open disc from $\mathbf{R}^2$ onto the region to fill in the space enclosed by the boundary component. Depending on whether l is separating or non-separating, this process yeilds two or one surfaces, with one or two boundary components on each surface. This process of cutting along curves and forming surfaces is instrumental in my method of determining the homotopy classes of loops on T_n.

2.4 Simple Loops on T_n

Now that we know what happens when we cut along a simple loop, we will examine the types of simple loops that are found on T_n.

Theorem 1 *On the once punctured, n-holed torus, T_n, there exists a homeomorphism that takes any simple closed loop, l, to one of the following:*

1. *a nonseparating loop, b_1,*
2. *a loop enclosing a disc,*
3. *a loop, Δ, enclosing a punctured disc, which can be described by the word $a_1b_1\overline{a_1}\overline{a_1}a_2b_2\overline{a_2}\overline{b_2}...a_nb_n\overline{a_n}\overline{b_n}$, or*
4. *a loop, λ_i, which separates a non-punctured i-holed torus from T_n, where $0 < i < n$. This loop can be described by the word $a_1b_1\overline{a_1}\overline{b_1}...a_ib_i\overline{a_i}\overline{b_i}$. If the puncture is on the torus separated off, the loop is described by the word $b_na_n\overline{b_n}\overline{a_n}\ldots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}}$.*

For the proof of this theorem, see [3]. See Figure 3 for the orientation of these simple loops.

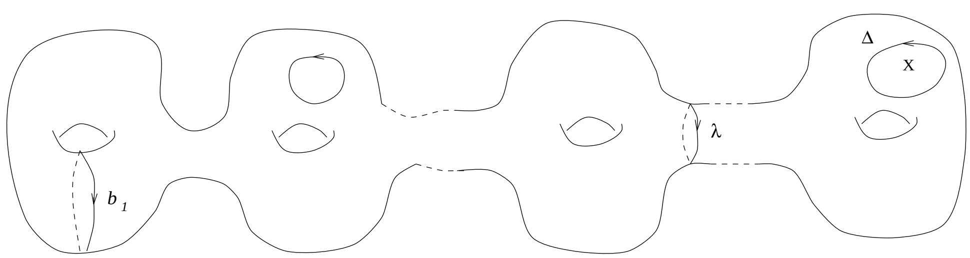


Figure 3

2.5 Loops with Self-intersections on T_n

Now we know the types of simple loops that can be found on T_n, and we have a tool for determining homeomorphism of simple loops using the cutting and gluing method for obtaining surfaces. Our next step is to find a way of applying this knowledge to loops with self-intersections, specifically, loops with two self-intersections. In order to do this, the following lemma is useful:

Lemma 1 *Up to a free homotopy, any loop l on T_n with k transverse self-intersection points can be formed as the composition of $k+1$ simple loops which intersect at only one point.*

The proof of this lemma on the one-holed, once-punctured torus can be found in [1]. The same argument holds for the two-holed once punctured torus. By this lemma, we can consider a loop with two self-intersections, l as the composition of three simple loops, l_1, l_2, and l_3, which intersect at only one point. This point of intersection is called the basepoint and this way of representing the curve is called the basepoint graph. Now that we have broken l down into simple loops, we can apply our cutting and gluing technique. Algebraically, the word for our loop l is the product of the words for each simple loop.

3 The Classification of Curves with Two Self-intersections on T_n

This section contains a detailed description of my classification techniques as well as the classification of all possible twice-intersecting curves on T_n.

3.1 Basepoint Graphs and the Euler Characteristic Technique

In this section, I will explain my classification techniques in detail. First, I will divide the loops into three general categories. Because we are considering more than one simple loop, there are different possible configurations for the loops to be in. Each possible configuration creates a different set of boundary components when the loop is cut along. This is important to my analysis because my technique rests upon the fact that a two curves creating homeomorphic surfaces are homeomorphic. This fact is true only up to the configuration of the curve. If two curves of different configurations separate off the same surfaces, there is not necessarily a homeomorphism between them.

Much of what follows is contained in [3]. To determine these possible configurations, consider a neighborhood around the two intersection points on our twice self-intersecting curve l (See Figure 4).

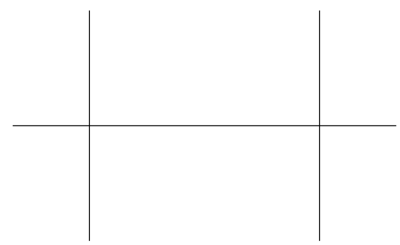


Figure 4

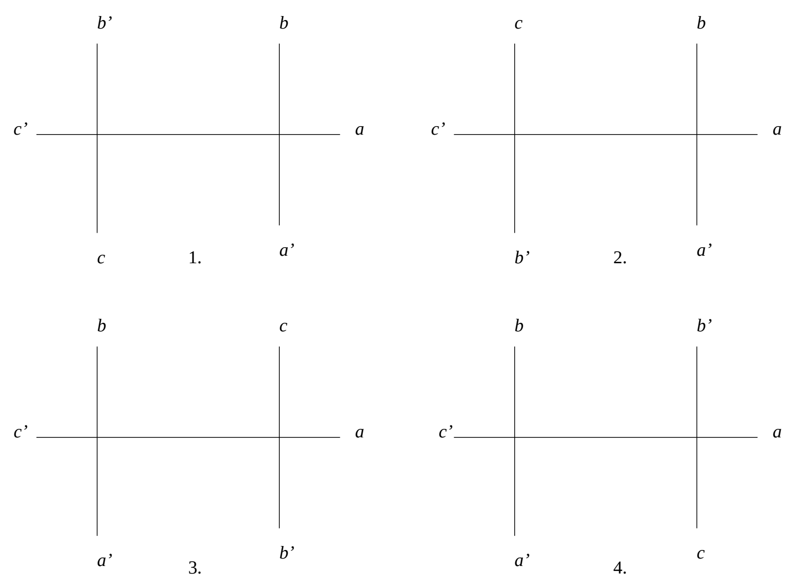


Figure 5

Considering symmetry arguments, there are four possible configurations, as shown in Figure 5. By Lemma 1, we know that these two intersections can be reduced to one, since l is equivalent to three simple loops that meet at a single point. This reduction is achieved by collapsing along some axis. For configurations 1 and 2, regardless of the axis chosen to collapse along, we obtain some rotation or reflection of basepoint graphs 1 and 2, respectively.

For configurations 3 and 4, regardless of the axis chosen to collapse along, some rotation or reflection of basepoint graph 3 is obtained.

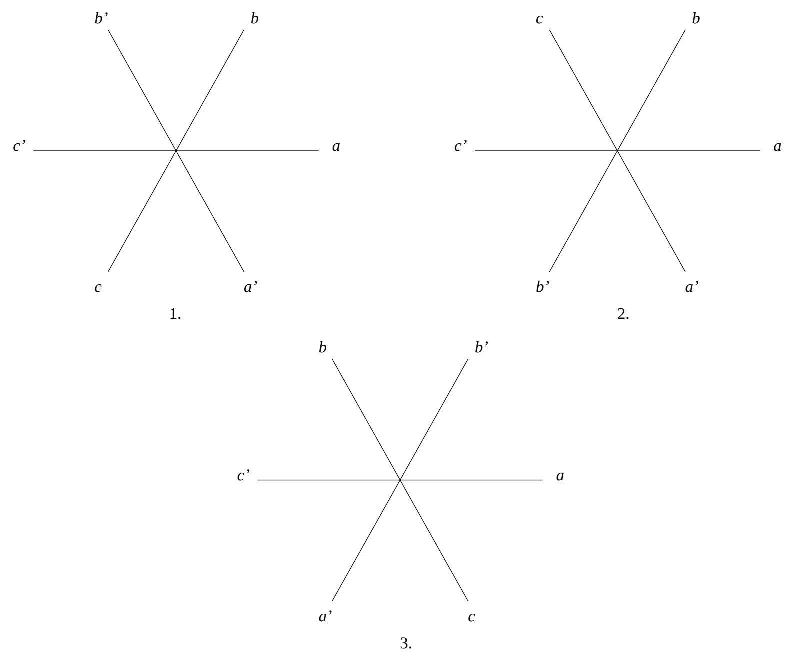


Figure 6

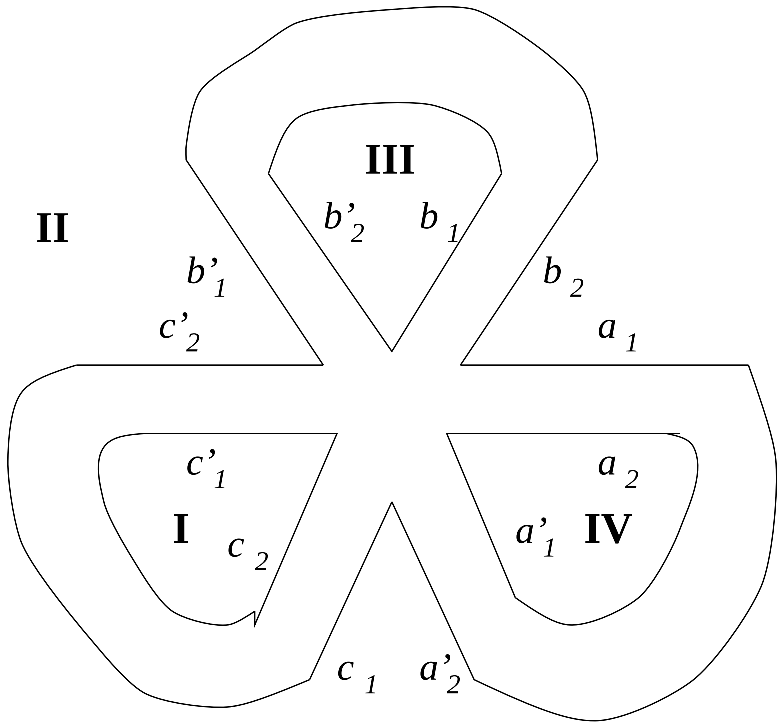


Figure 7

Now we will examine each basepoint graph more carefully.

For basepoint graph 1, after connecting the initial and final segments and cutting along the curve, four boundary components are formed: $C_1 = c_1' \to c_2$, $C_2 = c_1 \to c_2' \to b_1' \to b_2 \to a_1 \to a_2'$, $C_3 = b_2' \to b_1$, and $C_4 = a_2 \to a_1'$. We denote the four regions bounded by the curves as follows: C_1 bounds I, C_2 bounds II, C_3 bounds III, and C_4 bounds IV.

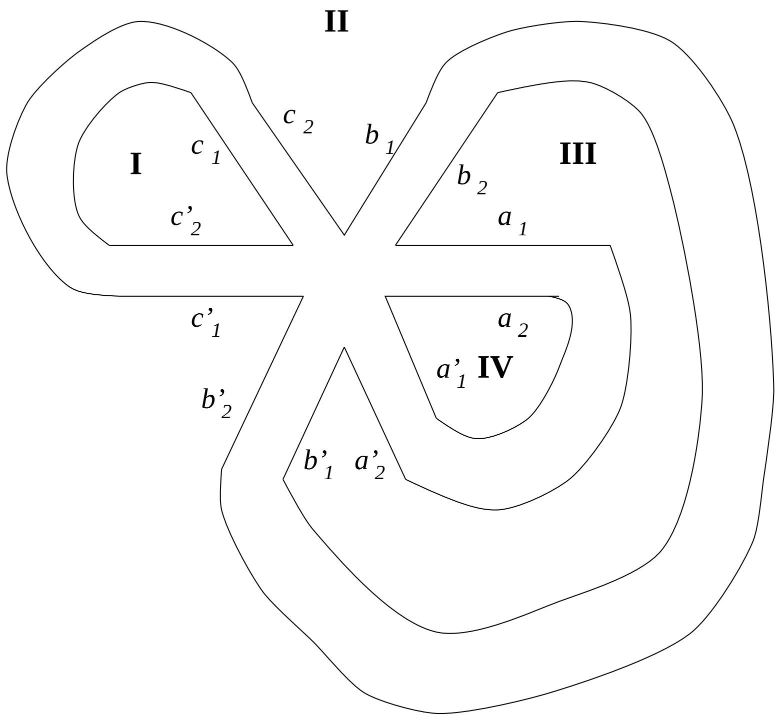


Figure 8

For basepoint graph 2, after connecting the initial and final segments and cutting along the curve, four boundary components are formed: $C_1 = c_1 \to c'_2$, $C_2 = c'_1 \to c_2 \to b_1 \to b'_2$, $C_3 = b_2 \to b'_1 \to a'_2 \to a_1$, $C_4 = a_2 \to a'_1$. We denote the four regions bounded by the curves as follows: C_1 bounds I, C_2 bounds II, C_3 bounds III, and C_4 bounds IV.

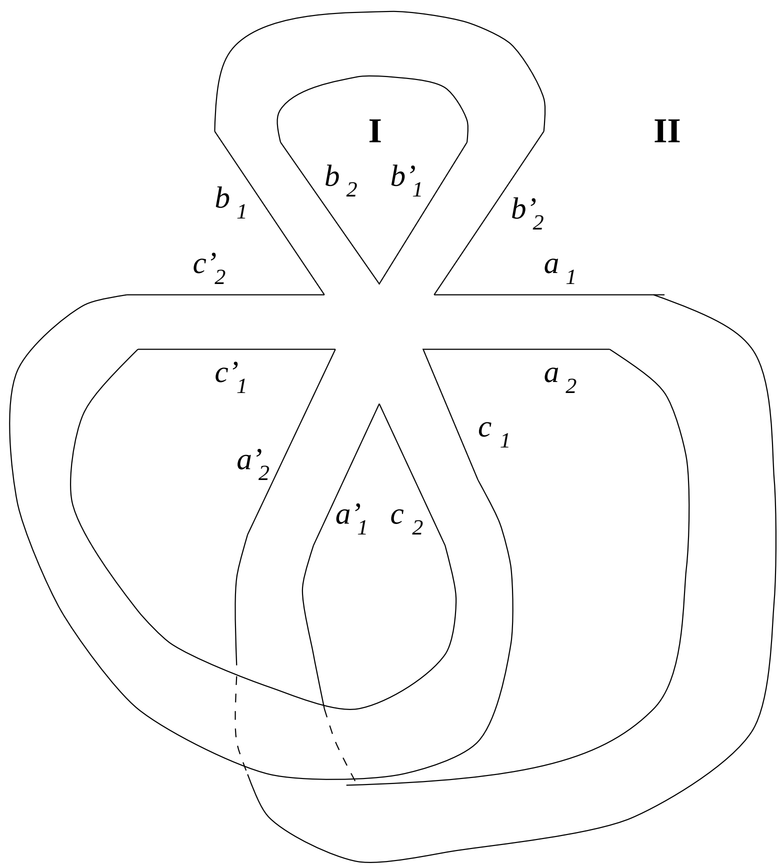


Figure 9

For basepoint graph 3, after connecting the initial and final segments and cutting along the curve, two boundary components are formed: $C_1 = b_2 \to b_1'$, and $C_2 = b_1 \to b_2' \to a_1 \to a_2' \to c_1' \to c_2 \to a_1' \to a_2 \to c_1 \to c_2'$. We denote the two regions bounded by the curves as follows: C_1 bounds I, and C_2 bounds II.

The Euler Characteristic method of classification is based upon the technique of cutting and gluing. We know that a loop l on the surface of T_n with two self-intersections is composed of three simple loops which intersect at a single point. The Euler characteristic of T_n is $2 - 2n$. What happens to this value when we cut along the simple loops and then glue to form surfaces? First, two new veritces are formed, which increases the Euler characteristic by two. Then, for each boundary component, the disc that we add to form a surface also increases the Euler characteristic by one. So when we are done cutting and gluing, χ_{new}=χ_{old}+2+# of boundary curves, where χ_{old}=$2-2n$. So for basepoint graph 1, since four boundary components are formed, $\chi_{new} = 8-2n$. Similarly, for basepoint graph 2, $\chi_{new} = 8-2n$. For basepoint graph 3, since only two boundary components are formed, $\chi_{new} = 6-2n$.

3.2 Table of Possible Surfaces

Now that we know the Euler characteristics produced by cutting along each of the three types of curves, $8-2n$ for basepoint graphs 1 and 2, and $6-2n$ for basepoint graph 3, our next step is to produce a table containing combinations of surfaces the sum of whose Euler characteristics gives us the desired value. In constructing such a table, it is important to keep in mind that any simple loop can be either separating or non-separating which implies that cutting along a curve corresponding to basepoint graph 1 or 2 can produce 1, 2, 3 or 4 surfaces, and cutting along a curve corresponding to basepoint graph 3 can produce 1 or 2 surfaces. We must consider all such cases in our analysis.

$\chi = 8-2n$: **4 Surfaces Formed**				
	A	B	C	D
1	S^2	S^2	S^2	T_n
2	S^2	S^2	T_n	S^2
3	S^2	T_n	S^2	S^2
4	T_n	S^2	S^2	S^2
5	S^2	S^2	T_i	T_{n-i}
6	S^2	T_i	S^2	T_{n-i}
7	T_i	S^2	S^2	T_{n-i}
8	S^2	T_i	T_{n-i}	S^2
9	T_i	S^2	T_{n-i}	S^2
10	T_i	T_{n-i}	S^2	S^2
11	S^2	T_i	T_j	T_{n-i-j}
12	T_i	S^2	T_j	T_{n-i-j}
13	T_i	T_j	S^2	T_{n-i-j}
14	T_i	T_j	T_{n-i-j}	S^2
15	T_i	T_j	T_k	$T_{n-i-j-k}$

$\chi = 8-2n$: **3 Surfaces Formed**			
	A	B	C
1	S^2	S^2	T_{n-1}
2	S^2	T_{n-1}	S^2
3	T_{n-1}	S^2	S^2
4	S^2	T_i	T_{n-i-1}
5	T_i	S^2	T_{n-i-1}
6	T_i	T_{n-i-1}	S^2
7	T_i	T_j	$T_{n-i-j-1}$

$\chi = 8-2n$: **2 Surfaces Formed**		
	A	B
1	S^2	T_{n-2}
2	T_{n-2}	S^2
3	T_i	T_{n-i-2}

In the case where $\chi = 8 - 2n$ and one surface is formed, that surface is T_{n-3}.

$\chi = 6 - 2n$: **2 Surfaces Formed**		
	A	B
1	S^2	T_{n-1}
2	T_{n-1}	S^2
3	T_i	T_{n-i-1}

In the case where $\chi = 6 - 2n$ and one surface is formed, that surface is T_{n-2}.

Now that we have all possible combinations of surfaces, it remains to determine whether curves for each basepoint configuration exist that form such surfaces when cut along. Since we know that any two curves of the same basepoint configuration that form the same surfaces are homeomorphic, we know that these are the only cases we need to consider.

Before we begin systematically anaylyzing the possible cases, I will introduce a lemma which will be useful in the analysis:

Lemma 2 *If a region bounded by a simple loop yields a sphere without a puncture, the region is bounded by a loop which is freely homotopic to the identity.*

Note that this lemma indicates that in Theorem 1 regarding simple loops on T_n, case two, a loop bounding a disc, is homotopic to the identity, and therefore will not be considered as a possibility for one of the three simple loops that make up our curve l. If it were one of the three simple loops, l would no longer have two self-intersections, but would only have one or zero. For this same reason, none of the simple loops making up l can bound a sphere without a puncture because by the lemma they would be homotopic to the identity and l would not have two self-intersections.

My general technique in this anaylsis is first to draw the curve on the surface of T_n according to the surfaces separated off by each curve, labeling the orientation so that l_1 is oriented forwards, and such that both self-intersections were transverse, thus determining the orientations of the other two curves. I then contract the curve along one of its axes so that it is in the form of a basepoint graph. In this form, it is possible to determine the word for each simple loop by examining the pattern of generators involved in forming each loop. Not every simple loop making up l is described exactly by one of the four words in Theorem 1- it depends on the loop's exact position on T_n relative to the other loops. In some cases, it was necessary to deform the curve by sliding it around on the surface of the torus in order to determine the word. Using these techniques and the above lemma, I was able to determine the words for all the possible curves for basepoint graphs 1 and 3, and most of the curves for basepoint graph 2 as well. I will give a somewhat detailed analysis of the first cases for basepoint graph one, and then continue with a listing of words corresponding to the remaining cases.

3.3 Analysis of Possibilities: Basepoint Graph 1

Now I will begin my case by case approach to determining whether basepoint graph 1 type curves exist which create each combination of surfaces.

3.3.1 4 Surfaces Formed

I will start with the case where four surfaces are formed. In this case, none of the simple loops can be non-separating because if one were, four regions would not be formed. Without loss of generality, we can let surface A correspond to region I, surface B correspond to region II, surface C correspond to region III, and surface D correspond to region IV.

Cases 1-4 These cases are impossible with basepoint graph one because each has more than one sphere bounded by a simple loop: in case 1, surfaces A and B; in case 2, surfaces A and D; in case 3, surfaces A, C, and D; and in case 4, surfaces C and D. Since there is only one puncture, at least one of these is an unpunctured sphere. So by the argument above, such a case would contradict the fact that l has two self intersections, and is therefore impossible.

Case 5 Since surface A is a sphere bounded by a simple loop, it must contain the puncture or else the loop would be homotopic to the identity. So $l_1 = \Delta$. Surface C is an i-torus, which implies that $l_2 = \lambda_i$, since the puncture must be betweeen l_2 and l_3 in order for surface B to be a sphere. Since region B is a sphere, l_3 must be λ_i'. So the word $l_1 l_2 l_3$ is

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_n b_n \overline{a_n}\,\overline{b_n} a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i}$$
$$b_n a_n \overline{b_n}\,\overline{a_n} \dots b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}}.$$

Case 6 Again, this case is impossible, since two spheres, A and C are bounded by simple loops.

Case 7 The puncture must be on region C because otherwise l_2 would be homotopic to the identity. So $l_2 = \Delta$. Surface A is an i-torus, which implies that $l_1 = \lambda_i$. Since B is a sphere and l_2 is between l_1 and l_3, l_3 must be $\overline{\lambda'_i}$. So the word $l_1 l_2 l_3$ is

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_i b_i \overline{a_i}\,\overline{b_i} a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_n b_n \overline{a_n}\,\overline{b_n}$$
$$a_{i+1} b_{i+1} \overline{a_{i+1}}\,\overline{b_{i+1}} \ldots a_n b_n \overline{a_n}\,\overline{b_n}.$$

Case 8 This case is impossible, since two spheres, A and D, are bounded by simple loops.

Case 9 There is a homeomorphism of the torus that takes l_1 to l_3, which makes this case equivalent to case 5.

Case 10 This case is impossible since two spheres, C and D, are bounded by simple loops.

Case 11 The puncture must be on A, so $l_1 = \Delta$. Since l_2 separates off a i- torus without a puncture, $l_2 = \lambda_i$. Since B is an i-torus, $l_3 = \lambda'_{i+j}$. So the word for this case is

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_n b_n \overline{a_n}\,\overline{b_n} a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_i b_i \overline{a_i}\,\overline{b_i} b_n a_n \overline{b_n}\,\overline{a_n} \ldots$$
$$b_{i+j+1} a_{i+j+1} \overline{b_{i+j+1} b_{i+j+1}}.$$

Case 12 In this case, the puncture can be on any of the four surfaces. If the puncture is on surface A, l_1 separates off a punctured i-torus, and therefore equals λ'_i. l_2 separates off the next j holes of the torus, and l_3 separates off the final $n-i-j$ and is therefore equal to $\overline{\lambda'_{i+j}}$. So with the puncture on A, the word is

$$b_n a_n \overline{b_n}\,\overline{a_n} \ldots b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}} b_{i+j} a_{i+j} \overline{b_{i+j}}\,\overline{a_{i+j}} \ldots$$
$$b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}} a_{i+j+1} b_{i+j+1} \overline{a_{i+j+1}}\,\overline{b_{i+j+1}} \ldots a_n b_n \overline{a_n}\,\overline{b_n}.$$

With the puncture on B, the torus l_1 separates off is no longer punctured, so $l_1 = \lambda_i$. l_2 and l_3 are unaffected by this different location of the puncture and remain unchanged. So the word for the case where the puncture is on B is

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_i b_i \overline{a_i}\,\overline{b_i} b_{i+j} a_{i+j} \overline{b_{i+j}}\,\overline{a_{i+j}} \ldots$$
$$b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}} a_{i+j+1} b_{i+j+1} \overline{a_{i+j+1}}\,\overline{b_{i+j+1}} \ldots a_n b_n \overline{a_n}\,\overline{b_n}.$$

With the puncture on C, l_1 and l_3 remain the same as above, but the word for l_2 changes. The new word for this case is

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_i b_i \overline{a_i}\,\overline{b_i} a_{i+j+1} b_{i+j+1} \overline{a_{i+j+1}}\,\overline{b_{i+j+1}} \ldots$$
$$a_n b_n \overline{a_n}\,\overline{b_n} a_1 b_1 \overline{a_1}\,\overline{b_1} \ldots a_i b_i \overline{a_i}\,\overline{b_i} a_{i+j+1} b_{i+j+1} \overline{a_{i+j+1}}\,\overline{b_{i+j+1}} \ldots a_n b_n \overline{a_n}\,\overline{b_n}.$$

The final case, where the puncture is on D, is equivalent to the first case where the puncture is on A because there is a homeomorphism which reverses the orientation of the curve which essentially takes the fourth case to the first case.

Case 13 The puncture must be on C, so $l_2 = \Delta$. l_1 separates off an unpunctured i-torus and is therefore equal to λ_i. There is a j-torus in the region between l_1 and l_3, so l_3 is equal to $\overline{\lambda'_{i+j}}$. So the actual word is:

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i} a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_n b_n \overline{a_n}\,\overline{b_n}$$
$$a_{i+j+1} b_{i+j+1} \overline{a_{i+j+1}}\,\overline{b_{i+j+1}} \dots a_n b_n \overline{a_n}\,\overline{b_n}.$$

Case 14 The puncture must be on D, so $l_3 = \overline{\Delta}$. l_1 separates off an unpunctured i-torus and is therefore equal to λ_i. There is a j-torus between l_1 and l_2, and the puncture is also located between them, so $l_2 = \lambda'_{i+j}$. So the word for this curve is:

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i} b_n a_n \overline{b_n}\,\overline{a_n} \dots$$
$$b_{i+j+1} a_{i+j+1} \overline{b_{i+j+1}}\,\overline{a_{i+j+1}} b_n a_n \overline{b_n}\,\overline{a_n} \dots b_1 a_1 \overline{b_1}\,\overline{a_1}.$$

Case 15 The puncture can be in four places, although the case where the puncture is on D is equivalent to the case where it is on A. If the puncture is on A, l_1 separates off a punctured i-torus and is therefore equal to λ'. l_2 separates off a k-torus, but there is also a j-torus enclosed by all three loops, so the word for l_2 is $b_{i+j+k} a_{i+j+k} \overline{b_{i+j+k}}\,\overline{a_{i+j+k}} \dots b_{i+j+1} a_{i+j+1} \overline{b_{i+j+1}}\,\overline{a_{i+j+1}}$. The word for l_3 is $\overline{\lambda'_{i+j+k}}$. So the word for the curve is:

$$b_n a_n \overline{b_n}\,\overline{a_n} \dots b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}} b_{i+j+k} a_{i+j+k} \overline{b_{i+j+k}}\,\overline{a_{i+j+k}} \dots$$
$$b_{i+j+1} ai + j + 1 \overline{b_{i+j+k}}\,\overline{a_{i+j+k}} a_{i+j+k+1} b_{i+j+k+1} \overline{a_{i+j+k+1}}\,\overline{b_{i+j+k+1}} \dots a_n b_n \overline{a_n}\,\overline{b_n}.$$

3.3.2 3 Surfaces Formed

For this case, there is more than one distinct assignment of regions I, II, III and IV to surfaces A, B and C. All seven cases must be analyzed for each possible assignment. We will start with the assignment A contains II and III, B contains I and C contains IV.

Case 1 and Case 2 These cases are in fact equivalent for this assignment since there is a homeomorphism of the torus which reverses orientation and thereby takes l_1 to l_3. I will give the analysis for case 2. The puncture must be on C, which makes l_3 equal to $\overline{\Delta}$. l_1 separates off a non-punctured $n-1$-torus and is therefore equal to λ_{n-1}. l_2 is a non-separating curve and in this case is equal to $\overline{a_n}$. So the word for this case is:

$$a_1 b_1 \overline{a_1}\overline{b_1} \ldots a_{n-1} b_{n-1} \overline{a_{n-1}}\overline{b_{n-1}}\overline{a_n} b_n a_n \overline{b_n}\overline{a_n} \ldots b_1 a_1 \overline{b_1}\overline{a_1}.$$

Case 3 This case is impossible since there are two spheres bounded by simple loops, B and C.

Case 4 The puncture can be in three places, on A, B or C. If the puncture is on A, the word is: $\lambda_i \overline{a_{i+1}}\overline{\lambda'_{i+1}} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \ldots a_i b_i \overline{a_i}\overline{b_i}\overline{a_{i+1}} a_{i+2} b_{i+2} \overline{a_{i+2}}\overline{b_{i+2}} \ldots a_n b_n \overline{a_n}\overline{b_n}.$$

If the puncture is on B, the word is $\lambda'_i \overline{a_{i+1}}\overline{\lambda'_{i+1}} =$

$$b_n a_n \overline{b_n}\overline{a_n} \ldots b_{i+1} a_{i+1} \overline{b_{i+1}}\overline{a_{i+1} a_{i+1}} a_{i+2} b_{i+2} \overline{a_{i+2}}\overline{b_{i+2}} \ldots a_n b_n \overline{a_n}\overline{b_n}.$$

If the puncture is on C, the word is: $\lambda_i \overline{a_{i+1}}\overline{\lambda_{i+1}} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \ldots a_i b_i \overline{a_i}\overline{b_i}\overline{a_{i+1}} b_{i+1} a_{i+1} \overline{b_{i+1}}\overline{a_{i+1}} \ldots b_1 a_1 \overline{b_1}\overline{a_1}.$$

Case 5 and Case 6 By the same argument as for cases 1 and 2, these cases are equivalent for this particular assignment of regions. I will give the word for case 6: $\lambda_i \overline{a_k}\overline{\Delta} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \ldots a_i b_i \overline{a_i}\overline{b_i}\overline{a_k} b_n a_n \overline{b_n}\overline{a_n} \ldots b_1 a_1 \overline{b_1}\overline{a_1}.$$

This is for $k > i$.

Case 7 There are three possible locations for the puncture. If the puncture is on A, the word is $\lambda'_i \overline{a_k}\overline{\lambda'_{i+j}} =$

$$b_n a_n \overline{b_n}\overline{a_n} \ldots b_{i+1} a_{i+1} \overline{b_{i+1}}\overline{a_{i+1} a_k} a_{i+j+1} b_{i+j+1} \overline{a_{i+j+1}}\overline{b_{i+j+1}} \ldots a_n b_n \overline{a_n}\overline{b_n}.$$

If the puncture is on B, the word is $\lambda_i \overline{a_k}\overline{\lambda'_{i+j}} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \ldots a_i b_i \overline{a_i}\overline{b_i}\overline{a_k} a_{i+j+1} b_{i+j+1} \overline{a_{i+j+1}}\overline{b_{i+j+1}} \ldots a_n b_n \overline{a_n}\overline{b_n}.$$

If the puncture is on C, the word is $\lambda_i \overline{a_k}\overline{\lambda_{i+j}} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} \ldots a_i b_i \overline{a_i}\overline{b_i}\overline{a_k} b_{i+j} a_{i+j} \overline{b_{i+j}}\overline{a_{i+j}}.$$

This is where $i < k < i + j$.

Now we will consider the case where A contains II and IV, B contains I and C contains III. Note that this is the same as if A contained II and I, B contained IV and C contained III since there is a homomorphism taking l_1 to l_3.

Case 1 The puncture must be on B, so the word is $\Delta\overline{\lambda_1' b_1} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_n b_n \overline{a_n}\,\overline{b_n} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_n b_n \overline{a_n}\,\overline{b_n b_1}.$$

Case 2 The puncture must be on C, so the word is $\lambda_{n-1}\Delta\overline{b_n} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} \ldots a_{n-1} b_{n-1} \overline{a_{n-1}}\,\overline{b_{n-1}} a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_n b_n \overline{a_n}\,\overline{b_n b_n}.$$

Case 3 This case is impossible since two spheres are bounded by simple loops, B and C.

Case 4 There are three possibilities for the puncture. If the puncture is on A, the word is $\lambda_i \lambda_{i+1}' a_{i+1} =$

$$b_n a_n \overline{b_n}\,\overline{a_n} \ldots b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}} b_n a_n \overline{b_n}\,\overline{a_n} \ldots b_{i+2} a_{i+2} \overline{b_{i+2}}\,\overline{a_{i+2}} a_{i+1}.$$

If the puncture is on B, the word is $\lambda_i \lambda' i + 1 a_{i+1} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_i b_i \overline{a_i}\,\overline{b_i} b_n a_n \overline{b_n}\,\overline{a_n} \ldots b_{i+2} a_{i+2} \overline{b_{i+2}}\,\overline{a_{i+2}} a_{i+1}.$$

If the puncture is on C, the word is $\lambda_i \lambda_{i+1} a_{i+1} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_i b_i \overline{a_i}\,\overline{b_i} a_1 b_1 \overline{a_1}\,\overline{b_1} a_{i+1} b_{i+1} \overline{a_{i+1}}\,\overline{b_{i+1}} a_{i+1}.$$

Case 5 The puncture must be on B. The word is $\Delta\overline{\lambda_i' b_j} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_n b_n \overline{a_n}\,\overline{b_n} a_{i+1} b_{i+1} \overline{a_{i+1}}\,\overline{b_{i+1}} \ldots a_n b_n \overline{a_n}\,\overline{b_n b_j}.$$

Here, $j < i$.

Case 6 The puncture must be on C. The word is $\lambda_i \Delta\overline{b_j} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_i b_i \overline{a_i}\,\overline{b_i} a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_n b_n \overline{a_n}\,\overline{b_n b_j}.$$

Here $j > i$.

Case 7 The puncture can be in three places. If the puncture is on A, the word is $\lambda_i \lambda_{i+j}' a_k =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_i b_i \overline{a_i}\,\overline{b_i} b_n a_n \overline{b_n}\,\overline{a_n} \ldots b_{i+j+1} a_{i+j+1} \overline{b_{i+j+1}}\,\overline{a_{i+j+1}} a_k.$$

If the puncture is on B, the word is $\lambda'_i\lambda'_{i+j}a_k =$

$$b_na_n\overline{b_n}\overline{a_n}\dots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}}b_na_n\overline{b_n}\overline{a_n}\dots b_{i+j+1}a_{i+j+1}\overline{b_{i+j+1}}\overline{a_{i+j+1}}a_k.$$

If the puncture is on C, the word is $\lambda_i\lambda_{i+j}a_k =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_i}a_1b_1\overline{a_1}\overline{b_1}\dots a_{i+j}b_{i+j}\overline{a_{i+j}}\overline{b_{i+j}}a_k.$$

Here, $i < k < i + j$.

Now we will consider the case where A contains III and IV, B contains I and C contains II. Again this is equivalent to A containing III and I, B containing IV and C containing II, as explained above.

Case 1 The puncture must be on B. The word is $\Delta\overline{a_n}b_na_nb_n =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_nb_n\overline{a_n}\overline{b_n}\overline{a_n}b_na_nb_n.$$

Case 2 The puncture can be in three places. If the puncture is on A, the word is $\lambda_{n-1}a_n\overline{b_n}\overline{a_n}\Delta\overline{b_n} =$

$$a_1b_1\overline{a_1}\overline{b_1}\dots a_{n-1}b_{n-1}\overline{a_{n-1}}\overline{b_{n-1}}a_n\overline{b_n}\overline{a_n}a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_nb_n\overline{a_n}\overline{b_nb_n}.$$

If the puncture is on B, the word is $\lambda'_{n-1}a_n\overline{b_n}\overline{a_n}\overline{b_n} =$

$$b_na_n\overline{b_nb_n}\overline{a_n}\overline{b_n}.$$

If the puncture is on C, the word is $\lambda_{n-1}a_n\overline{b_n}\overline{a_n}\overline{b_n} =$

$$a_1b_1\overline{a_1}\overline{b_1}\dots a_{n-1}b_{n-1}\overline{a_{n-1}}\overline{b_{n-1}}a_n\overline{b_n}\overline{a_n}\overline{b_n}.$$

Case 3 The puncture must be on B. The word is $\Delta\lambda_{n-1}b_nb_n =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_nb_n\overline{a_n}\overline{b_n}a_1b_1\overline{a_1}\overline{b_1}\dots a_{n-1}b_{n-1}\overline{a_{n-1}}\overline{b_{n-1}b_nb_n}.$$

Case 4 The puncture can be in three places. If the puncture is on A, the word is $\lambda_ia_n\overline{b_n}\overline{a_n}\Delta\overline{b_n} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_i}a_n\overline{b_n}\overline{a_n}a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_nb_n\overline{a_n}\overline{b_nb_n}.$$

If the puncture is on B, the word is $\lambda'_ia_n\overline{b_n}\overline{a_n}\overline{b_n} =$

$$b_na_n\overline{b_n}\overline{a_n}\dots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}}a_n\overline{b_n}\overline{a_n}\overline{b_n}.$$

If the puncture is on C, the word is $\lambda_i a_n \overline{b_n}\,\overline{a_n}\,\overline{b_n} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i} a_n \overline{b_n}\,\overline{a_n}\,\overline{b_n}.$$

Case 5 The puncture must be on B. The word is $\Delta b_{i+1} \overline{\lambda'_i} b_{i+1} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_n b_n \overline{a_n}\,\overline{b_n} b_{i+1} a_{i+1} b_{i+1} \overline{a_{i+1}}\,\overline{b_{i+1}} \dots a_n b_n \overline{a_n}\,\overline{b_n} b_{i+1}.$$

Case 6 The puncture can be in three places. If the puncture is on A, the word is $\lambda_i \lambda_i \overline{b_{i+1} b_{i+1}} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i} a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i b_{i+1} b_{i+1}}.$$

If the puncture is on B, the word is $\lambda'_i \lambda'_i \overline{b_{i+1} b_{i+1}} =$

$$b_n a_n \overline{b_n}\,\overline{a_n} \dots b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}} b_n a_n \overline{b_n}\,\overline{a_n} \dots b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}}\,\overline{b_{i+1} b_{i+1}}.$$

If the puncture is on C, the word is $\lambda_i \lambda'_i \overline{b_{i+1} b_{i+1}} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i} b_n a_n \overline{b_n}\,\overline{a_n} \dots b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}}\,\overline{b_{i+1} b_{i+1}}.$$

Case 7 The puncture can be in three places. If the puncture is on A, the word is $\lambda_i \lambda_{i+j} \overline{b_{i+j+1} b_{i+j+1}} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i} a_1 b_1 \overline{a_1}\,\overline{b_1} \dots a_{i+j} b_{i+j} \overline{a_{i+j}}\,\overline{b_{i+j} b_{i+j+1} b_{i+j+1}}.$$

If the puncture is on B, the word is $\lambda'_i \lambda'_{i+j} \overline{b_{i+j+1} b_{i+j+1}} =$

$$b_n a_n \overline{b_n}\,\overline{a_n} \dots b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}} b_n a_n \overline{b_n}\,\overline{a_n} \dots b_{i+j+1} a_{i+j+1} \overline{b_{i+j+1}}\,\overline{a_{i+j+1}}\,\overline{b_{i+j+1} b_{i+j+1}}.$$

If the puncture is on C, the word is $\lambda_i \lambda'_{i+j} \overline{b_{i+j+1} b_{i+j+1}} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i} b_n a_n \overline{b_n}\,\overline{a_n} b_{i+j+1} a_{i+j+1} \overline{b_{i+j+1}}\,\overline{a_{i+j+1}}\,\overline{b_{i+j+1} b_{i+j+1}}.$$

Now we will consider the case where A contains I and IV, B contains III, and C contains II.

Case 1 The puncture must be on B. The word is $\overline{a_1}\,\overline{b_1} a_1 \Delta b_1 =$

$$\overline{a_1}\,\overline{b_1} a_1 a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_n b_n \overline{a_n}\,\overline{b_n} b_1.$$

Case 2 The puncture can be in three places. If the puncture is on A, the word is $\overline{a_1}\overline{b_1}a_1\overline{\Delta}\lambda'_1 b_1 =$

$$\overline{a_1}\overline{b_1}a_1 b_n a_n \overline{b_n}\overline{a_n}\dots b_1 a_1 \overline{b_1}\overline{a_1} b_n a_n \overline{b_n}\overline{a_n}\dots b_2 a_2 \overline{b_2}\overline{a_2} b_1.$$

If the puncture is on B, the word is $\overline{a_1}\overline{b_1}a_1\lambda'_1 b_1 =$

$$\overline{a_1}\overline{b_1}a_1 b_n a_n \overline{b_n}\overline{a_n}\dots b_2 a_2 \overline{b_2}\overline{a_2} b_1.$$

If the puncture is on C, the word is $\overline{a_1}\overline{b_1}a_1\lambda_1 b_1 =$

$$\overline{a_1}\overline{b_1}a_1 a_1 b_1 \overline{a_1}\overline{b_1} b_1.$$

Case 3 The puncture is on B. The word is $\lambda_1\overline{b_1}\overline{\Delta}b_1 =$

$$a_1 b_1 \overline{a_1}\overline{b_1 b_1} b_n a_n \overline{b_n}\overline{a_n}\dots b_1 a_1 \overline{b_1}\overline{a_1} b_1.$$

Case 4 The puncture can be in three places. If the puncture is on A, the word is $\overline{a_1}\overline{b_1}a_1\overline{\Delta}\lambda'_i b_1 =$

$$\overline{a_1}\overline{b_1}a_1 b_n a_n \overline{b_n}\overline{a_n}\dots b_1 a_1 \overline{b_1}\overline{a_1} b_n a_n \overline{b_n}\overline{a_n}\dots b_{i+1} a_{i+1} \overline{b_{i+1}}\overline{a_{i+1}} b_1.$$

If the puncture is on B, the word is $\overline{a_1}\overline{b_1}a_1\lambda'_i b_1 =$

$$\overline{a_1}\overline{b_1}a_1 b_n a_n \overline{b_n}\overline{a_n}\dots b_{i+1} a_{i+1} \overline{b_{i+1}}\overline{a_{i+1}} b_1.$$

If the puncture is on C, the word is $\overline{a_1}\overline{b_1}a_1\lambda_i b_1 =$

$$\overline{a_1}\overline{b_1}a_1 a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2}\dots a_i b_i \overline{a_i}\overline{b_i} b_1.$$

Case 5 The puncture must be on B. The word is $\lambda_i\overline{b_{i+1}}\Delta b_{i+1} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2}\dots a_i b_i \overline{a_i}\overline{b_i b_{i+1}} a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2}\dots a_n b_n \overline{a_n}\overline{b_n} b_{i+1}.$$

Case 6 The puncture can be in three places. If the puncture is on A, the word is $\lambda'_i\overline{b_{i+1}}\lambda'_{i+1} b_{i+1} =$

$$b_n a_n \overline{b_n}\overline{a_n}\dots b_{i+1} a_{i+1} \overline{b_{i+1}}\overline{a_{i+1}}\overline{b_{i+1}} b_n a_n \overline{b_n}\overline{a_n}\dots b_{i+2} a_{i+2} \overline{b_{i+2}}\overline{a_{i+2}} b_{i+1}.$$

If the puncture is on B, the word is $\lambda_i\overline{b_{i+1}}\lambda' i + 1 b_{i+1} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2}\dots a_i b_i \overline{a_i}\overline{b_i b_{i+1}} b_n a_n \overline{b_n}\overline{a_n}\dots b_{i+2} a_{i+2} \overline{b_{i+2}}\overline{a_{i+2}} b_{i+1}.$$

If the puncture is on C, the word is $\lambda_i\overline{b_{i+1}}\lambda_{i+1}b_{i+1} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_i}\overline{b_{i+1}}a_1b_1\overline{a_1}\overline{b_1}\dots a_{i+1}b_{i+1}\overline{a_{i+1}}\overline{b_{i+1}}b_{i+1}.$$

Case 7 The puncture can be in three places. If the puncture is on A, the word is $\lambda'_i\overline{b_{i+1}}\lambda'_{i+j}b_{i+1} =$

$$b_na_n\overline{b_n}\overline{a_n}\dots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}}\overline{b_{i+1}}b_na_n\overline{b_n}\overline{a_n}\dots b_{i+j+1}a_{i+j+1}\overline{b_{i+j+1}}\overline{a_{i+j+1}}b_{i+1}.$$

If the puncture is on B, the word is $\lambda_i\overline{b_{i+1}}\lambda'_{i+j}b_{i+1} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_i}\overline{b_{i+1}}b_na_n\overline{b_n}\overline{a_n}\dots b_{i+j+1}a_{i+j+1}\overline{b_{i+j+1}}\overline{a_{i+j+1}}b_{i+1}.$$

If the puncture is on C, the word is $\lambda_i\overline{b_{i+1}}\lambda_{i+j}b_{i+1} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_i}\overline{b_{i+1}}a_1b_1\overline{a_1}\overline{b_1}\dots a_{i+j}b_{i+j}\overline{a_{i+j}}\overline{b_{i+j}}b_{i+1}.$$

3.3.3 2 Surfaces Formed

The first case we will consider is the one where A contains II, III and IV, and region B contains I.

Case 1 The puncture can be in three places. If the pucture is on A, the word is $\lambda_{n-2}\overline{a_{n-1}}a_n =$

$$a_ab_a\overline{a_a}\overline{b_a}\dots a_{n-2}b_{n-2}\overline{a_{n-2}}\overline{b_{n-2}}\overline{a_{n-1}}a_n.$$

If the puncture is on B, the word is $\lambda'_{n-2}\overline{a_{n-1}}a_n =$

$$b_na_n\overline{b_n}\overline{a_n}b_{n-1}a_{n-1}\overline{b_{n-1}}\overline{a_{n-1}a_{n-1}}a_n.$$

Case 2 The puncture is on B. The word is $\Delta a_i\overline{a_j} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_nb_n\overline{a_n}\overline{b_n}a_i\overline{a_j}.$$

Case 3 The puncture can be in two places. If the puncture is on A, the word is $\lambda_i\overline{a_j}a_k =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_i}\overline{a_j}a_k.$$

If the puncture is on B, the word is $\lambda'_i\overline{a_j}a_k =$

$$b_na_n\overline{b_n}\overline{a_n}\dots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}a_j}a_k.$$

Now we will consider the case where A contains I, II, and IV, and B contains III.

Case 1 There are two possibilities for the puncture. If the puncture is on A, the word is $b_n\lambda_{n-2}b_{n-1} =$

$$b_n a_1 b_1 \overline{a_1}\overline{b_1} \ldots a_{n-2} b_{n-2} \overline{a_{n-2}}\overline{b_{n-2}} b_{n-1}.$$

If the puncture is on B, the word is $b_n\lambda'_{n-2}b_{n-1} =$

$$b_n b_n a_n \overline{b_n}\overline{a_n} b_{n-1} a_{n-1} \overline{b_{n-1}}\overline{a_{n-1}} b_{n-1}.$$

Case 2 The puncture must be on B. The word is $\overline{b_i}\Delta b_j =$

$$\overline{b_i} b_n a_n \overline{b_n}\overline{a_n} \ldots b_1 a_1 \overline{b_1}\overline{a_1} b_j.$$

Case 3 The puncture can be in two places. If the puncture is on A, the word is $b_i\lambda_j b_k =$

$$b_i a_1 b_1 \overline{a_1}\overline{b_1} \ldots a_j b_j \overline{a_j}\overline{b_j} b_k.$$

If the puncture is on B, the word is $b_i\lambda'_j b_k =$

$$b_i b_n a_n \overline{b_n}\overline{a_n} \ldots b_{j+1} a_{j+1} \overline{b_{j+1}}\overline{a_{j+1}} b_k.$$

Here, $j < i.k$.

Now we will consider the case where A contains I and II and B contains III and IV. This is equivalent to the case where A contains I and III and B contains II and IV.

Case 1 The puncture can be in two places. If the puncture is on A, the word is $a_1\lambda'_1\overline{b_2 b_2} =$

$$a_1 b_n a_n \overline{b_n}\overline{a_n} \ldots b_2 a_2 \overline{b_2}\overline{a_2}\overline{b_2 b_2}.$$

If the puncture is on B, the word is $a_1\lambda_1\overline{b_2 b_2} =$

$$a_1 a_1 b_1 \overline{a_1}\overline{b_1 b_2 b_2}.$$

Case 2 The puncture can be in two places. If the puncture is on A, the word is $a_{n-1}\lambda'_{n-1}\overline{b_n b_n} =$

$$a_{n-1}\lambda'_{n-1}\overline{b_n b_n}.$$

If the puncture is on B, the word is $a_{n-1}\lambda_{n-1}\overline{b_n b_n} =$

$$a_{n-1} a_1 b_1 \overline{a_1}\overline{b_1} \ldots a_{n-1} b_{n-1} \overline{a_{n-1}}\overline{b_{n-1} b_n b_n}.$$

Case 3 The puncture can be in two places. If the puncture is on A, the word is $a_i\lambda'_i\overline{b_{i+1}b_{i+1}} =$

$$a_ib_na_n\overline{b_n}\,\overline{a_n}\dots b_{i+1}a_{i+1}\overline{b_{i+1}}\,\overline{a_{i+1}}\,\overline{b_{i+1}b_{i+1}}.$$

If the puncture is on B, the word is $a_i\lambda_i\overline{b_{i+1}b_{i+1}} =$

$$a_ia_1b_1\overline{a_1}\,\overline{b_1}a_2b_2\overline{a_2}\,\overline{b_2}\dots a_ib_i\overline{a_i}\,\overline{b_ib_{i+1}b_{i+1}}.$$

Now we will consider the case where A contains I and IV and B contains II and III.

Case 1 The puncture can be in two places. If the puncture is on A, the word is $\lambda_{n-1}\overline{b_n}a_{n-1}\overline{b_n} =$

$$a_1b_1\overline{a_1}\,\overline{b_1}\dots a_{n-1}b_{n-1}\overline{a_{n-1}}\,\overline{b_{n-1}b_n}a_{n-1}\overline{b_n}.$$

If the puncture is on B, the word is $\lambda' n-1\overline{b_n}a_{n-1}\overline{b_n} =$

$$b_na_n\overline{b_n}\,\overline{a_n}\,\overline{b_n}a_{n-1}\overline{b_n}.$$

Case 2 The puncture can be in two places. If the puncture is on A, the word is $\lambda_1\overline{b_2}a_1\overline{b_2} =$

$$a_1b_1\overline{a_1}\,\overline{b_1b_2}a_1\overline{b_2}.$$

If the puncture is on B, the word is $\lambda'_1\overline{b_2}a_1\overline{b_2} =$

$$b_na_n\overline{b_n}\,\overline{a_n}\dots b_2a_2\overline{b_2}\,\overline{a_2}\,\overline{b_2}a_1\overline{b_2}.$$

Case 3 The puncture can be in two places. If the puncture is on A, the word is $\lambda_i\overline{b_{i+1}}a_i\overline{b_{i+1}} =$

$$a_1b_1\overline{a_1}\,\overline{b_1}a_2b_2\overline{a_2}\,\overline{b_2}\dots a_ib_i\overline{a_i}\,\overline{b_ib_{i+1}}a_i\overline{b_{i+1}}.$$

If the puncture is on B, the word is $\lambda'_i\overline{b_{i+1}}a_i\overline{b_{i+1}} =$

$$b_na_n\overline{b_n}\,\overline{a_n}\dots b_{i+1}a_{i+1}\overline{b_{i+1}}\,\overline{a_{i+1}}\,\overline{b_{i+1}}a_i\overline{b_{i+1}}.$$

3.3.4 1 Surface Formed

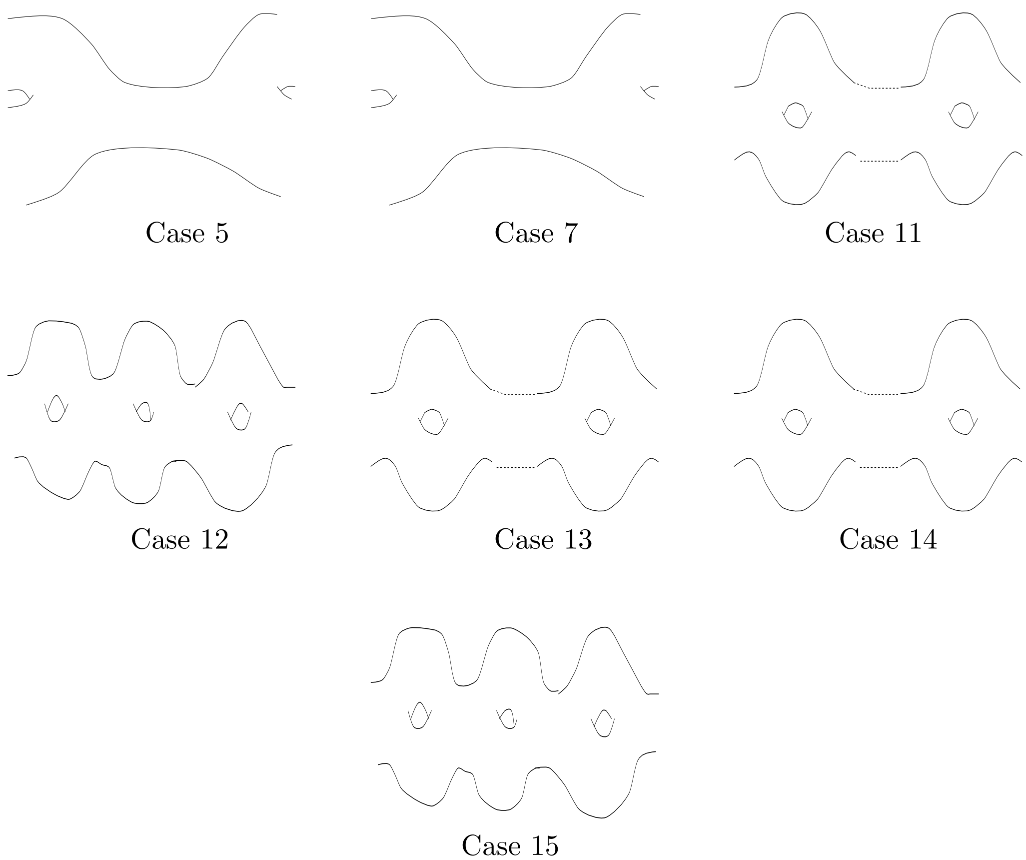
In this case, there is only one surface the puncture could be on. So there is one word for this case:

$$a_i\overline{a_j}a_k.$$

This concluses the analysis for basepoint graph 1.

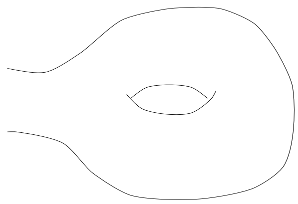
Figure 10: Curves with Basepoint Graph 1

Four Surfaces Formed

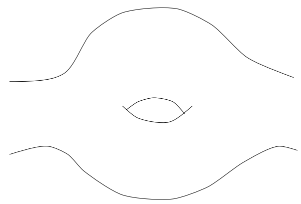


Three Surfaces Formed

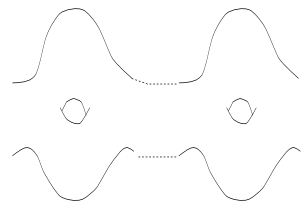
A contains II and III, B contains I and C contains IV.



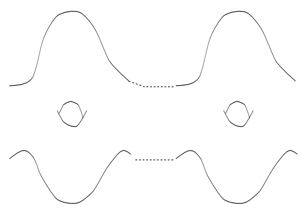
Case 2



Case 4

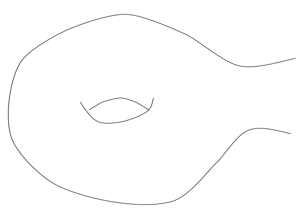


Case 6

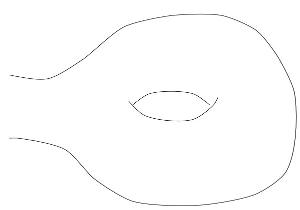


Case 7

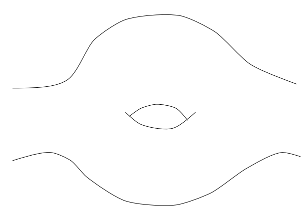
A contains II and IV, B contains I and C contains III.



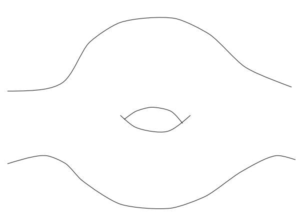
Case 1



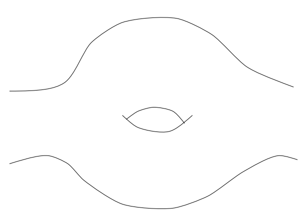
Case 2



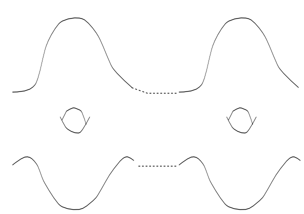
Case 4



Case 5

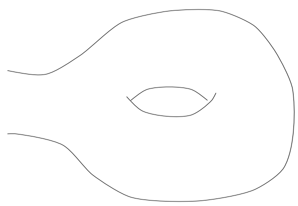


Case 6

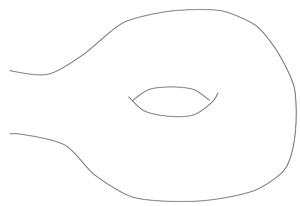


Case 7

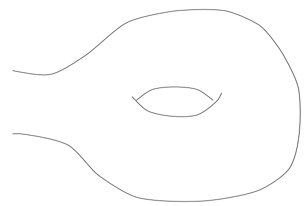
A contains III and IV, B contains I and C contains II.



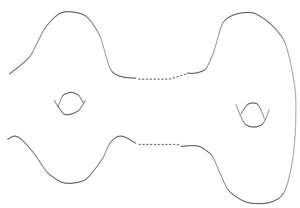
Case 1



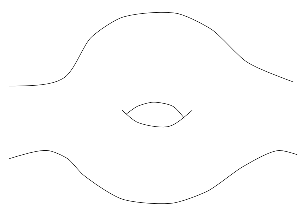
Case 2



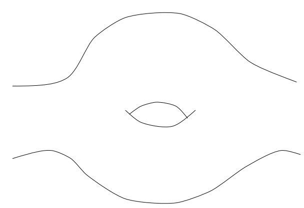
Case 3



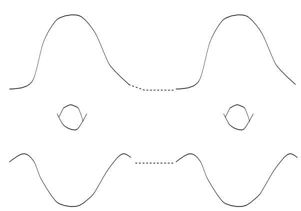
Case 4



Case 5

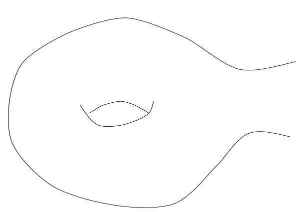


Case 6

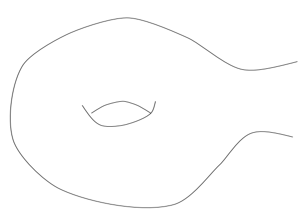


Case 7

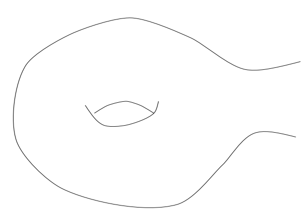
A contains I and IV, B contains III and C contains II.



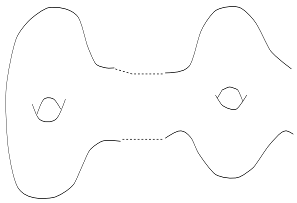
Case 1



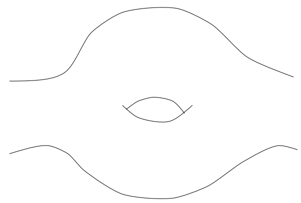
Case 2



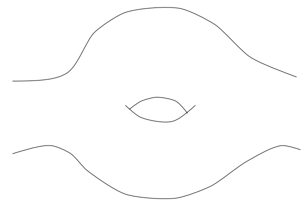
Case 3



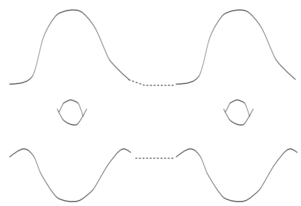
Case 4



Case 5



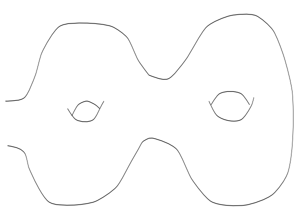
Case 6



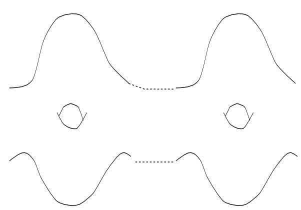
Case 7

Two Surfaces Formed

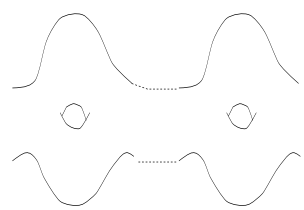
A contains II, III, and IV, B contains I.



Case 1

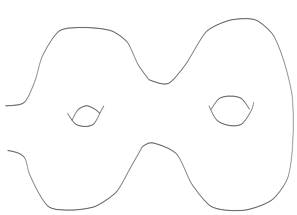


Case 2

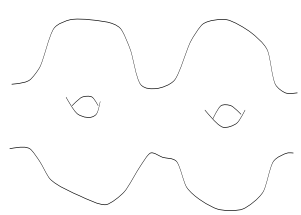


Case 3

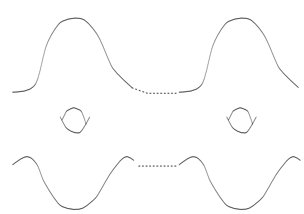
A contains I, II, and IV, B contains III.



Case 1

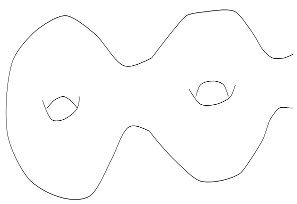


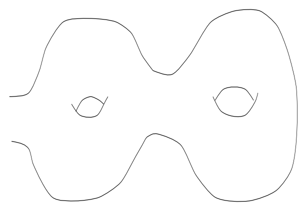
Case 2

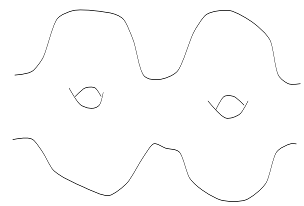


Case 3

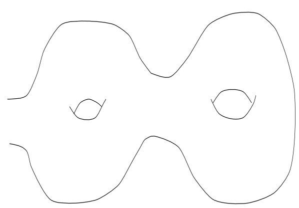
A contains I and II, B contains III and IV.

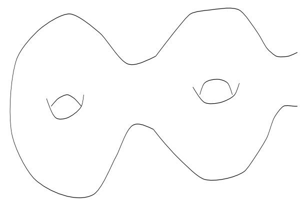

Case 1

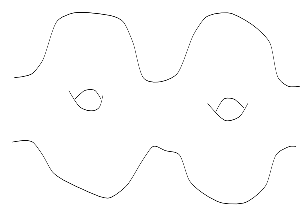

Case 2


Case 3

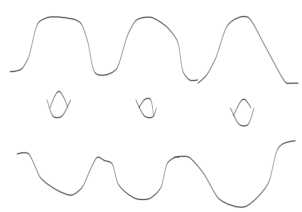
A contains I and IV, B contains II and III.


Case 1


Case 2


Case 3

One Surface Formed



3.4 Analysis of Possibilities: Basepoint Graph 2

3.4.1 4 Surfaces Formed

Without loss of generality we can let A contain I, B contain II, C contain III, and D contain IV.

Case 1 The puncture must be on A. The word is $(\Delta)^3 =$

$$(a_1 b_1 \overline{a_1} \overline{b_1} a_2 b_2 \overline{a_2} \overline{b_2} \dots a_n b_n \overline{a_n} \overline{b_n})^3.$$

Case 2 and Case 3 These cases are impossible because two surfaces are spheres with simple loops on them- surface A and surface D. So one of the simple loops would be the identity, leaving us with a curve with fewer than two self-intersections.

Case 4 This is equivalent to Case 1 with the simple loops traversed in the opposite order, i.e. with l_1 and l_3 switched.

Case 5 In this case, the puncture must be on A. The word is $(\Delta)^2\overline{\lambda'_i} =$

$$(a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_n b_n \overline{a_n}\,\overline{b_n})^2 a_{i+1} b_{i+1} \overline{a_{i+1}}\,\overline{b_{i+1}} \dots a_n b_n \overline{a_n}\,\overline{b_n}.$$

Case 6 The puncture must be on A. The word is $\Delta(\overline{\lambda'_i})^2 =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_n b_n \overline{a_n}\,\overline{b_n} (a_{i+1} b_{i+1} \overline{a_{i+1}}\,\overline{b_{i+1}} \dots a_n b_n \overline{a_n}\,\overline{b_n})^2.$$

Case 7 There are four possibilities for the puncture. If the puncture is on A, the word is $(\lambda'_i)^3 =$

$$(b_n a_n \overline{b_n}\,\overline{a_n} \dots b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}})^3.$$

If the puncture is on B, the word is $\lambda_i(\lambda'_i)^2 =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i} (b_n a_n \overline{b_n}\,\overline{a_n} \dots b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}})^2.$$

If the puncture is on C, the word is $(\lambda_i)^2\lambda'_i =$

$$(a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i})^2 b_n a_n \overline{b_n}\,\overline{a_n} \dots b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}}.$$

If the puncture is on D, the word is $(\lambda_i)^3 =$

$$(a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i})^3.$$

Case 8 This case is impossible because two surfaces are spheres with simple loops on them- surface A and surface D.

Case 9 In this case, the puncture must be on D. The word is $(\lambda_i)^2\Delta =$

$$(a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i})^2 a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_n b_n \overline{a_n}\,\overline{b_n}.$$

Case 10 The puncture must be on D. The word is $\lambda_i(\Delta)^2 =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_i b_i \overline{a_i}\,\overline{b_i} (a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \dots a_n b_n \overline{a_n}\,\overline{b_n})^2.$$

Case 11 The puncture must be on A. The word is $\Delta\overline{\lambda_i'\lambda_{i+j}'}$ =

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_nb_n\overline{a_n}\overline{b_n}a_{i+1}b_{i+1}\overline{a_{i+1}}\overline{b_{i+1}}\dots$$
$$a_nb_n\overline{a_n}\overline{b_n}a_{i+j+1}b_{i+j+1}\overline{a_{i+j+1}}\overline{b_{i+j+1}}\dots a_nb_n\overline{a_n}\overline{b_n}.$$

Case 12 The puncture can be in four places. If the puncture is on A, the word is $(la_i')^2\lambda' i+j =$

$$(b_na_n\overline{b_n}\overline{a_n}\dots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}})^2b_na_n\overline{b_n}\overline{a_n}\dots b_{i+j+1}a_{i+j+1}\overline{b_{i+j+1}}\overline{a_{i+j+1}}.$$

If the puncture is on B, the word is $\lambda_i\lambda_i'\lambda_{i+j}' =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_i}b_na_n\overline{b_n}\overline{a_n}\dots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}}b_na_n\overline{b_n}\overline{a_n}\dots$$
$$b_{i+j+1}a_{i+j+1}\overline{b_{i+j+1}}\overline{a_{i+j+1}}.$$

If the puncture is on C, the word is $(la_i)^2\lambda' i+j =$

$$(a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_i})^2b_na_n\overline{b_n}\overline{a_n}\dots b_{i+j+1}a_{i+j+1}\overline{b_{i+j+1}}\overline{a_{i+j+1}}.$$

If the puncture is on D, the word is $(la_i)^2\lambda_{i+j} =$

$$(a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_i})^2a_1b_1\overline{a_1}\overline{b_1}\dots a_{i+j}b_{i+j}\overline{a_{i+j}}\overline{b_{i+j}}.$$

Case 13 This case is the same as case 12 with the opposite orientation and relabeling l_1 and l_3.

Case 14 This case is the same as case 11 with the opposite orientation and relabeling l_1 and l_3.

Case 15 There are four possibilities for the puncture. If the puncture is on A, the word is $\lambda_i'\lambda_{i+j}'\lambda_{i+j+k}' =$

$$b_na_n\overline{b_n}\overline{a_n}\dots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}}b_na_n\overline{b_n}\overline{a_n}\dots$$
$$b_{i+j+1}a_{i+j+1}\overline{b_{i+j+1}}\overline{a_{i+j+1}}b_na_n\overline{b_n}\overline{a_n}\dots b_{i+j+k+1}a_{i+j+k+1}\overline{b_{i+j+k+1}}\overline{a_{i+j+k+1}}.$$

If the puncture is on B, the word is $\lambda_i\lambda_{i+j}'\lambda_{i+j+k}' =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_i}b_na_n\overline{b_n}\overline{a_n}\dots$$
$$b_{i+j+1}a_{i+j+1}\overline{b_{i+j+1}}\overline{a_{i+j+1}}b_na_n\overline{b_n}\overline{a_n}\dots b_{i+j+k+1}a_{i+j+k+1}\overline{b_{i+j+k+1}}\overline{a_{i+j+k+1}}.$$

If the puncture is on C, the word is $\lambda_i\lambda_{i+j}\lambda_{i+j+k}' =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_i}a_1b_1\overline{a_1}\overline{b_1}\dots$$
$$a_i+jb_i+j\overline{a_i+j}\overline{b_i+j}b_na_n\overline{b_n}\overline{a_n}\dots b_{i+j+k+1}a_{i+j+k+1}\overline{b_{i+j+k+1}}\overline{a_{i+j+k+1}}.$$

If the puncture is on D, the word is $\lambda_i\lambda_{i+j}\lambda_{i+j+k} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_i}a_1b_1\overline{a_1}\overline{b_1}\dots$$
$$a_{i+j}b_{i+j}\overline{a_{i+j}}\overline{b_{i+j}}a_1b_1\overline{a_1}\overline{b_1}\dots a_{i+j+k}b_{i+j+k}\overline{a_{i+j+k}}\overline{b_{i+j+k}}.$$

3.4.2 3 Surfaces Formed

First, I will consider the case where A contains II and III, B contains I and C contains IV.

Case 1 and Case 2 These two cases are equivalent since there is a map that takes l_1 to l_3. Using the surfaces for case 2, the puncture must be on C. The word is $\lambda_{n-1}b_n\Delta =$

$$a_1b_1\overline{a_1}\overline{b_1}\ldots a_{n-1}b_{n-1}\overline{a_{n-1}}\overline{b_{n-1}}b_na_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_nb_n\overline{a_n}\overline{b_n}.$$

Case 3 This case is impossible since two surfaces, B and C, are spheres containing only simple loops.

Case 4 The puncture can be in three places. If the puncture is on A, the word is $\lambda_ib_{i+1}\lambda'_{i+1} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_ib_i\overline{a_i}\overline{b_i}b_{i+1}b_na_n\overline{b_n}\overline{a_n}\ldots b_{i+2}a_{i+2}\overline{b_{i+2}}\overline{a_{i+2}}.$$

If the puncture is on B, the word is $\lambda'_ib_{i+1}\lambda'_{i+1} =$

$$b_na_n\overline{b_n}\overline{a_n}\ldots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}}b_{i+1}b_na_n\overline{b_n}\overline{a_n}\ldots b_{i+2}a_{i+2}\overline{b_{i+2}}\overline{a_{i+2}}.$$

If the puncture is on C, the word is $\lambda_ib_{i+1}\lambda_{i+1} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_ib_i\overline{a_i}\overline{b_i}b_{i+1}a_1b_1\overline{a_1}\overline{b_1}\ldots a_{i+1}b_{i+1}\overline{a_{i+1}}\overline{b_{i+1}}.$$

Case 5 The puncture must be on B. The word is $\Delta\overline{b_k\lambda'_{i+1}} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_nb_n\overline{a_n}\overline{b_nb_k}a_{i+2}b_{i+2}\overline{a_{i+2}}\overline{b_{i+2}}\ldots a_nb_n\overline{a_n}\overline{b_n}.$$

Note that this is for $k < i+1$.

Case 6 This case is equivalent to case 5.

Case 7 The puncture can be in three places. If the puncture is on A, the word is $\lambda_ib_k\lambda'_{i+j} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_ib_i\overline{a_i}\overline{b_i}b_kb_na_n\overline{b_n}\overline{a_n}\ldots b_{i+j+1}a_{i+j+1}\overline{b_{i+j+1}}\overline{a_{i+j+1}}.$$

If the puncture is on B, the word is $\lambda'_ib_k\lambda'_{i+j} =$

$$b_na_n\overline{b_n}\overline{a_n}\ldots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}}b_kb_na_n\overline{b_n}\overline{a_n}\ldots b_{i+j+1}a_{i+j+1}\overline{b_{i+j+1}}\overline{a_{i+j+1}}.$$

If the puncture is on C, the word is $\lambda_i b_k \lambda_{i+j} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_ib_i\overline{a_i}\overline{b_i}b_ka_1b_1\overline{a_1}\overline{b_1}\ldots a_{i+j}b_{i+j}\overline{a_{i+j}}\overline{b_{i+j}}.$$

The next case I will consider is the one where A contains III and IV, B contains I, and C contains II. This is equivalent to the case where A contains I and II, B contains IV and C contains III.

Case 1 The puncture must be on B. The word is $\Delta\overline{\lambda'_{n-1}b_n} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_nb_n\overline{a_n}\overline{b_n}a_nb_n\overline{a_n}\overline{b_nb_n}.$$

Case 2 The puncture can be in three places. If the puncture is on A, the word is $(\lambda_{n-1})^2b_n =$

$$(a_1b_1\overline{a_1}\overline{b_1}\ldots a_{n-1}b_{n-1}\overline{a_{n-1}}\overline{b_{n-1}})^2b_n.$$

If the puncture is on B, the word is $(\lambda'_{n-1})^2b_n =$

$$(b_na_n\overline{b_n}\overline{a_n})^2b_n.$$

If the puncture is on C, the word is $\lambda_{n-1}\lambda'_{n-1}b_n =$

$$a_1b_1\overline{a_1}\overline{b_1}\ldots a_{n-1}b_{n-1}\overline{a_{n-1}}\overline{b_{n-1}}b_na_n\overline{b_n}\overline{a_n}b_n.$$

Case 3 The puncture is on B. The word is $(\Delta)^2\overline{b_i} =$

$$(a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_nb_n\overline{a_n}\overline{b_n})^2\overline{b_i}.$$

Case 4 The puncture can be in three places. If the puncture is on A, the word is $\lambda_i\lambda_{n-1}b_n =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_ib_i\overline{a_i}\overline{b_i}a_1b_1\overline{a_1}\overline{b_1}\ldots a_{n-1}b_{n-1}\overline{a_{n-1}}\overline{b_{n-1}}b_n.$$

If the puncture is on B, the word is $\lambda'_i\lambda'_{n-1}b_n =$

$$b_na_n\overline{b_n}\overline{a_n}\ldots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}}b_na_n\overline{b_n}\overline{a_n}b_n.$$

If the puncture is on C, the word is $\lambda_i\lambda'_{n-1}b_n =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_ib_i\overline{a_i}\overline{b_i}b_na_n\overline{b_n}\overline{a_n}b_n.$$

Case 5 The puncture must be on B. The word is $\Delta\overline{\lambda_i'b_j} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_nb_n\overline{a_n}\overline{b_n}a_{i+1}b_{i+1}\overline{a_{i+1}}\overline{b_{i+1}}\ldots a_nb_n\overline{a_n}\overline{b_n}\overline{b_j}.$$

In this case, $j > i$.

Case 6 The puncture can be in three places. If the puncture is on A, the word is $(\lambda_i)^2b_k =$

$$(a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_ib_i\overline{a_i}\overline{b_i})^2b_k.$$

If the puncture is on B, the word is $(\lambda_i')^2b_k =$

$$(b_na_n\overline{b_n}\overline{a_n}\ldots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}})^2b_k.$$

If the puncture is on C, the word is $\lambda_i\lambda_i'b_k =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_ib_i\overline{a_i}\overline{b_i}b_na_n\overline{b_n}\overline{a_n}\ldots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}}b_k.$$

Case 7 The puncture can be in three places. If the puncture is on A, the word is $\lambda_i\lambda_{i+j}b_k =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_ib_i\overline{a_i}\overline{b_i}a_1b_1\overline{a_1}\overline{b_1}\ldots a_{i+j}b_{i+j}\overline{a_{i+j}}\overline{b_{i+j}}b_k.$$

If the puncture is on B, the word is $\lambda_i'\lambda_{i+j}'b_k =$

$$b_na_n\overline{b_n}\overline{a_n}\ldots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}}b_na_n\overline{b_n}\overline{a_n}\ldots b_{i+j+1}a_{i+j+1}\overline{b_{i+j+1}}\overline{a_{i+j+1}}b_k.$$

If the puncture is on C, the word is $\lambda_i\lambda_{i+j}'b_k =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\ldots a_ib_i\overline{a_i}\overline{b_i}b_na_n\overline{b_n}\overline{a_n}\ldots b_{i+j+1}a_{i+j+1}\overline{b_{i+j+1}}\overline{a_{i+j+1}}b_k.$$

Now we will consider the case where A contains II and IV, B contains I and C contains III. This is equivalent to the case where A contains III and I, B contains IV and C contains II.

Case 1 I was unable to determine the word for this case, although a representation of it is included along with the other figures. We do know the puncture must be on B.

Case 2 The puncture can be in three places. If the puncture is on A, the word is $\lambda_{n-1}\overline{a_n}\overline{a_n} =$

$$a_1b_1\overline{a_1}\overline{b_1}\ldots a_{n-1}b_{n-1}\overline{a_{n-1}}\overline{b_{n-1}}\overline{a_n}\overline{a_n}.$$

If the puncture is on B, the word is $\lambda'_{n-1}\overline{a_n a_n} =$

$$b_n a_n \overline{b_n} \overline{a_n a_n a_n}.$$

If the puncture is on C, the word is

$$\lambda_{n-1}\overline{a_n} a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \dots a_n b_n \overline{a_n}\overline{b_n}\overline{a_n}.$$

Case 3 The puncture must be on B. The word is $\Delta a_i a_i =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \dots a_n b_n \overline{a_n}\overline{b_n} a_i a_i.$$

Case 4 The puncture can be in three places. If the puncture is on A, the word is $\lambda_i \lambda'_{i+1}\overline{a_{i+1}} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \dots a_i b_i \overline{a_i}\overline{b_i} b_n a_n \overline{b_n}\overline{a_n} \dots b_{i+2} a_{i+2} \overline{b_{i+2}}\overline{a_{i+2} a_{i+1}}.$$

If the puncture is on B, the word is $\lambda'_i \lambda'_{i+1}\overline{a_{i+1}} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \dots a_i b_i \overline{a_i}\overline{b_i} b_n a_n \overline{b_n}\overline{a_n} \dots b_{i+2} a_{i+2} \overline{b_{i+2}}\overline{a_{i+2} a_{i+1}}.$$

If the puncture is on C, the word is $\lambda_i \lambda_{i+1}\overline{a_{i+1}} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \dots a_i b_i \overline{a_i}\overline{b_i} b_1 a_1 \overline{b_1}\overline{a_1} \dots b_{i+1} a_{i+1} \overline{b_{i+1}}\overline{a_{i+1} a_{i+1}}.$$

Case 5 The puncture must be on B. The word is $\Delta\overline{a_i}\overline{\lambda_i}\overline{a_i} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \dots a_n b_n \overline{a_n}\overline{b_n}\overline{a_i} b_i a_i \overline{b_i}\overline{a_i} \dots b_1 a_1 \overline{b_1}\overline{a_1 a_i}.$$

Case 6 The puncture can be in three places. If the puncture is on A, the word is $\lambda_i \overline{a_j a_j} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \dots a_i b_i \overline{a_i}\overline{b_i}\overline{a_j a_j}.$$

If the puncture is on B, the word is $\lambda'_i \overline{a_j a_j} =$

$$b_n a_n \overline{b_n}\overline{a_n} \dots b_{i+1} a_{i+1} \overline{b_{i+1}}\overline{a_{i+1} a_j a_j}.$$

If the puncture is on C, the word is $\lambda_i \overline{a_j}\Delta\overline{a_j} =$

$$a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \dots a_i b_i \overline{a_i}\overline{b_i}\overline{a_j} a_1 b_1 \overline{a_1}\overline{b_1} a_2 b_2 \overline{a_2}\overline{b_2} \dots a_n b_n \overline{a_n}\overline{b_n}\overline{a_j}.$$

Case 7 I was unable to determine the word for this case, although there is a representation along with the other figures.

Now I will consider the case where A contains I and IV, B contains II and C contains III.

Case 1 The puncture can be in three places. If the puncture is on A, the word is $a_n a_n \lambda_{n-1} a_n =$

$$a_n a_n a_1 b_1 \overline{a_1}\,\overline{b_1} \ldots a_{n-1} b_{n-1} \overline{a_{n-1}}\,\overline{b_{n-1}} a_n.$$

If the puncture is on B, the word is $a_n \overline{\Delta} a_n \lambda_{n-1} a_n =$

$$a_n b_n a_n \overline{b_n}\,\overline{a_n} \ldots b_1 a_1 \overline{b_1}\,\overline{a_1} a_n a_1 b_1 \overline{a_1}\,\overline{b_1} \ldots a_{n-1} b_{n-1} \overline{a_{n-1}}\,\overline{b_{n-1}} a_n.$$

If the puncture is on C, the word is $a_n a_n \lambda'_{n-1} a_n =$

$$a_n a_n b_n a_n \overline{b_n}\,\overline{a_n} a_n.$$

Case 2 This case is equivalent to case 1.

Case 3 The puncture can be in three places. If the puncture is on A, the word is $a_n a_n a_n$. If the puncture is on B, the word is $a_n \overline{\Delta} a_n a_n =$

$$a_n b_n a_n \overline{b_n}\,\overline{a_n} \ldots b_1 a_1 \overline{b_1}\,\overline{a_1} a_n a_n.$$

If the puncture is on C, the word is $a_n a_n \overline{\Delta} a_n =$

$$a_n a_n b_n a_n \overline{b_n}\,\overline{a_n} \ldots b_1 a_1 \overline{b_1}\,\overline{a_1} a_n.$$

Cases 4-7 I was unable to determine the words for these cases, although there are representations along with the other figures. Also note that cases 5 and 6 are equivalent.

3.4.3 2 Surfaces Formed

First, I will consider the case where A contains II, III, and IV and B contains I. This is equivalent to the case where A contains I, II and III and B contains IV.

Case 1 The puncture can be in two places. If the puncture is on A, the word is $\lambda_{n-2} \overline{a_{n-1} a_n a_n} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} \ldots a_{n-2} b_{n-2} \overline{a_{n-2}}\,\overline{b_{n-2}}\,\overline{a_{n-1} a_n a_n}.$$

If the puncture is on B, the word is $\lambda'_{n-2}\overline{a_{n-1}a_n a_n} =$

$$b_n a_n \overline{b_n}\,\overline{a_n} b_{n-1} a_{n-1} \overline{b_{n-1}}\,\overline{a_{n-1}a_{n-1}a_n a_n}.$$

Case 2 The puncture is on B. The word is $\Delta a_i a_{i+1} a_{i+1} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_n b_n \overline{a_n}\,\overline{b_n} a_i a_{i+1} a_{i+1}.$$

Case 3 The puncture can be in two places. If the puncture is on A, the word is $\lambda_i \overline{a_{i+1}a_{i+2}a_{i+2}} =$

$$a_1 b_1 \overline{a_1}\,\overline{b_1} a_2 b_2 \overline{a_2}\,\overline{b_2} \ldots a_i b_i \overline{a_i}\,\overline{b_i}\,\overline{a_{i+1}a_{i+2}a_{i+2}}.$$

If the puncture is on B, the word is $\lambda'_i \overline{a_{i+1}a_{i+2}a_{i+2}} =$

$$b_n a_n \overline{b_n}\,\overline{a_n} \ldots b_{i+1} a_{i+1} \overline{b_{i+1}}\,\overline{a_{i+1}a_{i+1}a_{i+2}a_{i+2}}.$$

The remaining cases are as follows:

1. A contains I, II, and IV, B contains III.
2. A contains I and II, B contains III and IV.
3. A contains I and III, B contains II and IV.
4. A contains I and IV, B contains II and III.

For each of these cases, I have drawn figures representing each possibility of regions formed, although I have not determined the words for these cases.

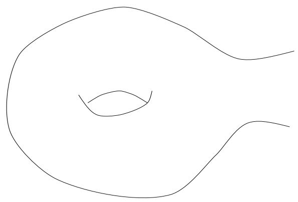
3.4.4 1 Surface Formed

Since there is only one surface, there is only one place for the puncture to be. The word for this case is $a_i a_{i+1} a_i a_{i+2}$.

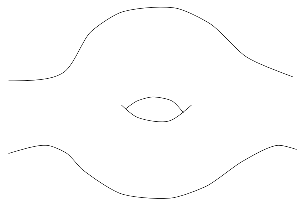
This concludes the analysis for basepoint graph 2.

Figure 11: Curves with Basepoint Graph 2

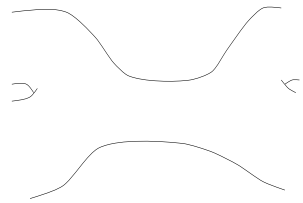
Four Surfaces Formed



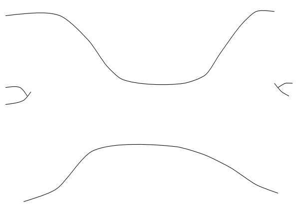
Case 1



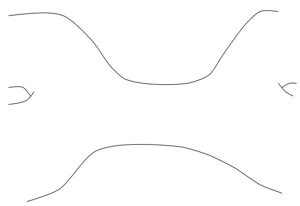
Case 5



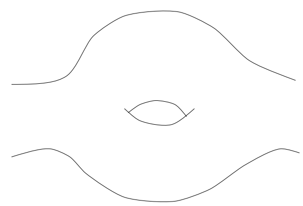
Case 6



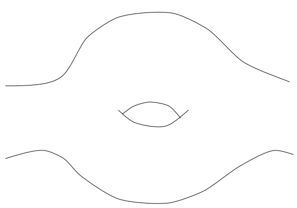
Case 7



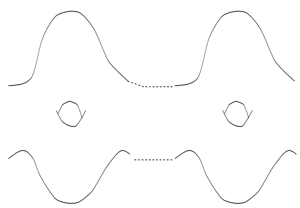
Case 9



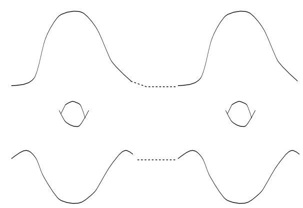
Case 10



Case 11

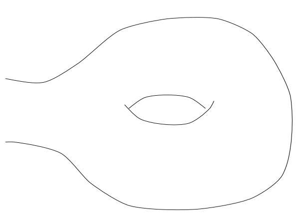


Case 12

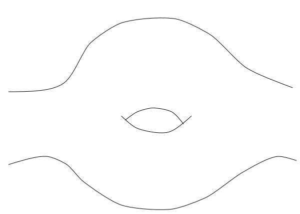


Case 15

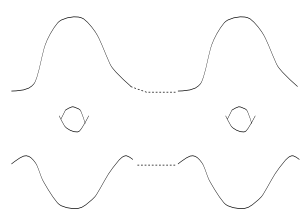
Three Surfaces Formed
A contains II and III, B contains I and C contains IV.



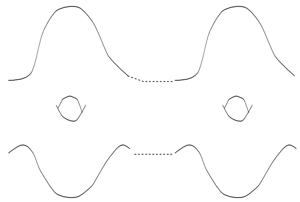
Case 2



Case 4

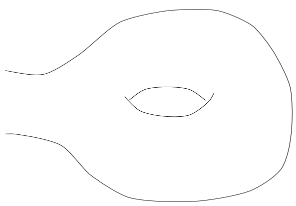


Case 5

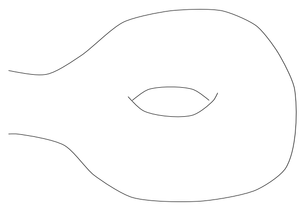


Case 7

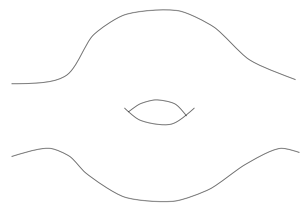
A contains III and IV, B contains I and C contains II.



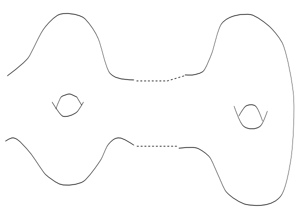
Case 1



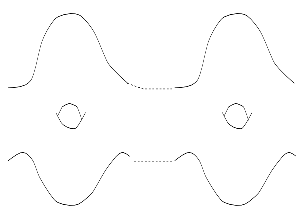
Case 2



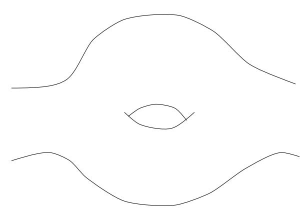
Case 3



Case 4



Case 5



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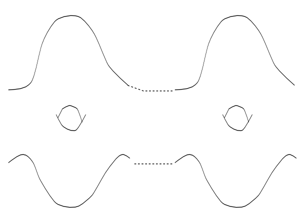
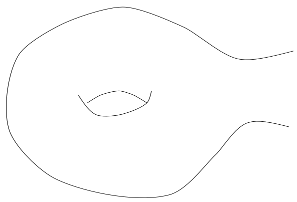
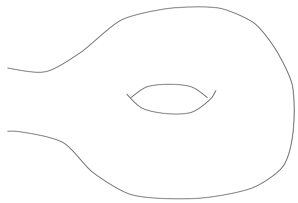


Case 7

A contains II and IV, B contains I and C contains III.



Case 1



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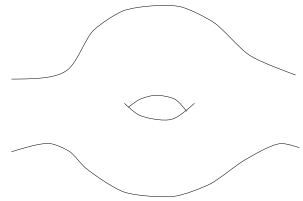
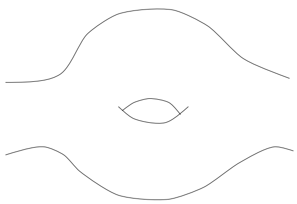
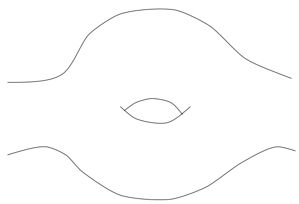


Case 3



Case 4



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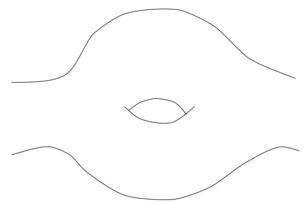
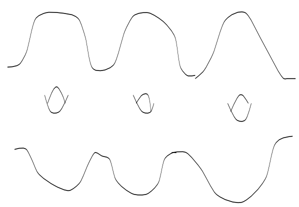
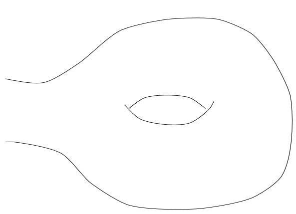


Case 6

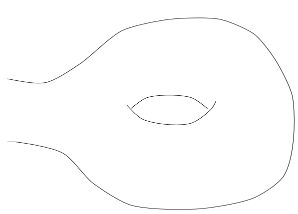


Case 7

A contains I and IV, B contains II and C contains III.



Case 1



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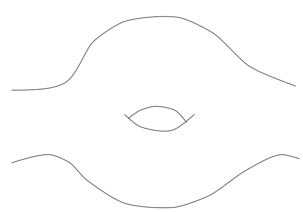
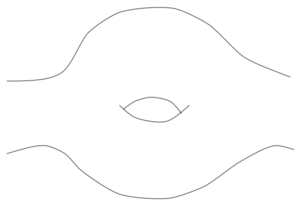
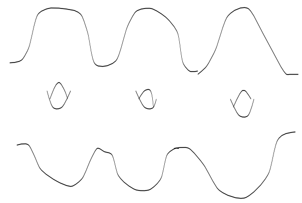


Case 4



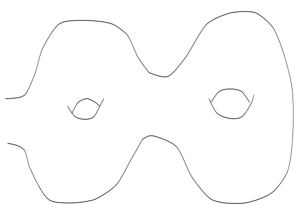
Case 5



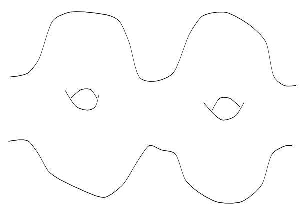
Case 7

Two Regions Formed

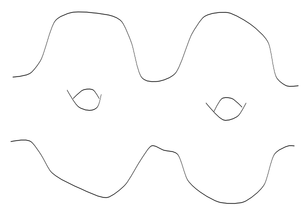
A contains II, III, and IV, B contains I.



Case 1

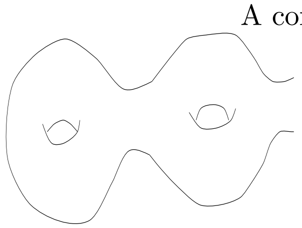


Case 2

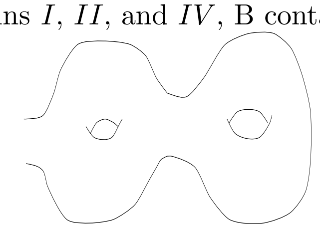


Case 3

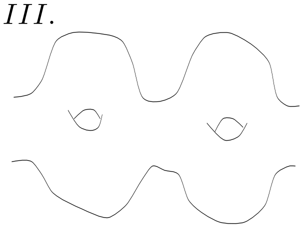
A contains I, II, and IV, B contains III.



Case 1

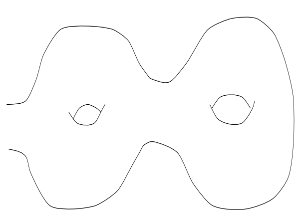


Case 2

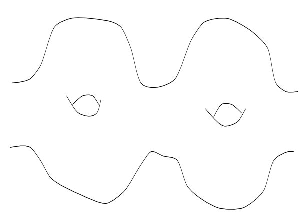


Case 3

A contains I and II, B contains III and IV.

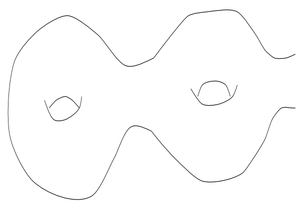


Case 1

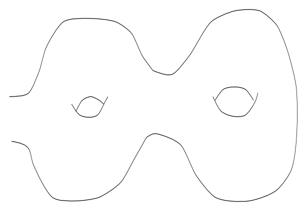


Case 3

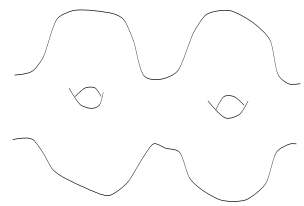
A contains I and III, B contains II and IV.



Case 1

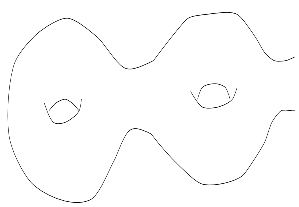


Case 2

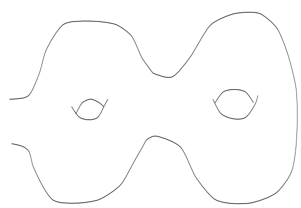


Case 3

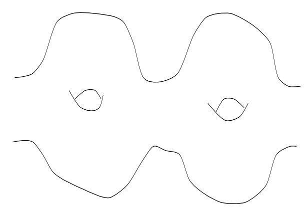
A contains I and IV, B contains II and III.



Case 1

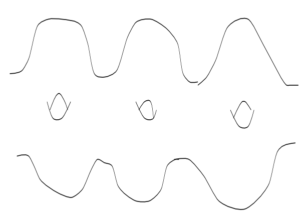


Case 2



Case 3

One Surface Formed



3.5 Analysis of Possibilities: Basepoint Graph 3

3.5.1 2 Surfaces Formed

Let A contain region I and B contain region II. **Case 1** The puncture is on A. The word is $\Delta a_i\overline{b_i} =$

$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_nb_n\overline{a_n}\overline{b_n}a_i\overline{b_i}.$$

Case 2 The puncture can be in two places. If the puncture is on A, the word is $\lambda'_{n-1}\overline{b_n}a_n =$

$$b_na_n\overline{b_n}\overline{a_n}\overline{b_n}a_n.$$

If the puncture is on B, the word is $\lambda_{n-1}\overline{b_n}a_n =$

$$a_1b_1\overline{a_1}\overline{b_1}\dots a_{n-1}b_{n-1}\overline{a_{n-1}}\overline{b_{n-1}b_n}a_n.$$

Case 3 The puncture can be in two places. If the puncture is on A, the word is $\lambda'_i\overline{b_{i+1}}a_{i+1} =$

$$b_na_n\overline{b_n}\overline{a_n}\dots b_{i+1}a_{i+1}\overline{b_{i+1}}\overline{a_{i+1}}\overline{b_{i+1}}a_{i+1}.$$

If the puncture is on B, the word is $\lambda_i\overline{b_{i+1}}a_{i+1} =$

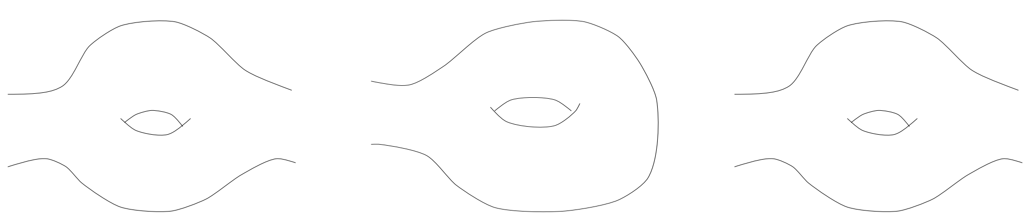
$$a_1b_1\overline{a_1}\overline{b_1}a_2b_2\overline{a_2}\overline{b_2}\dots a_ib_i\overline{a_i}\overline{b_ib_{i+1}}a_{i+1}.$$

3.5.2 1 Surface Formed

Since there is only one surface, there is only one place for the puncture to be. The word for this case is $b_i\overline{b_{i+1}}a_{i+1}$. This concludes the analysis for basepoint graph 3.

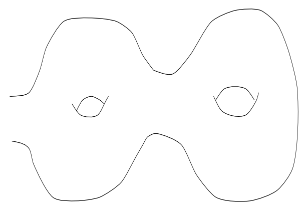
Figure 12: Words with Basepoint Graph 3

Two Surfaces Formed



Case 1 Case 2 Case 3

One Surface Formed



4 Determining Distinctness of Loops Using Whitehead Automorphisms

The above method of classification is one that assures us that we have a complete list of all possible homotopy classes of twice self intersecting curves on the surface of T_n. But it does not provide us with any assurance that the loops we have come up with are all distinct. Fortunately, the algebraic structure of the torus provides us with a method of determining distinctness. We know any curve on the surface is represented by a word consisting of elements of the free group on $2n$ letters. We also know that if there is a homeomorphism between two curves on the surface of T_n, this induces an automorphism on the free group that maps the word representing one curve to the word representing the other. Taking the contrapositive of this fact, we therefore know that if there does not exist an automorphism taking one word to another, there is not a homeomorphism that takes a curve represented by one of the words to the curve represented by the other. It is important to note, however, that an automorphism between two words does not imply topological equivalence of the curves. But determining whether or not there is an automorphism allows us to rule out many possible equivalencies.

So now we need a way of determining whether or not an automorphism exists that takes one word to another. Because the group of automorphisms on the free group of $2n$ letters is infinite, this might seem to be a difficult task. But due to an algorithm developed by J.H.C. Whitehead, there is a finite process which leads to an answer to this question. For more on Whitehead's algorithm, which relies on a certain subset of the group of automorphisms for the free group, see Michael Lau's Paper "A Computer Implementation of Whitehead's Algorithm." Whitehead's algorithm determines whether or not two words are equivalent in a series of steps. We will only concern ourselves with the first of these steps in this paper. An important aspect of Whitehead automorphisms is that they can be used to reduce a word to its minimal length, i.e. the length which the shortest word equivalent to the word in question has. Moreover, once we have reduced our words to minimal length, we know that two words can only be equivalent if they have the same minimal length.

Although I have not performed Whitehead's algorithm on many of the words I obtained in my analysis, I will make a few general comments on determining whether a word is of minimal length or not. First of all, it is interesting to observe that many of the words I found are made up of smaller words of the form $a_i b_i \overline{a_i} \overline{b_i}$ where the subscripts are either increasing or decreasing in order. So examining what the Whitehead automorphisms do to such a word is one step toward being able to tell if a word is of minimal length. So suppose we have a word $a_1 b_1 \overline{a_1} \overline{b_1} \ldots a_i b_i \overline{a_i} \overline{b_i}$. In order to reduce the length of the word, some cancellation must occur without anything else happening to compensate for the cancellation. So suppose we want to cancel a b_j. By the configuration of our word, we know we would want an automorphism that introduces a $\overline{b_j}$ to the right of a_j in order for the cancellation to occur. But any automorphism that introduces a $\overline{b_j}$ to the right of a_j also introduces a b_j to the left of $\overline{a_j}$. This introduction will balance out the cancellation unless another cancellation occurs. But such a cancellation would only be brought about by another $\overline{b_j}$ to the right of the a_j, which is impossible in a single automorphism. So it is impossible to reduce the length of the word by cancelling a b_j. It can be shown similarly that any cancellation on a word of this sort will not result in a word of length less than the original word. So we know that all words of the form $a_i b_i \overline{a_i} \overline{b_i}$ are of minimal length.

So in trying to establish whether some word that is a product of words of this type is minimal, the only possible cancellations that need to be considered are ones that might take place at the junction of two such strings, where the consecutive nature of the subscripts does not hold. So while I was not able to complete the analysis of the words I came up with, I have shown that it is certainly possible to establish fairly easily whether they are minimal or not.

5 Conclusion

Though my classification of twice self-intersecting curves on the surface of T_n is not entirely complete, I feel that I have succeeded in demonstrating the effectiveness of the Euler characteristic technique for systematically classifying curves. There is certainly room for further exploration into the determination of the distinctness of words of the type that I have come up with, though the basic ideas have now been established.

References

[1] D. Crisp. *The Markoff Spectrum and Geodesics on the Punctured Torus.* Phd thesis, University of Adelaide, 1993.

[2] D. Crisp et al. Closed curves and geodesics with two self-intersections on the punctured torus. REU Project follow-up paper, Oregon State University, Month = January, Year= 1997.

[3] J. Steiner R. Gould and B. Steinhoff. Classification of loops with self-intersections on the once punctured torus with genus n. REU Project paper, Oregon State University, August 1996.

[4] J.H.C. Whitehead. On equivalent sets of elements in a free group. *Proceedings of the London Math. Society*, pages 41:48–56, 1936.