Global Stability in Discrete One-Dimensional Population Models

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1 Introduction

Discrete one-dimensional difference equations can be used to model the growth and decay of a typical population. Characteristics of the models reflect the growth of a population until it reaches some environmental carrying capacity and the subsequent decay of the population. These models seem simple, but they can have surprisingly complex behavior. In "nicely" behaved models, the population will eventually reach a stable equilibrium at which the birth and death rates are equal, regardless of the initial population. This is called global stability. Figure 1 shows a "nicely" behaved function. Other possibilities with not-so-well behaved models are that the population can cycle infinitely through a number of sizes for some initial populations (Figure 2), or, for other initial populations near the equilibrium, the population will converge to a stable size. This is known as local stability. When applying a population model, it is important to know whether or not it is globally stable. Models which have the property of global stability are predictable while other models are not.

Figure 2. Unstable Behavior

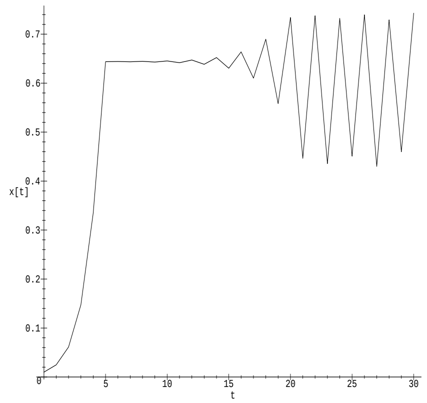
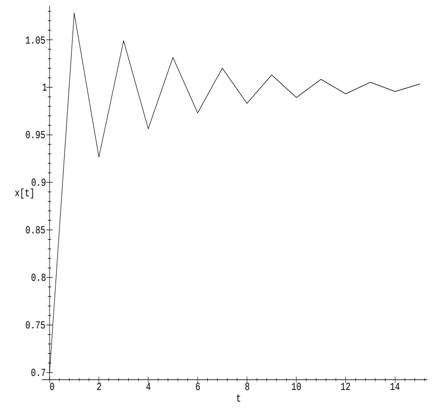


Figure 1. "Nice" Behavior



Global stability tends to be difficult to demonstrate. A traditional method for showing global stability of a model is the construction of a Lyapunov function [6]. Lyapanov functions are generally constructed for specific population models [2]. Sets of sufficient conditions for global stability that are simpler to check have been developed and are discussed by Cull [2] and Singer [13]. However, each of the three sets of conditions applies only to a limited number of the seven commonly studied models in which we are interested. Since one-dimensional models are special, it seems intuitive that there exists a single set of easy to test conditions for global stability which is satisfied by the seven population models of interest and could possibly be applied to other models.

In this paper, we look at the possibility of using enveloping curves to demonstrate global stability. We find that for all of these seven models of interest, enveloping curves of a linear fractional form can be determined. Similar to the construction of a Lyapanov function, the development of an enveloping curve is specific to each model. The symmetry of the linear fractional function proves to be a convenient characteristic in showing the global stability of the enveloped curve.

2 Definitions

A ***population model*** is a function of the form

$$x_{t+1} = f(x_t)$$

where f is a continuous function from the nonnegative reals to the nonnegative reals and there is a positive number $\overline{x}$, the equilibrium point, such that:

$$\begin{aligned} &f(0) = 0 \\ &f(x) > x \quad \text{for } 0 < x < \overline{x} \\ &f(x) = x \quad \text{for } x = \overline{x} \\ &f(x) < x \quad \text{for } x > \overline{x} \end{aligned}$$

and if $f'(x_m) = 0$ and $x_m \leq \overline{x}$ then

$$f'(x) > 0 \quad \text{for } 0 \leq x < x_m$$
$$f'(x) < 0 \quad \text{for } x > x_m \text{ such that } f(x) > 0.$$

We will allow the possibility that $f(x) = 0$ for all $x > \hat{x}$ and therefore, that $f(x)$ is not strictly differentiable at $\hat{x}$. Otherwise, we assume that f is three times continuously differentiable.

A population model is ***globally stable*** if and only if for all x_o such that $f(x_o) > 0$ we have

$$\lim_{t\to\infty} x_t = \overline{x}$$

where $\overline{x}$ is the unique equilibrium point of $x_{t+1} = f(x_t)$.

A population model is ***locally stable*** if and only if there exists a small enough neighborhood of $\overline{x}$ such that for all x_o in this neighborhood, x_t is in this neighborhood for all t, and

$$\lim_{t\to\infty} x_t = (\overline{x}).$$

Assuming that f is differentiable at $\overline{x}$, the neccessary condition for local stability is $-1 \leq f(\overline{x}) \leq 1$ (Cull).

The ***Schwarzian derivative*** of f at a point x is given by:

$$S(f,x) = \frac{f^{(3)}(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

for any real valued function f with at least three continuous derivatives.

3 Theorems

Theorem $\mathcal{I}$ [1] A population model is globally stable if and only if it has no cycles of period 2.

A population model has no cycles of period 2 if and only if $f(f(x)) > x$ for all $x \in [x_m, \overline{x})$.

Theorem $\mathcal{A}$ [1] If a population model has a maximum x_m in $(0, x_m)$ and satisfies:

1. $f''(x) < 0$ for x in $[x_m, \overline{x})$;
2. $f^{(3)}(x) \geq 0$ for all x such that $f''(x) < 0$ and f'' has at most one sign change; and
3. $|f'(x)| \leq 1$,

then the model is globally stable.

Theorem $\mathcal{B}$ [1] For a population model f, let $k(x) = \frac{x}{f(x)}$. Let the function g be defined by $\frac{k}{k'} = g(x) + Bx$ where B is a constant chosen to make $g(x)$ nonnegative. If the population model satisfies:

1. $f'(\overline{x}) = -1$;
2. $k' \leq 2$ on $[x_m, \overline{x})$;
3. $g(x) \geq 0$ on $[x_m, f(x_m)]$;
4. $g'(x) \leq 0$ on $[x_m, f(x_m)]$; and
5. $g''(x) \geq 0$ on $[x_m, f(x_m)]$,

then the model is globally stable.

Theorem $\mathcal{S}$ [6] Let $\mathcal{G}$ be the set of all endomorphisms which satisfy:

1. $f(0) = f(1) = 0$;
2. f has a unique critical point in $[0, 1]$; and
3. $S(f, x) < 0$ everywhere.

Then for any f in $\mathcal{G}$ there is at most one stable orbit in $(0, 1)$, namely $\{\overline{x},\overline{x},\overline{x}, \ldots\}$.

We find that one of the models in Singer's paper [3] does not quite satisfy the conditions of Theorem $\mathcal{S}$, so we propose a modified version.

Modified Theorem $\mathcal{S}$ Instead of $S(f, x) < 0$ everywhere, let $S(f, x) < 0$ for $x \in [x_m, f(x_m)]$.

An initial point x such that $0 < x < x_m$ or $x > f(x_m)$, where x_m is a critical point of the population model and $x_m < \overline{x}$, will eventually iterate into the interval $[x_m, f(x_m)]$. If stability exists on this interval, then global stability exists. This is why we are able to use a modified version of Theorem $\mathcal{S}$.

4 Previous Results

We have worked with a number of models which have been used to model population growth and regulation. The theorems can be used to show that each of the seven of these models is globally stable if it is population model according to the definition and if it satisfies the neccessary condition for local stability which is $-1 \leq f'(\overline{x}) < 1$. When $f'(\overline{x}) = 1$, we find that these models degenerate to $f(x) = x$ which is not a population model by our definition. Previous work with these models has shown that each of the theorems is satisfied by only some of the models. Here are the seven models:

Model	*Function*	*Parameters*	*Theorems*	*References*
II	$f_1(x) = xe^r(1 - \frac{x}{k})$	$0 < r \leq 2$	$\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$	[3], [7], [9], [12]
II	$f_2(x) = x(1 + r(1 - x))$	$0 < r \leq 2$	$\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$	[8], [14]
III	$f_3(x) = x(1 - r \ln x)$	$0 < r \leq 2$	$\mathcal{A}$	[10]
IV	$f_4(x) = x(\frac{1}{b+cx} - d)$	$\frac{d-1}{(d+1)^2} \leq b < \frac{1}{d+1}$	$\mathcal{A}$, $\mathcal{S}$	[16]
V	$f_5(x) = \frac{(1+ae^b)x}{1+ae^{bx}}$	$0 < a$, $0 < b$, $a(b-2)e^b \leq 2$	$\mathcal{B}$, $\mathcal{S}$	[11]
VI	$f_6(x) = \frac{(1+a)^b x}{(1+ax)^b}$	$0 < a$, $0 < b$, $a(b-2) \leq 2$	$\mathcal{B}$, Modified $\mathcal{S}$	[4]
VII	$f_7(x) = \frac{rx}{1+(r-1)x^c}$	$r(c-2) \leq c$	$\mathcal{B}$	[15]

Cull's paper [1] has indicated that Model V satisfies the conditions of Theorems $\mathcal{A}$ and $\mathcal{B}$. However, upon studying the details of the model, we find that Model V satisfies the conditions of Theorem $\mathcal{B}$, but not those of Theorem $\mathcal{A}$. Also, Singer's paper [13] states that Model VI satisfies the condition that the Schwarzian derivative must be negative for all x when $b > 1$, which is not true [5]. However, Model VI does satisfy the conditions of the modified Theorem $\mathcal{S}$.

5 Enveloping Functions

A curve, $\phi(x)$, is an ***enveloping function*** for a model $f(x)$ if the following conditions are satisfied:

1. $f(x) < \phi(x)$ for $0 < x < \overline{x}$;
2. $f(x) = \phi(x)$ for $x = \overline{x}$;
3. $f(x) > \phi(x)$ for $x > \overline{x}$.

Figure 3 shows a curve, $h(x)$, which envelops the curve $f(x)$.

It is hoped that enveloping curves can be found which indicate that the functions being enveloped are globally stable.

A ***linear fractional function*** is a function of the form:

$$\phi(x) = \frac{\alpha - x}{1 - (2 - \alpha)x}$$

where α is a constant.

A linear fractional function is a convenient way to try to envelop the models. Functions of this form are symmetric about the line $y = x$, so $\phi(\phi(x)) = x$ for all x. If a function, f, is enveloped by a linear fractional function, ϕ, then we know $f(f(x)) > x$ for $x \in [x_m, \overline{x})$ which implies global stability.

Figure 4 shows an example of a linear fractional function, $h(x)$, which envelops the function $f(x)$. Notice the symmetry of $h(x)$ about $y = x$.

Pleasantly, for each of the seven models of interest, there does exist an enveloping function of the linear fractional form.

Figure 4. Linear Fractional Curve

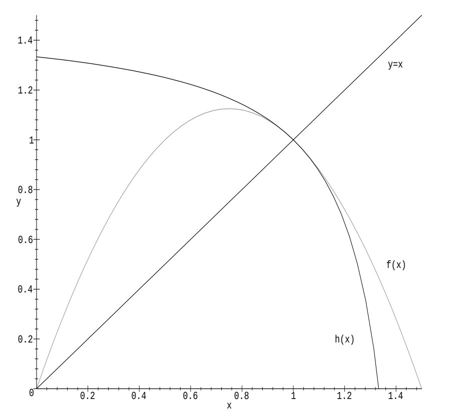
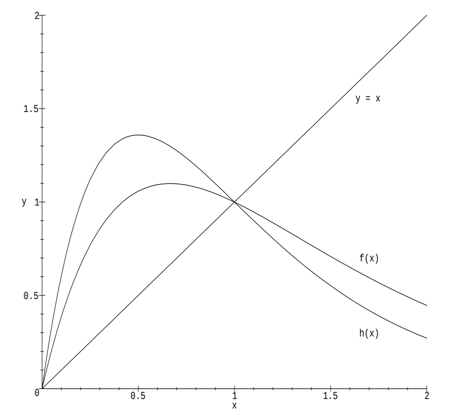


Figure 3. Enveloping Curve



Model	α	*Linear Fractional Function*
I	$\alpha = 2$	$\phi(x) = 2 - x$
II	$\alpha = \frac{4}{3}$	$\phi(x) = \frac{4-3x}{3-2x}$
III	$\alpha = \frac{3}{2}$	$\phi(x) = \frac{3-2x}{2-x}$
IV	$\alpha = \frac{4d}{3d-1}$	$\phi(x) = \frac{4d-3dx+x}{3d-1-2dx+2x}$
V	for $0 < b \leq 2$, $\alpha = 2$ for $b > 2$, $\alpha = \frac{b}{b-1}$	$\phi(x) = 2 - x$ $\phi(x) = \frac{b-(b-1)x}{b-1-(b-2)x}$
VI	for $0 < b \leq 2$, $\alpha \to \infty$ for $b > 2$, $\alpha = \frac{2(b-1)}{b-2}$	$\phi(x) = \frac{1}{x}$ $\phi(x) = \frac{2b-2-(b-2)x}{b-2+2x}$
VII	for $0 < c \leq 2$, $\alpha \to \infty$ for $c > 2$, $\alpha = \frac{c-1}{c-2}$	$\phi(x) = \frac{1}{x}$ $\phi(x) = \frac{c-1-(c-2)x}{c-2-(c-3)x}$

6 Models

Here we consider seven well known population models which have previously appeared in this article. The theorems discussed can be used to show that all of these models are globally stable if they are population models according to our definition and if they satisfy the neccessary condition for local stability, $-1 \leq f'(\overline{x}) \leq 1$. All of the following models, for convenience, have been normalized, meaning that they have had their parameters adjusted so that the equilibrium point is always at $x = 1$.

Cases in which the critical point, x_m, is not in $(0, \overline{x})$ are trivial. If the initial population is less than the equilibrium, the population will monotonically increase toward the equilibrium. If the initial population is greater than the equilibrium, the population will either monotonically decrease toward the equilibrium, or it will eventually iterate to a population less than the equilibrium at which point, it will begin monotonically increasing toward the equilibrium. The cases that prove to be interesting are those in which the critical point, x_m, falls in the interval $(0, \overline{x})$.

Model I:

$$f_1(x) = xe^{r(1-x)}$$
$$f_1'(x) = (1-rx)e^{r(1-x)}$$
$$f_1''(x) = -r(2-rx)e^{r(1-x)}$$
$$f_1^{(3)}(x) = r^2(3-rx)e^{r(1-x)}$$
$$k_1(x) = e^{r(x-1)}$$
$$k_1'(x) = re^{r(x-1)}$$
$$\frac{k_1}{k_1'} = \frac{1}{r} = g_1(x) + B_1 x$$
$$g_1(x) = \frac{1}{r} \quad B_1 = 0$$
$$g_1'(x) = 0$$
$$g_1''(x) = 0$$
$$\phi_1(x) = 2 - x$$
$$\phi_1'(x) = -1$$
$$\phi_1''(x) = 0$$

Parametric Region of Stability: $0 < r \leq 2$

Global stability of this model can be established in a number of ways. The necessary condition for local stability implies that $0 \leq r \leq 2$. For $r = 0$, the model degenerates to $f_1(x) = x$ which is not a population model by our definition. For $0 < r \leq 1$, the critical point, x_m, is greater than the equilibrium point, so we only need concern ourselves with the cases in which $1 < r \leq 2$. This model satisfies the conditions for Theorems $\mathcal{A}$ and $\mathcal{B}$ [1], and the conditions for Theorem $\mathcal{S}$ citekn:13. Also, the linear fractional function, $\phi_1(x) = 2 - x$, envelops this model when $0 < r \leq 2$. The only point at which $f_1(x)$ is equal to $\phi_1(x)$ is at $x = 1$. For $0 < r < 2$, $f_1'(1) - \phi_1'(1) > 0$, implying that ϕ satisfies the conditions that make it an enveloping curve. Now, for $r = 2$, $f_1'(1) = \phi_1'(1)$, so we must look further. Upon considering the second derivatives, we see that for $x < 1$, $f_1''(x) < \phi_1''(x)$, and for $x > 1$, $f_1''(x) > \phi_1''(x)$. Therefore, it is true that $\phi_1(x)$ is a linear fractional enveloping curve for this model.

Model II:

$f_2(x) = x(1 + r(1 - x))$

$f_2'(x) = 1 + r - 2rx$

$f_2''(x) = -2r$

$f_2^{(3)}(x) = 0$

$k_2(x) = \frac{1}{1+r(1-x)}$

$k_2'(x) = \frac{r}{(1+r(1-x))^2}$

$\frac{k_2}{k_2'} = \frac{1+r}{r} - x = g_2(x) + B_2 x$

$g_2(x) = \frac{1+r}{r} \quad B_2 = -1$

$g_2'(x) = 0$

$g_2''(x) = 0$

$\phi_1(x) = \frac{4-3x}{3-2x}$

$\phi_2'(x) = \frac{-1}{(3-2x)^2}$

$\phi_2''(x) = \frac{-4}{(3-2x)^3}$

Parametric Region of Stability: $0 < r \leq 2$

Similar to the previous model, there are several methods for establishing the global stability of Model II. Local stability, again, implies that $0 \leq r \leq 2$, but the model is degenerate for $r = 0$, and $0 < r \leq 1$ give functions that have no critical point in $(0, \overline{x})$. Model II satisfies the conditions of Theorems $\mathcal{A}$ and $\mathcal{B}$ [1], and the conditions of Theorem $\mathcal{S}$ [13]. The linear fractional function $\phi_2(x)$ envelops variations of Model II for $0 < r \leq 2$. The only real root of the equation $f_2(x) - \phi_2(x) = 0$ is $x = 1$. For $0 < r < 2$, $f_2'(1) - \phi_2'(1) > 0$ which implies that $f_2(x) > \phi_2(x)$ when $x > 1$ and $f_2(x) < \phi_2(x)$ when $x < 1$. When $r = 2$, $f_2'(1) = \phi_2'(1)$. For r=2, $f_2''(x) < \phi_2''(x)$ for $x < 1$ and $f_2''(x) > \phi_2''(x)$ when $x > 1$. Therefore, we know that it is true that ϕ_2 envelops this model when $0 < r \leq 2$, thereby establishing global stability.

Model III:

$f_3(x) = x(1 - r \ln x)$

$f_3'(x) = 1 - r - r \ln x$

$f_3''(x) = \frac{-r}{x}$

$f_3^{(3)}(x) = \frac{r}{x^2}$

$k_3(x) = \frac{1}{1-r\ln x}$

$k_3'(x) = \frac{r}{x(1-r\ln x)^2}$

$\frac{k_3}{k_3'} = \frac{x(1-r\ln x)}{r} = g_3(x) + B_3 x$

$g_3(x) = -x\ln x \quad B_3 = \frac{1}{r}$

$\phi_3(x) = \frac{3-2x}{2-x}$

$\phi_3'(x) = \frac{-1}{(2-x)^2}$

$\phi_3''(x) = \frac{-2}{(2-x)^3}$

Parametric Region of Stability: $0 < r \leq 2$

Again, local stability implies that $0 \leq r \leq 2$, but $r = 0$ gives a degenerate model, and $0 < r \leq 1$ has no critical point in $(0, \overline{x})$, giving trivial cases. When $1 < r \leq 2$, the conditions of Theorem $\mathcal{A}$ are satisfied, but the conditions of Theorem $\mathcal{B}$ are not satisfied since $g_3(x) < 0$ for some x in the interval $[x_m, f(x_m)]$ [1]. Model III does not satisfy the conditions of Theorem $\mathcal{S}$ [5]. $\phi_3(x)$ is a linear fractional function and $f_3(x) - \phi_3(x) = 0$ has only one real root at $x = 1$. As with models I and II, $f_3'(1) - \phi_3'(x) > 0$ for $0 < r < 2$. When $r = 2$, $f_3''(x) > \phi_3''(x)$ for $x > 1$ and $f_3''(x) < \phi_3''(x)$. Therefore, $\phi_3(x)$ is an enveloping curve which implies global stability for this model.

Model IV:

Since models with smaller values of b envelop models with larger values of b, let $b = \frac{d-1}{(d+1)^2}$, so that the model is in terms of one parameter.

$f_4(x) = x(\frac{(d+1)^2}{d-1+2x} - d)$

$f_4'(x) = \frac{(d+1)^2(d-1)}{(d-1+2x)^2}$

$f_4''(x) = \frac{-4(d+1)^2(d-1)}{(d-1+2x)^3}$

$f_4^{(x)}(x)\frac{24(d+1)^2(d-1)}{(d-1+2x)^4}$

$k_4(x) = \frac{d-1+2x}{3d+1-2dx}$

$k_4'(x) = \frac{2(d+1)^2}{(3d+1-2dx)^2}$

$\frac{k_4}{k_4'} = \frac{(d-1+2x)(3d+1-2dx)}{2(d+1)^2} = g(x) + Bx$

$g_4(x) = \frac{(d-1+2x)(3d+1-2dx)}{2(d+1)2} \quad B_4 = 0$

$g_4'(x) = \frac{-(d^2-4d-1+4dx)}{(d+1)^2}$

$g_4''(x) = \frac{-4d}{(d+1)^2}$

$\phi_4(x) = \frac{4d-3dx+x}{3d-1-2dx+2x}$

$\phi_4'(x) = \frac{-(d+1)^2}{(3d-1-2dx+2x)^2}$

$\phi_4''(x) = \frac{-4(d+1)^2(d-1)}{(3d-1-2dx+2x)^3}$

Parametric Region of Stability: $\frac{d-1}{(d+1)^2} \leq b < \frac{1}{d+1}, \quad d > 1$

For Model IV, local stability implies $\frac{d-1}{(d+1)^2} \leq b \leq \frac{1}{d+1}$, but the model degenerates for $b = \frac{1}{d+1}$. This model satisfies the conditions of Theorem $\mathcal{A}$ [1]. The conditions of Theorem $\mathcal{B}$ are not satisfied by this model since $g_4''(x)$ is always negative [1]. The Schwarzian derivative is not always negative for this model, so the conditions for Theorem $\mathcal{S}$ are not satisfied [5]. Look at a linear fractional function, $\phi_4(x) = \frac{4d-3dx+x}{3d-1-2dx+2x}$. The equation $f_4(x) - \phi_4(x) = 0$, when $f_4(x)$ and $\phi_4(x)$ are put over a common denominator, is a cubic function with a triple root at $x = 1$. Since $f_4'(1) = \phi_4'(1) = -1$, we need to look at the second derivatives. For $d > 1$, $f_4''(x) < \phi_4''(x)$ when $0 < x < 1$, and $f_4''(x) > \phi_4''(x)$ when $1 < x < \frac{3d-1}{2d-2}$. $\phi_4(x)$ has a pole at $x = \frac{3d-1}{2d-2}$, so we should not be concerned about comparing $f_4(x)$ and $\phi_4(x)$ for values of x larger than this. We have established that $\phi_4(x)$ is an enveloping function which implies global stability for Model IV.

Model V:

$f_5(x) = \frac{(1+ae^b)x}{1+ae^{bx}}$

$f_5'(x) = \frac{(1+ae^b)(1+(1-bx)ae^{bx})}{(1+ae^{bx})^2}$

$f_5''(x) = \frac{-(1+ae^b)ab^2e^{bx}[bx(1-ae^{bx})+2(1+ae^{bx})]}{(1+ae^{bx})^3}$

$f_5^{(3)}(x) = \frac{-(1+ae^b)ab^2e^{bx}[bx(1+a^2e^{2bx})+3(1-a^2e^{2bx})-4bxae^{bx}]}{(1+ae^{bx})^4}$

$k_5(x) = \frac{1+ae^{bx}}{1+ae^b}$

$k_5'(x) = \frac{abe^{bx}}{1+ae^b}$

$\frac{k_5}{k_5'} = \frac{e^{-bx}}{ab} + \frac{1}{b}$

$g_5(x) = \frac{k_5}{k_5'} \quad B_5 = 0$

$g_5'(x) = \frac{-e^{-bx}}{a}$

$g_5''(x) = \frac{be^{-bx}}{a}$

$\phi_{5a}(x) = 2 - x$ for $b \leq 2$

$\phi_{5b}(x) = \frac{b-(b-1)x}{b-1-(b-2)x}$ for $b > 2$

$\phi_{5b}'(x) = \frac{-1}{(b-1-(b-2)x)^2}$

$\phi_{5b}''(x) = \frac{-2(m-2)}{(m-1-(m-2)x)^3}$

Parametric Region of Stability: $a > 0, \quad b > 0, \quad a(b-2)e^b \leq 2$

Local stability gives the parametric region $a(b-2)e^b \leq 2, \; a \geq 0, \; b \geq 0$, but $a = 0$ or $b = 0$ give a degenerate model, $f(x) = x$. Model V, contrary to what has been stated in previous papers, does not satisfy the conditions for Theorem $\mathcal{A}$. For example, let $b = 5$, $c = 0.004$. We find that $x = 0.9$ is in the interval $[x_m, \overline{x})$. With these parameter values, $f_5^{(3)}(0.9) = -5.0865$ which does not satisfy the condition that the third derivative needs to be greater than or equal to zero on the interval $[x_m, \overline{x})$. The conditions for Theorem $\mathcal{B}$ [1] and for Theorem $\mathcal{S}$ are satisfied [13]. For $0 < b \leq 2$, $\phi_{5a}(x) = 2 - x$ envelops the model, and for $b > 2$, $\phi_{5b}(x) = \frac{b-(b-1)x}{b-1-(b-2)x}$ envelops the model. The only real root of $f_5(x) - \phi_{5a}(x) = 0$ is $x = 1$. To check that $\phi_{5a}(x)$ envelops $f_5(x)$ for $0 < b \leq 2$, we can look at $f_5'(1) - \phi_{5a}'(1)$ and see that this is a positive value. The only real root of $f_5(x) - \phi_{5b}(x) = 0$ is $x = 1$. To check that $\phi_{5b}(x)$ envelops $f_5(x)$ for $b > 2$, let $a = \frac{2}{(b-2)e^b}$ since models with larger values of a envelop models with smaller values of a. After this substitution, we have $f_5(x) = \frac{bx}{b-2+2e^{m(x-1)}}$. Now, we get $f_5(1) = \phi_{5b}(1)$, and the second derivatives of these functions are quite complicated, but with some simple algebraic manipulations, we find that $f_5(x) - \phi_{5b}(x) < 0$ for $x < 1$, and $f_5(x) - \phi_{5b}(x) > 0$ for $1 < x < \frac{m-1}{m-2}$. The function ϕ_{5b} has a pole at $x = \frac{m-1}{m-2}$, so we neen not be concerned with comparing $f_5(x)$ and $\phi_{5b}(x)$ for values of x larger than this.

Model VI:

$f_6(x) = \frac{(1+a)^b x}{(1+ax)^b}$

$f_6'(x) = \frac{(1+a)^b(1-a(b-1)x)}{(1+ax)^{b+1}}$

$f_6''(x) = \frac{-(1+a)^b ab(2-a(b-1)x)}{(1+ax)^{b+2}}$

$f_6^{(3)}(x) = \frac{(1+a)^b a^2 b(b+1)(3-a(b-1)x)}{(1+ax)^{b+3}}$

$k_6(x) = \frac{(1+ax)^b}{(1+a)^b}$

$k_6'(x) = \frac{ab(1+ax)^{b-1}}{(1+a)^b}$

$\frac{k_6}{k_6'} = \frac{1+ax}{ab} = g_6(x) + B_6 x$

$g_6(x) = \frac{1}{ab} \quad B_6 = \frac{1}{b}$

$g_6'(x) = 0$

$g_6''(x) = 0$

$\phi_{6a}(x) = \frac{1}{x}$ for $0 < b \leq 2$

$\phi_{6b}(x) = \frac{2b-2-(b-2)x}{b-2+2x}$ for $b > 0$

$\phi_{6b}'(x) = \frac{-b^2}{(b-2+2x)^2}$

$\phi_{6b}''(x) = \frac{4b^2}{(b-2+2x)^3}$

$S(f,x) = \frac{-a^2b(b-1)(x^2b^2a^2-4abx-3a^2bx^2+2a^2x^2+6+8ax)}{(1+ax)^2(1+ax-abx)^2}$

Parametric Region of Stability: $a(b-2) \leq 2, \quad a > 0, \quad b > 0$

The parametric region of stability agrees with the region implied by local stability except that $a = 0$ and $b = 0$ are not included because they cause the model to degenerate to $f(x) = x$. Theorem $\mathcal{A}$ cannot be used because there are parameter values which give stability but allow the second derivative to become negative on the interval $[x_m, \overline{x})$ [1]. Theorem $\mathcal{B}$ can be used on this model [1]. Model VI for $b > 1$ does not satisfy the conditmions for Theorem $\mathcal{S}$, contrary to what Singer stated. There are values (for example $x = 10.5$, $a = 1$, $b = 1.5$, which make the Schwarzian derivative positive. With these numbers, $S(f_6, x) = 0.000088 > 0$, contradicting Singer's claim [5]. However, if we look at the Schwarzian derivative, we find only for large values of x does Singer's claim not work. In order for $S(f,x)$ to be negative, the term in the second set of parenthesis in the numerator must be positive. This term can be rearranged to look like $a^2x^2[b^2 - 3b + 2] - 4ax[b - 2] + 6$. For certain values of b and large values of x, this term is negative. This is the reason for developing the Modified Theorem $\mathcal{S}$. The linear fractional function $\phi_{6a}(x)$ envelops Model VI for $0 < b \leq 2$, and $\phi_{6b}(x)$ envelops Model VI for $b > 2$. The only real root of $f_6(x) - \phi_{6a}(x) = 0$ is $x = 1$. We see that $f_6'(1) - \phi_{6a}'(1)$ is positive, so $\phi_{6a}(x)$ envelops Model VI for $0 < b \leq 2$. Models with smaller values of a are enveloped by models with larger values of a. Because of this, we let $a = \frac{2}{b-2}$. Now, we have $f_6(x) = \frac{b^b x}{(b-2+2x)^b}$. Since the real roots of $f_6(x) - \phi_{6b}(x) = 0$ are $x = 1$, and $f_6'(1) = \phi_{6b}(1) = -1$, look at the second and third derivatives of $f_6(x)$ and $\phi_{6b}(x)$ to determine that $\phi_{6b}(x)$ envelops $f_6(x)$ for $b > 2$.

Model VII:

$f_7(x) = \frac{rc}{1+(r-1)x^c}$

$f_7'(x) = \frac{r(1-(r-1)(c-1)x^c)}{(1+(r-1)x^c)^2}$

$f_7''(x) = \frac{-rc(r-1)x^{c-1}(c+1-(r-1)(c-1)x^c)}{(1+(r-1)x^c)^2}$

$f_7^{(3)}(x) = \frac{r(r-1)cx^{c-2}[(r-1)(c-1)x^c(c-2-(r-1)(c+1))-(c+1)(c-1-3c(r-1)x^c)}{(1+(r-1)x^c)^4}$

$k_7(x) = \frac{1+(r-1)x^c}{r}$

$k_7'(x) = \frac{c(r-1)x^{c-1}}{r}$

$\frac{k_7}{k_7'} = \frac{1}{c(r-1)x^{c-1}} + \frac{x}{c} = g_7(x) + B_7 x$

$g_7(x) = \frac{1}{c(r-1)x^{c-1}} \quad B_7 = \frac{1}{c}$

$g_7'(x) = \frac{-(c-1)}{c(r-1)x^c}$

$g_7''(x) = \frac{c-1}{(r-1)x^{c+1}}$

$\phi_{7a}(x) = \frac{1}{7}x$

$\phi_{7b}(x) = \frac{c-1-(c-2)x}{c-2-(c-3)x}$

Parametric Region of Stability: $c > 0, \quad r > 1, \quad r(c-2) \leq c$

The parametric region of stability is determined from the necessary condition for local stability. $r = 1$ and $c = 0$ cause a degenerate model. Theorem $\mathcal{A}$ cannot be used with this model, but Theorem $\mathcal{B}$ can be used [1]Theorem

$\mathcal{S}$ does not work with this model [5]. For $0 < c \leq 2$, $\phi_{7a}(x)$ is an enveloping curve for Model VII, and for $c > 2$, $\phi_{7b}(x)$ is an enveloping curve. To determine that these are enveloping curves, the first derivative of $\phi_{7a}(x)$ must be used, and the second and third derivatives of $\phi_{7b}(x)$ must be used.

7 Discussion

Theoretical biologists are aware of the conditions for local stability. Local stability is not terribly difficult to establish. Sometimes, after considering local stability, biologists imply global stability. However, local stability implying global stability is not always the case. Demonstrating global stability often is not simple. A good understanding of the behavior of a model depends on knowing whether or not the model is globally stable. Behavior of locally stable and globally stable models can be drastically different.

A standard way to prove global stability is to determine a Lyapanov function specific to each model [6]. This method can be troublesome, and one-dimensional population models are special enough that there is another simpler neccessary and sufficient condition from which we can conclude global stability. This condition is that the model must have no cycle of period 2 (i.e. $f(f(x)) > x$ for $x \in [x_m, \overline{x})$). There are a number of mathematical computer programs that make it simple to test to see whether or not, for a specific model, there exists a cycle of period 2. For fixed parameters, a biologist can easily plot $f(f(x))$ and determine whether $f(f(x)) > x$ on the correct interval.

However, there are times when the parameters are not known, and even times when the functions are not known. The lack of cycles of period 2 is not easily detectable in many models. For this reason, three theorems have previously been developed which have fairly simple sets of conditions with which a person can easily test for global stability. The problem is that none of these three sets of conditions could be applied to all seven of the common population models that were of interest in this article. This made it unreasonable to attempt to use any of the sets of conditions to determine whether other models might be globally stable.

Now, we have discovered that linear fractional functions serve as nice enveloping functions. Since the linear fractional functions are symmetric about the line $y = x$, it is obvious that $\phi(\phi(x)) = x$. From this fact, it is easy to see that a curve enveloped by a linear fractional function must have the property $f(f(x)) > x$ for $x \in [x_m, \overline{x})$. An enveloping function of the linear fractional form is a fairly simple method of determining global stability. It is not as simple a method as plotting the curve of $f(f(x))$ with a mathematical program, but with an enveloping linear fractional function, it is not neccessary to know the parameters or even the function. If a biologist has a set of data points, it can be tested to see if the points are enveloped by a linear fractional curve, in which case, one could infer that the system being dealt with was a globally stable one.

We find that there does exist a set of conditions with which to test for global stability that can be applied to all seven of these well known, common population models. All seven of these models can be labeled as simple. The next question open for consideration is: How simple must a model be to still be globally stable? Also, how complex of a model could still satisfy the conditions to have an enveloping curve which will imply global stability?

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