Point-Source Geometric Tomography

Dorothy Lam
Wellesley College

Andrea Perrine
Oregon State University

Advisor*: Professor Donald Solmon
NSF sponsored REU Program
Oregon State University

August 24, 2003

Abstract

A point source x-ray detects rays on which singularities of a convex body lies. From this information, we can successively determine the location of the singularities and hence determine convex polygons from three point sources. We give a new proof of a special case of a theorem

*We would like to thank Dr. Solmon for his guidance, advice, and encouragement. We would also like to thank Dr. Weidemann for his help with the reconstruction algorithm.

by Gardner stating that with the exception of parallel wedges, convex polygons can be uniquely determined by one point source x-ray and an algorithm for reconstructing convex polygons from a single directed x-ray.

1 Background

Geometric tomography is a field of mathematics involved with the retrieval of information about a geometric body via idealized x-rays. X-rays, in the medical sense, show where dense matter like teeth and bones occur in the body. An x-ray beam is attenuated by the material it passes through. In geometric tomography, the material is assumed to have unifrom density 1 and an x-ray gives the chord length of the intersection of the ray with the body [**?**].

Let's begin by establishing common definitions.

Definition 1.1 *A compact convex subset of the Euclidean plane is a closed and bounded subset such that the line joining any two points of the subset is included in the subset.*

Definition 1.2 *A convex body is a compact convex subset of the plane with non-empty interior.*

The boundary of a convex body is composed of two types of functions, concave and convex.

Proposition 1.1 *If ϕ is convex on (a, b), then ϕ is absolutely continuous on each closed subinterval of (a, b). The right- and left-hand derivatives of ϕ exist at each point of (a, b) and are equal to each other except on a countable set. The left- and right-hand derivatives are monotone increasing functions, and at each point the left-hand derivative is less than or equal to the right-hand derivative [**?**].*

So a convex function's right- and left-hand derivatives exist and are equal except at a countable number of points, which we will call the non-smooth points. The derivative is monotone increasing, which means the tangent lines are always below the graph of the function. Since a concave function is the negative of a convex function, the same conditions apply to a concave function's derivatives, except the derivative of a concave function is monotone decreasing and the tangent lines are always above the graph.

The divergent x-ray transform, also known as a point source x-ray or a fan-beam x-ray, gives the chord length of the convex body along the particular direction chosen. The characteristic function of the convex body, χ, equals 1 inside the body, 0 outside the body. The function $D_o\chi(\phi)$ gives the length of the directed x-ray of a geometric body at angle ϕ from a point o.

2 Detecting singularities of convex bodies by directed X-rays

Lemma 2.1 *Rotation of the $x-$ and $y-$axes does not alter the x-ray transform, $D_o\chi(\phi)$*

Proof If $\theta = (cos(\phi), sin(\phi)) = \begin{pmatrix} cos(\phi) \\ sin(\phi) \end{pmatrix}$,

$D_o\chi f(\theta) = \int_0^\infty f(tcos(\phi), tsin(\phi))d\phi$

We define the operator Υ_α: $\Upsilon_\alpha = \begin{pmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & -cos(\alpha) \end{pmatrix}$

such that $\Upsilon_\alpha f(\theta) = f(\Upsilon_\alpha\theta)$.

$D_o\chi(\Upsilon_\alpha f)(\theta) = \int_0^\infty (\Upsilon_\alpha(t\theta)dt$

$(\Upsilon_\alpha f)(t\theta) = f(\Upsilon_\alpha(t\theta))$

$= f(t\Upsilon_\alpha(\theta))$

$= f[t(cos(\alpha)cos(\phi) - sin(\alpha)sin(\phi), sin(\alpha)cos(\phi) + cos(\alpha)sin(\phi)]$

$= f(t(cos(\alpha + \phi), sin(\alpha + \phi)))$

$\int_0^\infty f(tcos(\alpha + \phi), tsin(\alpha + \phi))dt = D_o\chi f(\Upsilon_\alpha(\theta)) = \Upsilon_\alpha[D_o\chi f](\theta)$ ■

Theorem 2.1 *All rays on which non-smooth points of a convex body lie can be detected by directed x-rays emanating from one point source.*

Proof Given an arbitrary convex body and a ray passing through its interior, we may choose coordinates so that the ray emanates from $(0, 0)$. The convex body lies in the upper half plane, and in a neighborhood of the points of intersection of the ray with the convex body, its boundary is given by a convex function $G(x)$ and a concave function $F(x)$. The left- and right-hand derivatives of either function with respect to x will be labeled as $H'_-(x)$ and $H'_+(x)$, respectively. For our computations, both $F(x)$ and $G(x)$ will need to be written as polar functions. These functions will be $f(\theta)$ and $g(\theta)$, where θ is the normal polar angle measured counterclockwise from the x-axis. Their derivatives will be noted by $h'_-(\theta)$ for the left-hand, and $h'_+(\theta)$ for the right-hand derivative. We want to show that the x-ray data $D_0\chi(\theta) = f(\theta) - g(\theta)$ has a discontinuity in its derivative at a point θ_o if and only if, F, or G, or both have a discontinuity in its derivative along the ray making angle θ_o with the positive x-axis. Note that:

$H'(x) = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$, where $y = h(\theta)sin\theta$, $x = h(\theta)cos\theta$.

This becomes $\frac{dy}{dx} = H'(x) = \frac{h'(\theta)sin\theta + h(\theta)cos\theta}{h'(\theta)cos\theta - h(\theta)sin\theta}$

by implementation of the chain rule.

By Lemma 2.1, we may rotate the coordinates such that $\theta = \frac{\pi}{2}$. So the expression then becomes:

$H'(0) = \frac{h'(\frac{\pi}{2})}{-h(\frac{\pi}{2})}$.

Since $F(x)$ is a concave function with decreasing derivative,
$F'_-(0) \geq F'_+(0)$. Because the relation between derivatives contains a negative, the inequality in polar form becomes $f'_-(\frac{\pi}{2}) \leq f'_+(\frac{\pi}{2})$. Thus
$f'_-(\frac{\pi}{2}) - f'_+(\frac{\pi}{2}) \leq 0$.

So $(D_0\chi)'_-(\frac{\pi}{2}) = f'_-(\frac{\pi}{2}) - g'_-(\frac{\pi}{2}) = -f(\frac{\pi}{2})F'_-(0) + g(\frac{\pi}{2})G'_-(0)$.

Similarly, $(D_0\chi)'_+(\frac{\pi}{2}) = -f(\frac{\pi}{2}F'_+(0) + g(\frac{\pi}{2})G'_+(0)$.

Hence
$(D_0\chi)'_+(\frac{\pi}{2}) - (D_0\chi)'_-(\frac{\pi}{2}) = g(\frac{\pi}{2})(G'_+(0) - G'_-(0)) - f(\frac{\pi}{2})(F'_+(0) - F'_-(0))$.

Since $g(\frac{\pi}{2}) < f(\frac{\pi}{2})$ and $G'_+(0) - G'_-(0) \geq 0$ while $F'_+(0) - F'_-(0) \leq 0$. It follows that

$(D_0\chi)'_+(\frac{\pi}{2}) - (D_0\chi)'_-(\frac{\pi}{2}) = 0$

if and only if both F' and G' are continuous at 0. ■

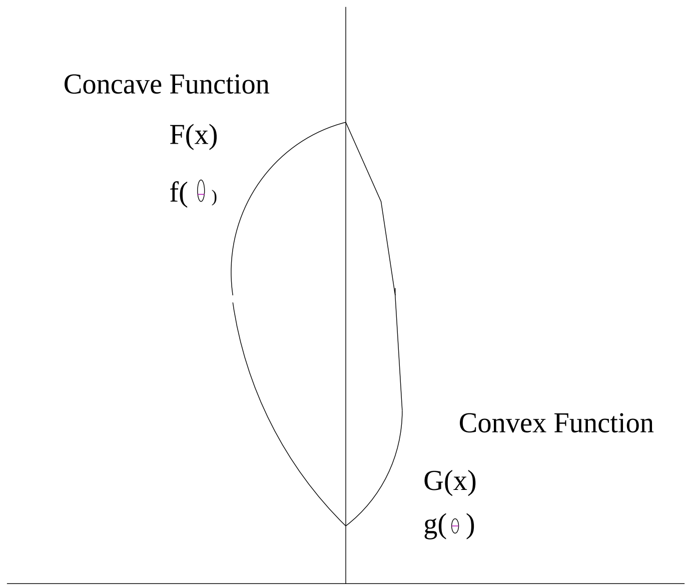

Figure 2.1 Convex body with x-ray at line x=0.

2.1 Detecting vertices in convex polygons

The following theorem is a special case of the previous theorem. We first proved this result for convex polygons, then expanded it to the case of all convex bodies.

If the geometric object is a convex polygon, we can determine which chords the vertices lie upon by a discontinuity in the derivative of the x-ray lengths.

$D_o\chi(\phi) = f(\phi)$ - $n(\phi)$, where f is the length from the point source to the far side of the polygon and n is the length from the point source to the near side of the polygon.

Theorem 2.2 *There is a discontinuity in the derivative of $D_o\chi(\phi)$ if and only if there is a vertex along the ray corresponding to ϕ.*

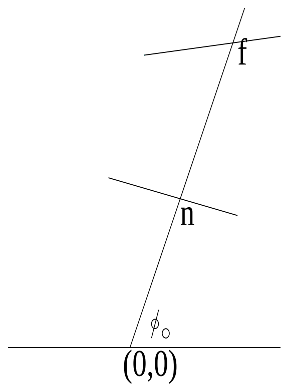

Figure 2.2 X-ray intersecting far side and near side of polygon

Proof Suppose for an angle ϕ, the far side of the polygon is given by the line $\frac{x}{a_f} + \frac{y}{b_f} = 1$, for some a_f and $b_f \neq 0$. The point where the x-ray and $\frac{x}{a_f} + \frac{y}{b_f} = 1$ intersect is $(f(\phi)cos(\phi), f(\phi)sin(\phi))$.

So $\frac{f(\phi)cos(\phi)}{a_f} + \frac{f(\phi)sin(\phi)}{b_f} = 1$.

$f(\phi) = \frac{a_f b_f}{b_f cos(\phi) + a_f sin(\phi)}$.

Thus, $\frac{1}{f(\phi)} = \frac{b_f cos(\phi) + a_f sin(\phi))}{a_f b_f} = \frac{cos(\phi)}{a_f} + \frac{sin(\phi)}{b_f}$.

$(\frac{1}{f(\phi)})' = \frac{-f'(\phi)}{(f(\phi))^2} = \frac{-sin(\phi)}{a_f} + \frac{cos(\phi)}{b_f}$

$f'(\phi) = (f(\phi))^2(\frac{sin(\phi)}{a_f} - \frac{cos(\phi)}{b_f})$

Similarly, if the near side is given by $\frac{x}{a_n} + \frac{y}{b_n} = 1$ for some a_n and $b_n \neq 0$, then

$n(\phi) = \frac{a_n b_n}{(b_n cos(\phi) + a_n sin(\phi))}$.

$n'(\phi) = (n(\phi))^2(\frac{sin(\phi)}{a_n} - \frac{cos(\phi)}{b_n})$.

Since the length of the x-ray is given by $D_o\chi(\phi) = f(\phi)$ - $n(\phi)$, then $D'_o\chi(\phi) = f'(\phi)$ - $n'(\phi)$.

If a vertex occurs at an angle ϕ_0, we will call the far side function: f_1 for $\phi > \phi_0$ and f_2 for $\phi < \phi_0$, the near side function: n_1 for $\phi > \phi_0$ and n_2 for $\phi < \phi_0$, and the length function: $D_{o1}\chi$ for $\phi > \phi_0$ and $D_{o2}\chi$ for $\phi < \phi_0$.

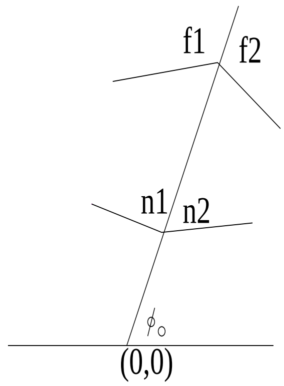

Figure 2.3 X-ray intersecting vertices on the far side and near side of polygon.

We conjecture that $D'_o\chi(\phi)$ is discontinuous if one or two vertices occur at ϕ_0, i.e. $D'_{o1}\chi(\phi_0) \neq D'_{o2}\chi(\phi_0)$.

Since $f(\phi)$, $n(\phi)$, and $D_o\chi(\phi)$ are continuous, $f_1(\phi_0) = f_2(\phi_0)$ and $n_1(\phi_0) = n_2(\phi_0)$.

$f'_1(\phi_0) = (f_1(\phi_0))^2(\frac{sin(\phi_0)}{a_{f1}} - \frac{cos(\phi_0)}{b_{f1}})$
$f'_2(\phi_0) = (f_1(\phi_0))^2(\frac{sin(\phi_0)}{a_{f2}} - \frac{cos(\phi_0)}{b_{f2}})$
$n'_1(\phi_0) = (n_1(\phi_0))^2(\frac{sin(\phi_0)}{a_{n1}} - \frac{cos(\phi_0)}{b_{n1}})$
$n'_2(\phi_0) = (n_2(\phi_0))^2(\frac{sin(\phi_0)}{a_{n2}} - \frac{cos(\phi_0)}{b_{n2}})$
By Lemma 2.1, we can rotate the coordinates to assume that $\phi_o = \frac{\pi}{2}$. This reduces $f'(\phi_0)$, $n'(\phi_0)$, and $D'_o\chi(\phi_0)$ as follows:

$f'_1(\phi_0) = (f_1(\phi_0))^2(\frac{1}{a_{f1}})$
$f'_2(\phi_0) = (f_1(\phi_0))^2(\frac{1}{a_{f2}})$
$n'_1(\phi_0) = (n_1(\phi_0))^2(\frac{1}{a_{n1}})$
$n'_2(\phi_0) = (n_1(\phi_0))^2(\frac{1}{a_{n2}})$
$D'_{o1}\chi(\phi_0) = (f_1(\phi_0))^2(\frac{1}{a_{f1}}) - (n_1(\phi_0))^2(\frac{1}{a_{n1}})$
$D'_{o2}\chi(\phi_0) = (f_1(\phi_0))^2(\frac{1}{a_{f2}}) - (n_1(\phi_0))^2(\frac{1}{a_{n2}})$

Suppose one vertex occurs at ϕ_0 and also suppose $D'_{o1}\chi(\phi_0) = D'_{o2}\chi(\phi_0)$.

Then $(f_1(\phi_0))^2(\frac{1}{a_{f1}}) - (n_1(\phi_0))^2(\frac{1}{a_{n1}}) = (f_1(\phi_0))^2(\frac{1}{a_{f2}}) - (n_1(\phi_0))^2(\frac{1}{a_{n2}})$

We get
$(f_1(\phi_0))^2(\frac{1}{a_{f1}} - \frac{1}{a_{f2}}) = (n_1(\phi_0))^2(\frac{1}{a_{n1}} - \frac{1}{a_{n2}})$ where
$0 < (n_1(\phi_0))^2 < (f_1(\phi_0))^2$.

If a vertex occurs only in the far side, then $a_{n1} = a_{n2}$, so

$$(f_1(\phi_0))^2(\frac{1}{a_{f1}} - \frac{1}{a_{f2}}) = 0,$$

which is a contradiction. Similarly, if a vertex occurs only in the near side, then $a_{f1} = a_{f2}$, so

$$(n_1(\phi_0))^2(\frac{1}{a_{n1}} - \frac{1}{a_{n2}}) = 0,$$

which is also a contradiction. Hence, there is a discontinuity in $D'_o\chi(\phi_0)$ if one vertex occurs at ϕ_0.

If a vertex occurs in the far side, we have only five possible types of vertices because of convexity.

If $a_{f1} < 0$ and $a_{f2} > 0$, then $\frac{1}{a_{f1}} - \frac{1}{a_{f2}} < 0$.
If $a_{f1} = \infty$ and $a_{f2} > 0$, then $\frac{1}{a_{f1}} - \frac{1}{a_{f2}} < 0$.
If $a_{f1} < 0$ and $a_{f2} = \infty$, then $\frac{1}{a_{f1}} - \frac{1}{a_{f2}} < 0$.
If $a_{f1} > 0$, $a_{f2} > 0$, and $a_{f1} > a_{f2}$, then $\frac{1}{a_{f1}} - \frac{1}{a_{f2}} < 0$.
If $a_{f1} < 0$, $a_{f2} < 0$, and $a_{f1} > a_{f2}$, then $\frac{1}{a_{f1}} - \frac{1}{a_{f2}} < 0$.

If a vertex occurs in the near side, we have only five possible types of vertices because of convexity.

If $a_{n1} > 0$ and $a_{n2} < 0$, then $\frac{1}{a_{n1}} - \frac{1}{a_{n2}} > 0$.
If $a_{n1} = \infty$ and $a_{n2} < 0$, then $\frac{1}{a_{n1}} - \frac{1}{a_{n2}} > 0$.
If $a_{n1} > 0$ and $a_{n2} = \infty$, then $\frac{1}{a_{n1}} - \frac{1}{a_{n2}} > 0$.
If $a_{n1} > 0$, $a_{n2} > 0$, and $a_{n1} < a_{n2}$, then $\frac{1}{a_{n1}} - \frac{1}{a_{n2}} > 0$.
If $a_{n1} < 0$, $a_{f2} < 0$, and $a_{n1} < a_{n2}$, then $\frac{1}{a_{n1}} - \frac{1}{a_{n2}} > 0$.

If $a_{f1}\ a_{n1} \neq a_{f2}\ a_{n2}$, then

$(f_1(\phi_0))^2(\frac{1}{a_{f1}} - \frac{1}{a_{f2}}) \neq (n_1(\phi_0))^2(\frac{1}{a_{n1}} - \frac{1}{a_{n2}})$.

Therefore $D'_o\chi(\phi_0)$ does not exist.

If no vertices occur at ϕ_0, then $a_{f1}\ a_{n1} = a_{f2}\ a_{n2}$, and $D'_o\chi(\phi_0)$ exists. ∎

3 Successive reconstruction

Definition 3.1 *A convex body can be successively determined if it can be distinguished from any other convex body by its x-rays taken in the following way. After the first x-ray is taken, the x-ray information is used to choose which direction the second x-ray will be taken. If necessary, the information from the second x-ray is used to choose the third direction, and so on.[**?**]*

Theorem 3.1 *Convex polygons can be successively determined by x-rays from three collinear points.*

Proof Successive determination allows us the prior knowledge that the object being x-rayed is a convex polygon. Let K be a convex polygon in E^2 located above the x-axis. Let p, q be two points on the x-axis. From Theorem 1.1, we can detect which angles relative to p and q vertices occur along. Since there are a finite number of vertices, there would be a finite number of angles. If we extend rays from these angles, we get a finite grid of n possible vertex points. If we draw a line between every pair of possible vertex points, we would have at most ${}_nC_2 = \frac{n!}{(n-2)!2!}$, lines which would intersect the x-axis at at most ${}_nC_2$ points. Then a third point along the x-axis can now be chosen so that it does not coincide with any of these lines intersecting the x-axis. X-ray infomation from this third source will now identify angles at which vertices occur. Intersection of rays from all three points will be the vertices of the polygon. A convex polygon is reconstructable from its vertices. ∎

4 Reconstruction Algorithm for one point source

4.1 Terminology and Formulas

Definition 4.1 *If P is a polygon, we can partition P into a finite set of cones with its vertex at our x-ray source such that no vertices of P lie within* *the interior of these cones. Let each cone be represented by* $C(A,B) = (r,\theta) : A \leq \theta \leq B$, *where* $0 < A < B < \pi$.*[**?**]*

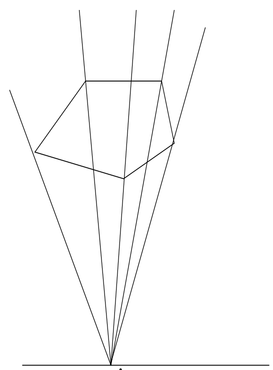

Figure 4.1 Convex polygon partitioned into cones with vertex at point source.

We will introduce a different formula for the length of an x-ray at an angle T.

Theorem 4.1 $D_o\chi(T) = \frac{xy(sin(B-A))}{ysin(B-T)+xsin(T-A)} - \frac{wz(sin(B-A))}{wsin(B-T)+zsin(T-A)}$ *[**?**]*

Proof Let's look at $C(A,B)$ whose vertex is at the origin. Consider the triangle formed by the polar coordinates $(0,0),(x,A)$, and (y,B), where (x,A) and (y,B) are points on the far side of the polygon. Suppose we have an x-ray at angle T, where $A < T < B$. The x-ray intersects the line created by (x,A) and (y,B) at the point (f,T). By the equation of a line, we get:

$$(fsin(T) - xsin(A)) = (\frac{xsin(A)-ysin(B)}{xcos(A)-ycos(B)})(fcos(T) - xcos(A))$$

$$f = \frac{xy(sin(B-A)}{ysin(B-T)+xsin(T-A)}$$

Similarly, consider the triangle formed by the polar coordinates $(0,0),(z,A)$, and (w,B), where (z,A) and (w,B) are points on the near side of the polygon. Our x-ray at angle T, where $A < T < B$, intersects the line created by (z,A) and (w,B) at the point (n,T). By the equation of a line, we get:

$(nsin(T) - zsin(A)) = (\frac{zsin(A)-wsin(B)}{zcos(A)-wcos(B)})(ncos(T) - zcos(A))$

$n = \frac{wz(sin(B-A)}{wsin(B-T)+zsin(T-A)}$

Thus, the length of an x-ray is given by:

$D_o\chi(T) = \frac{xy(sin(B-A))}{ysin(B-T)+xsin(T-A)} - \frac{wz(sin(B-A))}{wsin(B-T)+zsin(T-A)}$ ■

Definition 4.2 *We will refer to the quadrilateral formed by the points* x, y, z, w *as a wedge. The wedge is contained within* $C(A, B)$.

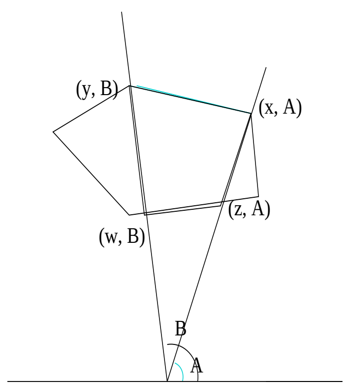

Figure 4.2 Wedge with corners
(x, A), (y, B), (z, A), (w, B)

4.2 Determining convex polygons from a single point source x-ray

Theorem 4.2 *Any convex polygon is uniquely determined by one point source x-rays with the exception of parallel strips.*

Proof Gardner provides a proof in his book.[**?**] ■

We proved a special case of the previous theorem using the equation for $D_o\chi(T)$.

Theorem 4.3 *Special types of convex polygons are uniquely determined by one point source x-rays with the exception of parallel strips.*

Proof Assume we have two wedges contained in a cone defined by angles A and B, where the x-rays are equal for every angle T such that $A < T < B$. The following equation would hold:

$$\frac{xy(sin(B-A))}{ysin(B-T)+xsin(A-T)} - \frac{wz(sin(B-A))}{wsin(B-T)+zsin(T-A)} = \frac{(x+a)(y+b)(sin(B-A))}{(y+b)sin(B-T)+(x+a)sin(A-T)} - \frac{(w+b)(z+a)(sin(B-A))}{(w+b)sin(B-T)+(z+a)sin(T-A)}$$

where the line segment $\overline{xz}$ is translated a units, and the line segment $\overline{yw}$ is translated b units. $a > -z$ and $b > -w$.

After some algebraic manipulation, we get the following:

$$sin(T-A)(xw(-aw+2bz)(x+a)+yz(ay-2bx)(z+a)+b^2xz(x-z)) =$$
$$sin(B-T)(yz(-bz+2aw)(y+b)+xw(bx-2az)(w+b)+a^2yw(y-w)).$$

If T is allowed to assume infinitely many values from A to B, then the previous equation only holds true when:

$xw(-aw+2bz)(x+a)+yz(ay-2bx)(z+a)+b^2xz(x-z) = 0$ (1)
and $yz(-bz+2aw)(y+b)+xw(bx-2az)(w+b)+a^2yw(y-w) = 0.$ (2)

Notice the symmetry of the two equations. x is interchanged with y, z with w, and a with b.

Equation (1) holds true if:
$(i) aw^2 = bz(2w+b)$ and $ay^2 = bx(2y+b)$
or $(ii) \frac{z+a}{x+a} = \frac{xw(aw-2bz)-b^2xz}{yz(ay-2bx)-b^2xz}$.

Equation (2) holds true if:
$(iii) bz^2 = aw(2z+a)$ and $bx^2 = ay(2x+a)$
or $(iv) \frac{w+b}{y+b} = \frac{yz(bz-2aw)-a^2wy}{wx(bx-2ay)-a^2wy}$.

We have four cases to consider:
I. (i) and (iii)
II. (i) and (iv)
III. (ii) and (iii)

IV. (*ii*) and (*iv*).

In turn, there are several possible geometric configurations:

(A) parallel strips (all four lines parallel)
(B) triangles
(C) two points on one edge of the cone fixed
(D) parallel strips that are not parallel to each other
(E) one side of a wedge parallel to one side of the other wedge
(F) two wedges, all four lines not parallel

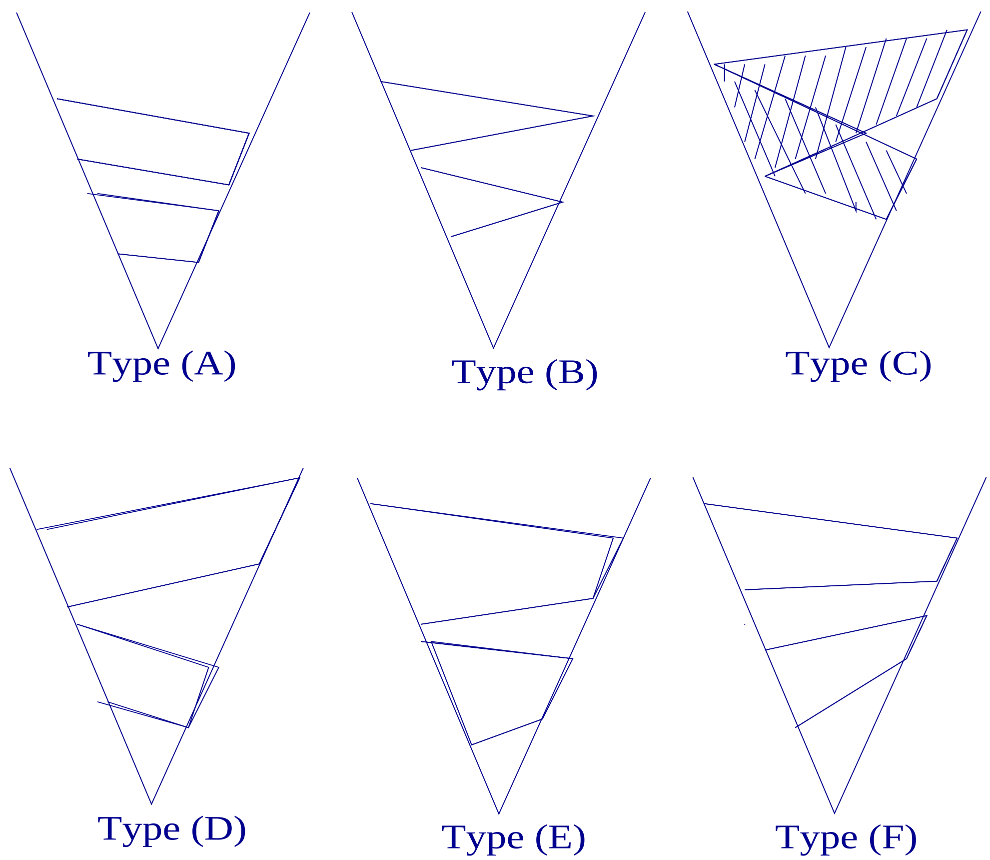

Figure 4.3 Wedges of Types (A), (B), (C), (D), (E), (F)

Let's examine wedges of type (A), parallel strips with all four line parallel. We know

$\frac{w}{y} = \frac{z}{x}$

from properties of similar triangles. Condition (i) states that

$\frac{w^2}{y^2} = \frac{z(2w+b)}{x(2y+b)}$.

b must equal zero, which forces a equal to zero. Condition (iii) similarly shows $a, b = 0$. So, cases I, II and III imply uniqueness in the x-rays.

Let's look at case IV. We also know that

$\frac{w+b}{y+b} = \frac{z+a}{x+a}$ and $\frac{x}{y} = \frac{z}{w} = \frac{x+a}{y+b} = \frac{z+a}{w+b} = \frac{a}{b}$.

So, we have:
$\frac{z+a}{x+a} = \frac{w+b}{y+b} = \frac{xw(aw-2bz)-b^2xz}{yz(ay-2bx)-b^2xz} = \frac{yz(bz-2aw)-a^2wy}{wx(bx-2ay)-a^2wy}$.

After some algebraic manipulation, we get the left hand side equal to the right hand side. So therefore, the x-rays for a parallel strip are not unique. Parallel strips satisfy Case IV and have equal x-rays for all angles T for $A < T < B$.

Let's look at Case I for all types of wedges. From (i) and (iii), we have: $\frac{w^2}{y^2} = \frac{z(2w+b)}{x(2y+b)}$ (3) and $\frac{z^2}{w^2} = \frac{w(2z+a)}{y(2x+a)}$ (4)

Assume $\frac{w}{y} \neq \frac{z}{x}$. Without loss of generality, say $\frac{w}{y} > \frac{z}{x}$. Then from equations (3) and (4), $\frac{2z+a}{2x+a} < \frac{z}{x} < \frac{w}{y} < \frac{2w+b}{2y+b}$. Thus, $b > 0$ and $a < 0$. This is impossible since one of our equations states $aw^2 = bz(2w+b)$ and we know $b > -w$. Therefore, $\frac{w}{y} = \frac{z}{x}$ and $a, b = 0$.

Now let's consider wedges of type (B), triangles. We will assume $y = w$. So, looking at case II, we have

$ay^2 = aw^2 = bz(2w+b) = bx(2y+b)$.
This implies either $x = z$ or $a, b = 0$. Looking at case III, we have

$\frac{z^2}{x^2} = \frac{2z+a}{2x+a}$.

$2x^2z + ax^2 - 2xz^2 - az^2 = 0$

$2xz(x-z) + a(x+z)(x-z) = 0$ This implies $x = z$.

Case IV gives us the following:

$z(bz - 2ay) - ay^2 = x(bx - 2ay) - a^2y$ (5)

$(z+a)(yz(ay-2bx) - b^2xz) = (x+a)(xy(ay-2bz) - b^2xz)$ (6)

Equation (5) gives us $z(bz - 2ay) = x(bx - 2ay)$. Equation (6) gives us $z(ay^2z + a^2y^2 - 2bxyz - b^2xz) = x(axy^2 + a^2y^2 - 2bxyz - b^2xz)$.

So we have

$\frac{bx-2ay}{bz-2ay} = \frac{ay^2z+a^2y^2-2bxyz-b^2xz}{axy^2+a^2y^2-2bxyz-b^2xz}$

$a^2by^2z - 2b^2xyz^2 - b^3xz^2 - 2a^2y^3x = a^2bxy^2 - 2b^2x^2yz - b^3x^2z - 2a^2y^3z$ (7)

$(a^2y^2 - b^2xz)(2y + b)(x - z) = 0$

We know $(2y + b) \neq 0$ because $b > -w$. $(a^2y^2 - b^2xz) \neq 0$ because if we plug $(a^2y^2 - b^2xz) = 0$ back into equation (7), we get both sides of equation (7) $= 0$, which is a contradiction. Thus, $x - z = 0$, which indicates that the x-rays of triangular wedges are unique.

Next, let's compare wedges of type (C), where two points are fixed on one edge of the cone. We will assume that $b = 0$. Condition (i) now states
$0 = aw(2z + a)$ and $0 = ay(2x + a)$
This implies $a = 0$ because $a > -z$.

Condition (iii) states

$ay^2 = 0$ and $aw^2 = 0$

This also implies $a = 0$. So, cases II and III are settled.

Case IV now gives us

$\frac{z+a}{x+a} = \frac{axw^2}{ay^2z}$ and $\frac{w}{y} = \frac{2az+a^2}{2ax+a^2}$

So, $\frac{(2az+a^2)^2}{(2ax+a^2)^2} = \frac{az(z+a)}{ax(x+a)}$.

Hence, $a^6x = a^6z$.

Either $x = z$, which would imply $w = y$, or $a = 0$. Thus, the x-rays of wedges who share the same points on one side of the cone are unique.

Now we will compare wedges of type (D), parallel strips that are not parallel to each other. Suppose wedge α and wedge β are parallel strips not parallel to each other, but have the same x-rays along angles A and B. Then, we can translate wedge α so that it shares the same points as wedge β on one side of the cone. We know we can translate a parallel wedge without changing its x-ray data from the proof of type (A) wedges. Now, we have reduced the wedges to type (C), which we know has unique x-rays.

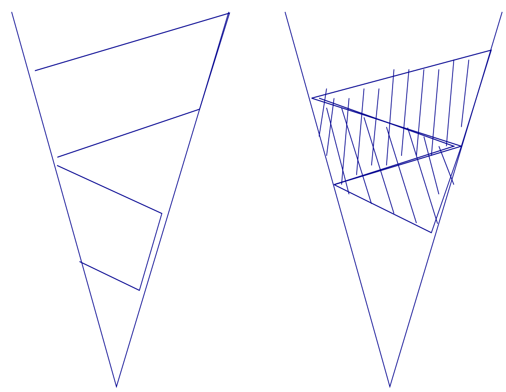

Figure 4.4 Translation of wedges of Type (D)

Let's now examine wedges of type (E), non parallel wedges with one side of a wedge parallel to one side of the other wedge. Suppose wedge κ and wedge λ are both non-parallel wedges, but one side of wedge κ is parallel to one side of wedge λ. Then, we can decompose wedge κ into a parallel strip and a triangle. Similarly, wedge λ decomposes into a parallel strip and a triangle. So the x-rays of each of these wedges is equal to the sum of the x-rays of its parallel strip and the x-rays of its triangle. Since, one side of each of the wedges is parallel, both are parallel, and their parallel strips are equal. However, we know from wedges of type (B), the x-rays of their triangles are not equal. Thus the x-rays of wedges κ and λ are not equal.

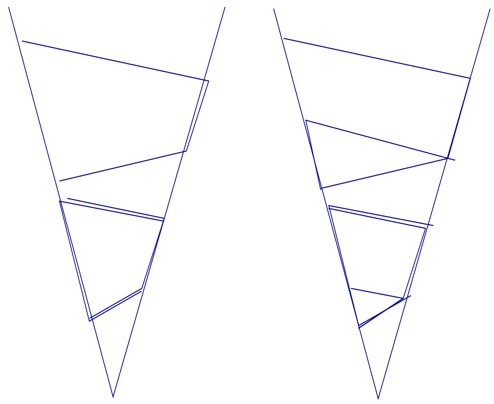

Figure 4.5 Decomposition of wedges of type (E)

We have not yet had success proving that the x-rays of type (F) wedges are unique. ■

4.3 Approximating Vertices

Our function for the length of an x-ray is non-linear and difficult to work with. So, we will reduce the function to a more workable form.

$$
\begin{aligned}
D_o\chi(T) &= \frac{xy(sin(B-A)}{ysin(B-T)+xsin(A-T)} - \frac{(wz(sin(B-A)}{wsin(B-T)+zsin(T-A)} \\
&= sin(B-A)[\frac{1}{\frac{1}{x}sin(B-T)+\frac{1}{y}sin(T-A)} - \frac{1}{\frac{1}{z}sin(B-T)+\frac{1}{w}sin(T-A)}] \\
&= sin(B-A)[\frac{1}{\frac{1}{x}(sin(B)cos(T)-cos(B)sin(T))+\frac{1}{y}sin(T)cos(A)-cos(T)sin(A)} - \\
&\quad \frac{1}{\frac{1}{z}(sin(B)cos(T)-cos(B)sin(T))+\frac{1}{w}sin(T)cos(A)-cos(T)sin(A))}] \\
&= sin(B-A)[\frac{1}{(\frac{1}{x}sin(B)-\frac{1}{y}sin(A))cos(T)+(-\frac{1}{x}cos(B)+\frac{1}{y}cos(A))sin(T)} - \\
&\quad \frac{1}{\frac{1}{z}sin(B)-\frac{1}{w}sin(A))cos(T)+(-\frac{1}{z}cos(B)+\frac{1}{w}cos(A))sin(T)}]
\end{aligned}
$$

Let's define:

$p = \frac{1}{x}sin(B) - \frac{1}{y}sin(A)$
$q = -\frac{1}{x}cos(B) + \frac{1}{y}cos(A)$
$r = \frac{1}{z}sin(B) - \frac{1}{w}sin(A)$

$s = -\frac{1}{z}cos(B) + \frac{1}{w}cos(A)$

Since we know what angles define the cone, $sin(B-A)$ is a constant. So, we can divide the x-ray data by $sin(B-A)$. Our function for our modified x-ray data is:

$L(T) = \frac{D_o\chi(T)}{sin(B-A)} = \frac{1}{pcos(T)+qsin(T)} - \frac{1}{rcos(T)+ssin(T)}$.

We can compute the coefficient matrix of the system of linear equations p, q, r, s with respect to x, y, z, w.

$$M = \begin{pmatrix} sin(B) & -sin(A) & 0 & 0 \\ -cos(B) & cos(A) & 0 & 0 \\ 0 & 0 & sin(B) & -sin(A) \\ 0 & 0 & -cos(B) & cos(A) \end{pmatrix}$$

Thus, $det(M) = 2sin(A-B) \neq 0$ since $A \neq B$ and $A \neq B + \pi$. p, q, r, s are uniquely determined by x, y, z, w.

We have written a program in Matlab that approximates p, q, r, s using the least squares approximation. Our program is shown below:

```
function E=vertices(data)
T=[A:0.02:B]
F=[modified x-ray data for corresponding T values]
p=data(1);q=data(2);r=data(3);s=data(4);
E=F - (1/(p*cos(T)+q*sin(T)) - 1/(r*cos(T)+s*sin(T)));
```

In Matlab, we type "leastsq('vertices',[*initial guesses for p,q,r,s*])". Matlab sums up the square of the errors, E, and minimizes the sum. p, q, r, s usually is less than one, unless x, y, z, w are less than one. We can arrive at a reasonable initial guess by looking at the length of the x-ray data. For example, if $L(T) = m$, a good initial guess might be $\frac{1}{m}$.

Our idea for an algorithm to reconstruct convex polygons is as follows. After the x-ray data is collected, compute divided differences to find discontinuities in the derivative of the x-rays. Then divide the upper plane into cones according to where the discontinities occur. Starting at the left-most or right-most cone, use the least squares approximation to determine p, q, r, s.

With probability one, this wedge should be a triangle. Then proceed to the right or to the left. Two points are already determined from the previous wedge. Then p, q, r, s will determine x, y, z, w by the equations:

$$x = \frac{sin(B-A)}{pcos(A)+qsin(A)}$$
$$y = \frac{sin(B-A)}{pcos(B)+qsin(B)}$$
$$z = \frac{sin(B-A)}{rcos(A)+ssin(A)}$$
$$w = \frac{sin(B-A)}{rcos(B)+ssin(B)}$$

In the case that both the furthest right and furthest left wedges are of type (F), Gardner's theorem states that all situations are unique except parallel wedges. If there is only one cone whose x-rays do not give unique p, q, r, s values, then the polygon is a parallel strip, and another x-ray must be taken. The parallel strip is guaranteed to be determined by a second x-ray because the polygon will not be a parallel strip from a second distinct point.

5 Questions for further investigation

We confirmed the non-uniqueness of x-rays of type (A) wedges from a single point source. We have also established the uniqueness of one source x-rays of wedges of type (B), (C), (D), and (E) through an alternate proof from Gradner's. The uniqueness of x-rays of type (F) wedges still remains unproven by our methods.

We devised an algorithm for determining the vertices of a convex polygon from a single point source. For reconstruction purposes, we would like to know if a wedge is uniquely determined by a finite number of x-ray values from a single source, and if so, how many values are needed for uniqueness.

Another point of interest may be to prove or disprove that the x-ray data of a convex geometric object is a convex body itself.

We know that the non-smooth points in a convex subset can be detected by x-rays from one point source. A further topic to investigate is determining how many point sources are needed to successfully reconstruct the convex subset. We would like to create an algorithm for successive determination of a general convex body. We would like to investigate whether our algorithm for convex polygon reconstruction can be expanded to include the general case.

The stability (effect of small perturbations) of the algorithm also should be investigated.

References