Symmetry Reduction of Maxwell's Equations
A Case Study

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Abstract

We apply the symmetry reduction process to Maxwell's Equations in order to construct solutions invariant under various subgroups of the inhomogeneous Poincaré group. We investigate the physical properties of the solutions. Of particular interest are their applications to the Goursat problem.

1 Introduction

The Poincaré group is generated by translations, rotations, and Lorentz boosts. Homogeneous and isotropic space is invariant under the full Poincaré group. In practical problems, this symmetry may be broken by factors such as charges, currents, and boundary conditions. The resulting electromagnetic field, however, might remain invariant under a subgroup of the Poincaré group. If a field carries a Poincaré subgroup as a symmetry group, we can then observe the motion of a test charged particle in that field. Also, if a group preserves boundary conditions, we can then use group-invariant solutions to help solve boundary value problems for the corresponding electromagnetic field. Of particular interest is the Goursat problem, which arises in a new formulation of electromagnetics advocated by Dirac, in which boundary condition data on two hyperplanes, rather than the classical charge and current data, is used to solve Maxwell's Equations.

The symmetry reduction process is used to determine the fields invariant under a specific symmetry group. We note that the classical problems of finding rotationally and axially symmetric solutions are special cases of this process. For example, the symmetry group of an electromagnetic field created by a point source with a charge is the full rotational (about the point) symmetry group, since a point source is rotationally invariant. The field created by a line source with a charge carries a symmetry group consisting of rotations about an infinite line, and, provided that the charge is homogeneous, translations along the line.

This problem has already been investigated for a charged system in a constant and uniform electromagnetic field and for the electromagnetic field of a circularly polarized plane wave. The first of these studies has led to derivation of additivity of charge and the superselection rule for the electric charge, and the second has produced interesting results about an electron interacting with a laser beam. Work has also been done in the classification of electromagnetic fields invariant under connected electromagnetic Poincaré subgroups of dimensions five and six. Further, it has been proven that if a Poincaré subgroup is a symmetry group of a non-trivial electromagnetic field, then the subgroup must be of degree six or less $_2$.

Here we study various electromagetic fields invariant under subgroups of degree one, two, and three. We select several invariance Poincaré subgroups and perform symmetry reduction in order to find explicit solutions to Maxwell's Equations. We investigate the physical properties of these solutions, in particular their applications to the Goursat problem. This work was done as part of the Research Experiences for Undergraduates (REU) Program at Oregon State University during the summer of 1999 under the direction of Juha Pohjanpelto.

2 Maxwell's Equations

We list the basic laws of electromagnetism, known as Maxwell's Equations, which describe the behavior of electric and magnetic fields and their interaction. Here we consider the empty-space Maxwell equations, which are applicable in vacuum:

$$\operatorname{div} \vec{E} = \frac{\partial E^1}{\partial x} + \frac{\partial E^2}{\partial y} + \frac{\partial E^3}{\partial z} = 0 \qquad \operatorname{div} \vec{B} = \frac{\partial B^1}{\partial x} + \frac{\partial B^2}{\partial y} + \frac{\partial B^3}{\partial z} = 0 \quad (1)$$

$$\frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B} \qquad \frac{\partial \vec{B}}{\partial t} = - \nabla \times \vec{E} \quad (2)$$

Equations (1) are known as the divergence equations. Equations (2), the time-evolution equations, are equivalent to the six equations below:

$$\frac{\partial E^1}{\partial t} = \frac{\partial B^3}{\partial y} - \frac{\partial B^2}{\partial z} \tag{3}$$

$$\frac{\partial B^1}{\partial t} = -\frac{\partial E^3}{\partial y} + \frac{\partial E^2}{\partial z} \tag{4}$$

$$\frac{\partial E^2}{\partial t} = \frac{\partial B^1}{\partial z} - \frac{\partial B^3}{\partial x} \tag{5}$$

$$\frac{\partial B^2}{\partial t} = -\frac{\partial E^1}{\partial z} + \frac{\partial E^3}{\partial x} \tag{6}$$

$$\frac{\partial E^3}{\partial t} = \frac{\partial B^2}{\partial x} - \frac{\partial B^1}{\partial y} \tag{7}$$

$$\frac{\partial B^3}{\partial t} = -\frac{\partial E^2}{\partial x} + \frac{\partial E^1}{\partial y} \tag{8}$$

We observe that charge conjugation is preserved by the Maxwell equations:

$$\sigma(E, B) = (B, -E)$$

We will take full advantage of charge conjugation in carrying out our calculations, defining our quantities so as to preserve this symmetry. Thus, once an equation involving electric fields is derived, the symmetrical equation for magnetic fields can easily be derived without doing the corresponding calculations.

3 Group-Invariant Solutions

Given a complicated system of partial differential equations, such as the Maxwell Equations, it is often a difficult matter to find explicit solutions. However, a wide variety of exact solutions can often be obtained by performing symmetry reduction on the system. The symmetry reduction process (detailed below) is a procedure for constructing group-invariant solutions to a system of partial differential equations. This algorithm dates back to S. Lie, and has recently been used successfully to construct exact solutions to various nonlinear partial differential equations such as Yang-Mills, KdV, and PKP. In this process we find explicit solutions which are invariant under some symmetry group of the system.

Definition 1 *A symmetry group of a system of differential equations Δ is a group of transformations G such that if u=f(x) solves Δ and gf (the transformation of f by g) is defined for $g \in G$, then u=gf(x) also solves Δ.*

We call the solution f of the system invariant under the action of the symmetry group if the infinitesimal generator of the symmetry group is tangent to the graph of f at every point. Using the symmetry reduction process, it is possible to reduce a system of partial differential equations to a system which may be easier to solve. The reduced system yields a complete set of group-invariant solutions.

3.1 The Symmetry Reduction Process

The basic steps involved in symmetry reduction are outlined below. We assume that our original system has p independent variables and q dependent variables.

1. Find and classify infinitesimal generators of symmetry groups of the system of equations.
2. Choose a degree of symmetry, s, for the invariant solutions. If a solution is invariant under an s-dimensional symmetry group, then it satisfies a system of reduced equations that involves s less independent variables than the original system. Choosing $s = p - 1$ will result in a reduced system of ordinary differential equations. Note that the projected action of the subgroup must be regular, in other words, that the space-time components of the generators of the subgroup must be functionally independent.
3. Fixing the symmetry group G, find a set of functionally independent invariants, $p - s$ of these designated as the new independent variables, and the remaining q invariants designated as the new dependent variables. Assume that these q variables are dependent on the new $p - s$ independent variables.
4. Solve for the original dependent variables in terms of the new dependent variables and the original independent variables .
5. Substitute these expressions into the original system of equations. A reduced system of equations will emerge which depends entirely on the new independent variables, thereby simplifying the problem by eliminating s independent variables.
6. Solve the reduced system, if possible.

3.2 Conjugate Subgroups and Classification

We observe that Lie groups, in our case the Poincaré group, typically have an infinite number of s-dimensional subgroups, making it impossible to consider every such subgroup individually. However, the classification of group-invariant solutions can be effectively simplified by noting the following:

Theorem 2 *Let $H \subset G$ be a subgroup of a group G, and let function f be invariant under H, and $g \in G$. Then $f_g = gf$ is invariant under gHg^{-1}.*

Proof. $ghg^{-1}(f_g) = ghg^{-1}(gf) = ghf = gf = f_g$. ■

It follows that to classify all invariant solutions, it is enough to classify one representative of each conjugacy class of the Poincaré group.

Winternitz et al. of CRM in Montreal have classified these conjugacy classes$_3$. We use his classification to begin carrying out the symmetry reduction process.

4 Applying the Process to Various Subgroups

We apply the symmetry reduction process to various Poincaré subgroups of one, two, and three dimensions found in Winternitz' classification. Write

$u_0 = t\partial_z + z\partial_t - B^1\partial_{E^2} - E^2\partial_{B^1} + E^1\partial_{B^2} + B^2\partial_{E^1}$,

$u_1 = 2(y\partial_x - x\partial_y) - \frac{1}{2}(\partial_t + \partial_z) + 2(E^2\partial_{E^1} - E^1\partial_{E^2}) + 2(B^2\partial_{B^1} - B^1\partial_{B^2})$,

$v_1 = \partial_y$ for spatial translation along y-axis,

$v_2 = \frac{1}{2}(\partial_t - \partial_z)$ for spatial translation along z-axis and time translation,

$v_3 = -\partial_x$ for spatial translation along x-axis, and

$v_4 = \frac{1}{2}(\partial_t + \partial_z)$ for spatial translation along z-axis and time translation.

4.1 Subgroup generated by u_0

We first consider a one-dimensional subgroup generated by u_0. This generator represents a Lorentz boost lifted into a symmetry of Maxwell's equations. The Lorentz boost is of interest because it preserves boundary conditions on $t+z=0$ and $t-z=0$ and is thus relevant to the Goursat problem.

By the algorithm, we know that a one-dimensional subgroup will reduce the number of independent variables from the original system by one. We choose our invariants as follows:

$$x, y, s \qquad \text{where } s = \sqrt{t^2 - z^2}$$

$$\begin{aligned} e_1 &= tE^1 - zB^2 & b_1 &= tB^1 + zE^2 \\ e_2 &= zB^1 + tE^2 & b_2 &= -zE^1 + tB^2 \\ e_3 &= E^3 & b_3 &= B^3 \end{aligned}$$

Note that we choose and name our invariants so that charge conjugation is preserved.

We then use Cramer's Rule to solve for E^i and B^i:

$$\begin{aligned} E^1 &= \frac{te_1 + zb_2}{s^2} & B^1 &= \frac{tb_1 - ze_2}{s^2} \\ E^2 &= \frac{te_2 - zb_1}{s^2} & B^2 &= \frac{tb_2 + ze_1}{s^2} \\ E^3 &= e_3 & B^3 &= b_3 \end{aligned}$$

Choosing e_i, b_i to be dependent on x, y and s, we compute the partial derivatives necessary to substitute back into the original system:

$$\frac{\partial E^1}{\partial x} = \frac{t}{s^2}\frac{\partial e_1}{\partial x} + \frac{z}{s^2}\frac{\partial b_2}{\partial x} \qquad \frac{\partial E^1}{\partial y} = \frac{t}{s^2}\frac{\partial e_1}{\partial y} + \frac{z}{s^2}\frac{\partial b_2}{\partial y}$$

$$\frac{\partial E^1}{\partial z} = \frac{2zt}{s^4}e_1 - \frac{zt}{s^3}\frac{\partial e_1}{\partial s} + \left(\frac{1}{s^2} + \frac{2z^2}{s^4}\right)b_2 - \frac{z^2}{s^3}\frac{\partial b_2}{\partial s}$$

$$\frac{\partial E^1}{\partial t} = \left(\frac{1}{s^2} - \frac{2t^2}{s^4}\right)e_1 + \frac{t^2}{s^3}\frac{\partial e_1}{\partial s} - \frac{2zt}{s^4}b_2 + \frac{zt}{s^3}\frac{\partial b_2}{\partial s}$$

$$\frac{\partial E^2}{\partial x} = \frac{t}{s^2}\frac{\partial e_2}{\partial x} - \frac{z}{s^2}\frac{\partial b_1}{\partial x} \qquad \frac{\partial E^2}{\partial y} = \frac{t}{s^2}\frac{\partial e_2}{\partial y} - \frac{z}{s^2}\frac{\partial b_1}{\partial y}$$

$$\frac{\partial E^2}{\partial z} = \frac{2zt}{s^4}e_2 - \frac{zt}{s^3}\frac{\partial e_2}{\partial s} - \left(\frac{1}{s^2} + \frac{2z^2}{s^4}\right)b_1 + \frac{z^2}{s^3}\frac{\partial b_1}{\partial s}$$

$$\frac{\partial E^2}{\partial t} = \left(\frac{1}{s^2} - \frac{2t^2}{s^4}\right)e_2 + \frac{t^2}{s^3}\frac{\partial e_2}{\partial s} + \frac{2zt}{s^4}b_1 - \frac{zt}{s^3}\frac{\partial b_1}{\partial s}$$

$$\frac{\partial E^3}{\partial x} = \frac{\partial e_3}{\partial x} \qquad \frac{\partial E^3}{\partial y} = \frac{\partial e_3}{\partial y}$$

$$\frac{\partial E^3}{\partial z} = \frac{-z}{s}\frac{\partial e_3}{\partial s} \qquad \frac{\partial E^3}{\partial t} = \frac{t}{s}\frac{\partial e_3}{\partial s}$$

The partial derivatives for the components of $\vec{B}$ follow by charge conjugation.

Substituting these new expressions for the partial derivatives into Maxwell's equations yields the following system of equations:

$$\frac{t}{s^2}\frac{\partial e_1}{\partial x}+\frac{t}{s^2}\frac{\partial e_2}{\partial y}-\frac{z}{s}\frac{\partial e_3}{\partial s}-\frac{z}{s^2}\frac{\partial b_1}{\partial y}+\frac{z}{s^2}\frac{\partial b_2}{\partial x}=0 \qquad (9)$$

$$\frac{t}{s^2}\frac{\partial b_1}{\partial x}+\frac{t}{s^2}\frac{\partial b_2}{\partial y}-\frac{z}{s}\frac{\partial b_3}{\partial s}+\frac{z}{s^2}\frac{\partial e_1}{\partial y}-\frac{z}{s^2}\frac{\partial e_2}{\partial x}=0 \qquad (10)$$

$$\frac{1}{s}\frac{\partial e_1}{\partial s}=\frac{\partial b_3}{\partial y} \qquad \frac{1}{s}\frac{\partial b_1}{\partial s}=-\frac{\partial e_3}{\partial y} \qquad (11),(12)$$

$$\frac{1}{s}\frac{\partial e_2}{\partial s}=-\frac{\partial b_3}{\partial x} \qquad \frac{1}{s}\frac{\partial b_2}{\partial s}=\frac{\partial e_3}{\partial x} \qquad (13),(14)$$

$$\frac{z}{s^2}\frac{\partial e_1}{\partial x}+\frac{z}{s^2}\frac{\partial e_2}{\partial y}-\frac{t}{s}\frac{\partial e_3}{\partial s}-\frac{t}{s^2}\frac{\partial b_1}{\partial y}+\frac{t}{s^2}\frac{\partial b_2}{\partial x}=0 \qquad (15)$$

$$\frac{z}{s^2}\frac{\partial b_1}{\partial x}+\frac{z}{s^2}\frac{\partial b_2}{\partial y}-\frac{t}{s}\frac{\partial b_3}{\partial s}+\frac{t}{s^2}\frac{\partial e_1}{\partial y}-\frac{t}{s^2}\frac{\partial e_2}{\partial x}=0 \qquad (16)$$

We see that we can manipulate equations (9), (10), (15) and (16) so as to be left with equations only in terms of x, y and s as follows:

$$t*(9)-z*(15) \rightarrow \quad \frac{\partial e_1}{\partial x}+\frac{\partial e_2}{\partial y}=0 \qquad (17)$$

$$z*(9)-t*(15) \rightarrow \quad \frac{\partial b_1}{\partial y}-\frac{\partial b_2}{\partial x}=-s\frac{\partial e_3}{\partial s} \qquad (18)$$

$$t*(10)-z*(16) \rightarrow \quad \frac{\partial b_1}{\partial x}+\frac{\partial b_2}{\partial y}=0 \qquad (19)$$

$$z*(10)-t*(16) \rightarrow \quad \frac{\partial e_1}{\partial y}-\frac{\partial e_2}{\partial x}=s\frac{\partial b_3}{\partial s} \qquad (20)$$

Equations (11) through (14) and the four above equations make up a new system of equations that we can attempt to solve. Equation (19) implies that locally there is an $A = A(x, y, s)$ such that

$$b_1=\frac{\partial A}{\partial y} \quad \text{and} \quad b_2=-\frac{\partial A}{\partial x}.$$

Substituting these values into (18), we see that

$$\Delta A=-s\frac{\partial e_3}{\partial s}.$$

By (12),

$$\frac{\partial e_3}{\partial y}=-\frac{1}{s}\frac{\partial^2 A}{\partial y\partial s}=\frac{\partial}{\partial y}(-\frac{1}{s}\frac{\partial A}{\partial s}) \Rightarrow e_3=-\frac{1}{s}\frac{\partial A}{\partial s}+B(x,s).$$

$$\Rightarrow \frac{\partial e_3}{\partial x}=-\frac{1}{s}\frac{\partial^2 A}{\partial x\partial s}+\frac{\partial B}{\partial x}$$

From (14),

$$\frac{\partial e_3}{\partial x} = -\frac{1}{s}\frac{\partial^2 A}{\partial x \partial s}$$

so

$$\frac{\partial B}{\partial x} = 0 \Rightarrow B = B(s) \Rightarrow e_3 = -\frac{1}{s}\frac{\partial A}{\partial s} + B(s).$$

Hence,

$$\Delta A = -s\frac{\partial}{\partial s}\left(-\frac{1}{s}\frac{\partial A}{\partial s} + B(s)\right) = s\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial A}{\partial s} - s\frac{\partial}{\partial s}B(s).$$

We define A_0, A_1 by

$$(*) \quad A_0(s) \equiv \int^s sB(s)ds \quad \text{and} \qquad A \equiv A_0 + A_1.$$

From above,

$$\begin{aligned} s\frac{\partial B}{\partial s} &= s\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial A}{\partial s} - \Delta A = s\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial}{\partial s}(A_0 + A_1) - \Delta(A_0 + A_1) = s\frac{\partial B(s)}{\partial s} + s\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial A_1}{\partial s} - \Delta A_1 \\ &\Rightarrow \quad (**) \quad s\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial A_1}{\partial s} - \Delta A_1 = 0. \end{aligned}$$

A general solution may thus be obtained as follows: We let B be any function of s, then A_0 is determined by $(*)$. Further, we let A_1 be any solution of $(**)$, and define A$\equiv A_0 + A_1$. Then

$$e_3 = -\frac{1}{s}\frac{\partial A}{\partial s} + B(s), \quad b_1 = \frac{\partial A}{\partial y} \quad \text{and} \quad b_2 = -\frac{\partial A}{\partial x}.$$

By symmetry, there are functions C(x,y,s) and D(s) such that

$$b_3 = \frac{1}{s}\frac{\partial C}{\partial s} + D(s), \qquad e_1 = \frac{\partial C}{\partial y} \quad \text{and} \quad e_2 = -\frac{\partial C}{\partial x}.$$

We could attempt to solve $(**)$ for A_1 by using separation of variables, or we could perform further symmetry reduction in order to find solutions.

We note that there is no change of variables $p = p(s)$ that would bring

$$(**) \quad s\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial A_1}{\partial s} - \Delta A_1 = 0$$

into the wave equation. In face, suppose $p = p(s)$ is such a change of variables. We compute

$$\frac{\partial}{\partial p} = \frac{\partial s}{\partial p}\frac{\partial}{\partial s}$$

$$\Rightarrow \frac{\partial^2}{\partial p^2} = \frac{\partial}{\partial p}\left(\frac{\partial s}{\partial p}\frac{\partial}{\partial s}\right) = \frac{\partial^2 s}{\partial p^2}\frac{\partial}{\partial s} + \left(\frac{\partial s}{\partial p}\right)^2 \frac{\partial^2}{\partial s^2}.$$

Since $p(s)$ takes $(**)$ into the wave equation, we have that

$$(***) \quad s\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial}{\partial s} = \frac{\partial^2}{\partial p^2} \qquad \text{and}$$

$$s\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial}{\partial s} = \frac{\partial^2 s}{\partial p^2}\frac{\partial}{\partial s} + \left(\frac{\partial s}{\partial p}\right)^2 \frac{\partial^2}{\partial s^2},$$

from which

$$\left(\frac{\partial s}{\partial p}\right)^2 = 1.$$

Thus, it follows that

$$\frac{\partial^2 s}{\partial p^2} = 0.$$

However, from $(* * *)$ we see that

$$\frac{\partial^2 s}{\partial p^2} = s\frac{\partial}{\partial s}\frac{1}{s} = -\frac{1}{s} \Rightarrow -\frac{1}{s} = 0, \quad \text{which is a contradiction.}$$

4.2 Subgroup generated by u_0 and v_1

Under a two-dimensional subgroup, the number of independent variables from the original system is reduced by two. The invariants used in the subgroup generated by u_0 alone are also invariant under v_1. We use these same invariants in our calculations for this two-dimensional subgroup, simply eliminating y.

Excluding y from the independent variables forces all partial derivatives with respect to y to equal 0, while all other partial derivatives remain the same as shown previously. The resulting system of equations derived from substituting the partial derivatives into Maxwell's equations is simpler, and reduces to the following:

$$\frac{1}{s}\frac{\partial e_1}{\partial s} = 0 \qquad \frac{1}{s}\frac{\partial b_1}{\partial s} = 0 \qquad (21),(22)$$

$$\frac{\partial e_3}{\partial x} = \frac{1}{s}\frac{\partial b_2}{\partial s} \qquad \frac{\partial b_3}{\partial x} = -\frac{1}{s}\frac{\partial e_2}{\partial s} \qquad (23),(24)$$

$$\frac{t}{s^2}\frac{\partial e_1}{\partial x} + \frac{z}{s^2}\frac{\partial b_2}{\partial x} - \frac{z}{s}\frac{\partial e_3}{\partial s} = 0 \qquad (25)$$

$$\frac{t}{s}\frac{\partial e_3}{\partial s} - \frac{t}{s^2}\frac{\partial b_2}{\partial x} - \frac{z}{s^2}\frac{\partial e_1}{\partial x} = 0 \qquad (26)$$

$$-\frac{z}{s^2}\frac{\partial e_2}{\partial x} + \frac{t}{s^2}\frac{\partial b_1}{\partial x} - \frac{z}{s}\frac{\partial b_3}{\partial s} = 0 \qquad (27)$$

$$\frac{t}{s}\frac{\partial b_3}{\partial s} + \frac{t}{s^2}\frac{\partial e_2}{\partial x} - \frac{z}{s^2}\frac{\partial b_1}{\partial x} = 0 \qquad (28)$$

Equations (21)-(24) tell us that e_1 and b_1 are constants. From equations (25)-(28) it follows that

$$\begin{aligned}\frac{\partial e_3}{\partial x} &= \frac{1}{s}\frac{\partial b_2}{\partial s} \qquad \frac{\partial e_3}{\partial s} = \frac{1}{s}\frac{\partial b_2}{\partial x}\\ \frac{\partial b_3}{\partial x} &= -\frac{1}{s}\frac{\partial e_2}{\partial s} \qquad \frac{\partial b_3}{\partial s} = -\frac{1}{s}\frac{\partial e_2}{\partial x}.\end{aligned}$$

From the partial derivatives of e_3, we see that

$$\frac{\partial^2 e_3}{\partial x^2} = \frac{1}{s}\frac{\partial e_3}{\partial s} + \frac{\partial^2 e_3}{\partial s^2}.$$

We can solve this by separation of variables. Let $e_3(x,s) = f(x)g(s)$. Then,

$$\frac{1}{f}\frac{d^2 f}{dx^2} = \frac{1}{s}\frac{1}{g}\frac{dg}{ds} + \frac{1}{g}\frac{d^2 g}{ds^2} = -\lambda^2.$$

We can find explicit solutions to Maxwell's equations by solving

$$\frac{f''}{f} = -\lambda^2 \qquad s^2 g'' + sg' + \lambda^2 s^2 g = 0.$$

We note that $f(x)$ will be in the form of trigonometric functions, and that $g(s)$ satisfies Bessel's equation of order 0. Hence, for example, on the interval $-\pi \le x \le \pi$ the component e_3 can be represented by the infinite series

$$e_3(x,s) = \sum_{n=0}^{\infty}(A_n \cos nx + B_n \sin nx) J_0(ns).$$

4.3 Subgroup generated by u_0 and v_2

Considering the two-dimensional subgroup generated by u_0 and v_2, we choose our invariants as follows:

$$x, \qquad y$$

$$\begin{aligned} e_1 &= \frac{E^1 + B^2}{\sigma} & b_1 &= \frac{B^1 - E^2}{\sigma}\\ e_2 &= (E^2 + B^1)\sigma & b_2 &= (B^2 - E^1)\sigma\\ e_3 &= E^3 & b_3 &= B^3\end{aligned}$$

where $\sigma = t + z$. Then

$$\begin{aligned} E^1 &= \frac{\sigma e_1 - \frac{1}{\sigma} b_2}{2} & B^1 &= \frac{\sigma b_1 + \frac{1}{\sigma} e_2}{2}\\ E^2 &= \frac{-\sigma b_1 + \frac{1}{\sigma} e_2}{2} & B^2 &= \frac{\sigma e_1 + \frac{1}{\sigma} b_2}{2}\\ E^3 &= e_3 & B^3 &= b_3\end{aligned}$$

After calculating the partial derivatives and substituting them into Maxwell's equations, we have the following system:

$$\frac{\partial e_1}{\partial x} - \frac{\partial b_1}{\partial y} = 0 \qquad \frac{\partial e_1}{\partial y} + \frac{\partial b_1}{\partial x} = 0$$

$$\frac{\partial e_2}{\partial x} + \frac{\partial b_2}{\partial y} = 0 \qquad \frac{\partial e_2}{\partial y} - \frac{\partial b_2}{\partial x} = 0$$

$$e_1 = \frac{\partial e_3}{\partial x} = \frac{\partial b_3}{\partial y} \qquad b_1 = -\frac{\partial e_3}{\partial y} = \frac{\partial b_3}{\partial x}$$

Each pair e_α, b_α satisfies the Cauchy-Riemann equations, so $e_1 + ib_1$, $b_2 + ie_2$ and $e_3 + ib_3$ are complex analytic functions in $z = x + iy$. In the next section, we give examples of solutions obtained from these equations by first letting $e_3 + ib_3 = e^{nz}$, then letting $b_2 + ie_2 = \frac{1}{z}$.

We note the somewhat surprising fact that under symmetry reduction, the type of a system of partial differential equations can change. In fact, the above equations are elliptic as opposed to Maxwell's equations, which are hyperbolic.

4.3.1 Example Solution 1

Let $e_2 = 0$, $\quad b_2 = 0$, $\quad e_3 = e^{nx}\cos ny$, $\quad b_3 = e^{nx}\sin ny$. It follows that $e_1 = ne^{nx}\cos ny$ and $b_1 = ne^{nx}\sin ny$. Then

$$\begin{aligned} E^1 &= \frac{(t+z)ne^{nx}\cos ny}{2} & B^1 &= \frac{(t+z)ne^{nx}\sin ny}{2} \\ E^2 &= -\frac{(t+z)ne^{nx}\sin ny}{2} & B^2 &= \frac{(t+z)ne^{nx}\cos ny}{2} \\ E^3 &= e^{nx}\cos ny & B^3 &= e^{nx}\sin ny \end{aligned}$$

is a solution for Maxwell's equations that is invariant under the subgroup generated by u_0 and v_2.

We can obtain further invariant solutions by forming the infinite series

$$E^3 = \sum_{n=0}^{\infty} a_n e^{nx}\cos\ ny$$

where a_n satisfy a suitable convergence condition. It follows that

$$\begin{aligned} E^1 &= \sum_{n=0}^{\infty} a_n \frac{(t+z)n}{2} e^{nx}\cos\ ny & B^1 &= \sum_{n=1}^{\infty} a_n \frac{(t+z)n}{2} e^{nx}\sin\ ny \\ E^2 &= -\sum_{n=1}^{\infty} a_n \frac{(t+z)n}{2} e^{nx}\sin\ ny & B^2 &= \sum_{n=0}^{\infty} a_n \frac{(t+z)n}{2} e^{nx}\cos\ ny \end{aligned}$$

$$B^3 = \sum_{n=1}^{\infty} a_n e^{nx} \sin ny.$$

Imposing a boundary condition on E^3 will determine a_n for each series. We can use these solutions to solve the Goursat problem with boundary data on $t+z=0$, in which case the solution reduces to

$$\begin{aligned} E^1 &= 0 & B^1 &= 0 \\ E^2 &= 0 & B^2 &= 0 \\ E^3 &= \sum_{n=0}^{\infty} a_n e^{nx} \cos ny & B^3 &= \sum_{n=1}^{\infty} a_n e^{nx} \sin ny. \end{aligned}$$

4.3.2 Example Solution 2

$$\text{Let} \qquad e_2 = -\frac{y}{x^2+y^2}, \quad b_2 = \frac{x}{x^2+y^2}, \qquad e_3 = 0, \quad b_3 = 0.$$

It follows that $e_1 = 0$ and $b_1 = 0$. Then

$$\begin{aligned} E^1 &= \frac{x}{2(t+z)(x^2+y^2)} & B^1 &= \frac{y}{2(t+z)(x^2+y^2)} \\ E^2 &= \frac{y}{2(t+z)(x^2+y^2)} & B^2 &= -\frac{x}{2(t+z)(x^2+y^2)} \\ E^3 &= 0 & B^3 &= 0 \end{aligned}$$

is another solution for Maxwell's equations that is invariant under this subgroup. There is a singularity along the z-axis, where $x = y = 0$, so we can determine whether or not this electromagnetic field arises from a line source on the z-axis.

Recall that Gauss's Law says:

$$\int_S \vec{E} \cdot \vec{n}\, ds = Q_{enc}$$

By this, we can see that the charge density, $\rho(t+z)$, of a wire along the z-axis is $\rho = \frac{\pi}{t+z}$. Further, by Ampére's Law,

$$\oint \vec{B} \cdot d\vec{s} = \frac{\partial \Phi_E}{\partial t} + i,$$

we can determine the current along the wire, $i(t+z)$, to be $i = \frac{-\pi}{t+z}$. For a physical interpretation of this solution, we assume $z \geq 0$ in order to avoid the singularity at $z = -t$. Otherwise, we can say that at z=-t, there is an infinite charge travelling at the speed of light in the negative z direction on the wire.

Furthermore, this solution lends itself to the Goursat problem. We see that on the hyperplane $t = z$ the solution reduces to

$$\begin{aligned} E^1 &= \frac{x}{4z(x^2+y^2)} & B^1 &= \frac{y}{4z(x^2+y^2)} \\ E^2 &= \frac{y}{4z(x^2+y^2)} & B^2 &= -\frac{x}{4z(x^2+y^2)} \\ E^3 &= 0 & B^3 &= 0 \end{aligned}$$

4.4 Subgroup generated by u_0, v_1, and v_2

In a three-dimensional subgroup, there will only be one independent variable. The invariants used for the subgroup generated by u_0 and v_1 are also invariant under v_2. We use these same invariants in our calculations for the three-dimensional subgroup generated by u_0, v_1, and v_2, but eliminating y. After the necessary computations, we find that the following hold:

$$e_1 = b_1 = 0$$

$$e_2, b_2, e_3, b_3 \quad \text{are constants}$$

So

$$E^1 = \frac{-b_2}{2(t+z)} \qquad E^2 = \frac{e_2}{2(t+z)} \qquad E^3 = e_3$$

$$B^1 = \frac{e_2}{2(t+z)} \qquad B^2 = \frac{b_2}{2(t+z)} \qquad B^3 = b_3$$

is an explicit solution to Maxwell's equations that is invariant under the subgroup generated by u_0, v_1, and v_2. For further applications, we can choose specific values for e_2, e_3, b_2 and b_3 and interpret the results.

4.5 Subgroup generated by u_0, v_1, and v_3

The invariants used for the subgroup generated by u_0 and v_1 are also invariant under v_3, so keeping only $s = \sqrt{t^2 - z^2}$ from the independent variables, the rest of our invariants are again

$$\begin{aligned} e_1 &= tE^1 - zB^2 & b_1 &= tB^1 + zE^2 \\ e_2 &= zB^1 + tE^2 & b_2 &= -zE^1 + tB^2 \\ e_3 &= E^3 & b_3 &= B^3 \end{aligned} .$$

Carrying out the symmetry reduction process reveals that

$$\begin{aligned} E^1 &= \frac{te_1 + zb_2}{t^2 - z^2} & B^1 &= \frac{tb_1 - ze_2}{t^2 - z^2} \\ E^2 &= \frac{te_2 - zb_1}{t^2 - z^2} & B^2 &= \frac{tb_2 + ze_1}{t^2 - z^2} \\ E^3 &= e_3 & B^3 &= b_3 \end{aligned}$$

where all e_α and b_α are constants. As in the previous subgroup, we can choose specific values for the functions e_α and b_α and interpret out results for further applications.

4.6 Subgroup generated by u_1

We now construct a one-dimensional subgroup using the following infinitesimal generator:

$$u_1 = 2(y\partial x - x\partial y) - \frac{1}{2}(\partial t + \partial z) + 2(E^2\partial E^1 - E^1\partial E^2) + 2(B^2\partial B^1 - B^1\partial B^2)$$

u_1 generates a subgroup of the Poincaré group representing a combination of rotation about the z-axis, spatial translation (along the z axis), and time translation.

By the algorithm, we expect that a one-dimensional subgroup will eliminate one independent variable from the original system. We choose our invariants as follows:

$$\eta = t - z,\ \ \rho = \sqrt{x^2 + y^2},\ \ \theta = -2(t + z) + \arctan\frac{y}{x}$$

$$\begin{aligned} e_1 &= xE^1 + yE^2 & b_1 &= xB^1 + yB^2 \\ e_2 &= yE^1 - xE^2 & b_2 &= yB^1 - xB^2 \\ e_3 &= E^3 & b_3 &= B^3 \end{aligned}$$

Solving for E^i and B^i using Cramer's Rule, we get

$$\begin{aligned} E^1 &= \frac{xe_1 + ye_2}{x^2 + y^2} & B^1 &= \frac{xb_1 + yb_2}{x^2 + y^2} \\ E^2 &= \frac{ye_1 - xe_2}{x^2 + y^2} & B^2 &= \frac{yb_1 - xb_2}{x^2 + y^2} \\ E^3 &= e_3 & B^3 &= b_3 \end{aligned}$$

Assuming that e_i, b_i are new variables dependent on η, ρ, and θ, we now compute the partial derivatives necessary to substitute back into the original system:

$$
\begin{aligned}
\frac{\partial E^1}{\partial x} &= (\frac{1}{\rho^2} - \frac{2x^2}{\rho^4})e_1 + \frac{x^2}{\rho^3}\partial_\rho e_1 - \frac{xy}{\rho^4}\partial_\theta e_1 - \frac{2xy}{\rho^4}e_2 + \frac{xy}{\rho^3}\partial_\rho e_2 - \frac{y^2}{\rho^4}\partial_\theta e_2 \\
\frac{\partial E^1}{\partial y} &= -\frac{2xy}{\rho^4}e_1 + \frac{xy}{\rho^3}\partial_\rho e_1 + \frac{x^2}{\rho^4}\partial_\theta e_1 + (\frac{1}{\rho^2} - \frac{2y^2}{\rho^4})e_2 + \frac{y^2}{\rho^3}\partial_\rho e_2 + \frac{xy}{\rho^4}\partial_\theta e_2 \\
\frac{\partial E^1}{\partial z} &= -\frac{x}{\rho^2}\partial_\eta e_1 - \frac{2x}{\rho^2}\partial_\theta e_1 - \frac{y}{\rho^2}\partial_\eta e_2 - \frac{2y}{\rho^2}\partial_\theta e_2 \\
\frac{\partial E^1}{\partial t} &= \frac{x}{\rho^2}\partial_\eta e_1 - \frac{2x}{\rho^2}\partial_\theta e_1 + \frac{y}{\rho^2}\partial_\eta e_2 - \frac{2y}{\rho^2}\partial_\theta e_2 \\
\frac{\partial E^2}{\partial x} &= -\frac{2xy}{\rho^4}e_1 + \frac{xy}{\rho^3}\partial_\rho e_1 - \frac{y^2}{\rho^4}\partial_\theta e_1 + (-\frac{1}{\rho^2} + \frac{2x^2}{\rho^4})e_2 - \frac{x^2}{\rho^3}\partial_\rho e_2 + \frac{xy}{\rho^4}\partial_\theta e_2 \\
\frac{\partial E^2}{\partial y} &= (\frac{1}{\rho^2} - \frac{2y^2}{\rho^4})e_1 + \frac{y^2}{\rho^3}\partial_\rho e_1 + \frac{xy}{\rho^4}\partial_\theta e_1 + \frac{2xy}{\rho^4}e_2 - \frac{xy}{\rho^3}\partial_\rho e_2 - \frac{x^2}{\rho^4}\partial_\theta e_2 \\
\frac{\partial E^2}{\partial z} &= -\frac{y}{\rho^2}\partial_\eta e_1 - \frac{2y}{\rho^2}\partial_\theta e_1 + \frac{x}{\rho^2}\partial_\eta e_2 + \frac{2x}{\rho^2}\partial_\theta e_2 \\
\frac{\partial E^2}{\partial t} &= \frac{y}{\rho^2}\partial_\eta e_1 - \frac{2y}{\rho^2}\partial_\theta e_1 - \frac{x}{\rho^2}\partial_\eta e_2 + \frac{2x}{\rho^2}\partial_\theta e_2 \\
\frac{\partial E^3}{\partial x} &= \left(\frac{x}{\rho}\right)\partial_\rho e_3 - \frac{y}{\rho^2}\partial_\theta e_3 \\
\frac{\partial E^3}{\partial y} &= \left(\frac{y}{\rho}\right)\partial_\rho e_3 + \frac{x}{\rho^2}\partial_\theta e_3 \\
\frac{\partial E^3}{\partial z} &= -\partial_\eta e_3 - 2\partial_\theta e_3 \\
\frac{\partial E^3}{\partial t} &= \partial_\eta e_3 - 2\partial_\theta e_3
\end{aligned}
$$

We compute the partial derivatives for $\vec{B}$ in an identical manner.

We now substitute these partial derivatives back into Maxwell's Equations. We observe that our equations are greatly simplified by both cancellation and summation of terms. Since v_1 generates a symmetry group of Maxwell's Equations, we expect that at the end of our simplifications, the original independent variables x, y, z, t will be completely eliminated, leaving a system entirely dependent on the new independent variables η, ρ, θ.

We start with the divergence equations.

$$
\begin{aligned}
0 = \mathrm{div}\vec{E} &= \frac{\partial E^1}{\partial x} + \frac{\partial E^2}{\partial y} + \frac{\partial E^3}{\partial z} \\
&= (\tfrac{1}{\rho^2} - \tfrac{2x^2}{\rho^4})e_1 + \left(\tfrac{x^2}{\rho^3}\right)\partial_\rho e_1 - \left(\tfrac{xy}{\rho^4}\right)\partial_\theta e_1 - \left(\tfrac{2xy}{\rho^4}\right)e_2 + \left(\tfrac{xy}{\rho^3}\right)\partial_\rho e_2 - \left(\tfrac{y^2}{\rho^4}\right)\partial_\theta e_2 \\
&\quad + (\tfrac{1}{\rho^2} - \tfrac{2y^2}{\rho^4})e_1 + \left(\tfrac{y^2}{\rho^3}\right)\partial_\rho e_1 + \left(\tfrac{xy}{\rho^4}\right)\partial_\theta e_1 + \left(\tfrac{2xy}{\rho^4}\right)e_2 - \left(\tfrac{xy}{\rho^3}\right)\partial_\rho e_2 - \left(\tfrac{x^2}{\rho^4}\right)\partial_\theta e_2
\end{aligned}
$$

$-\partial_\eta e_3 - 2\partial_\theta e_3,$

which simplifies to the equation

$$\frac{1}{\rho}\partial_\rho e_1 - \frac{1}{\rho^2}\partial_\theta e_2 - \partial_\eta e_3 - 2\partial_\theta e_3 = 0. \tag{29}$$

Likewise, from the divergence equation for magnetic fields, we get

$$\frac{1}{\rho}\partial_\rho b_1 - \frac{1}{\rho^2}\partial_\theta b_2 - \partial_\eta b_3 - 2\partial_\theta b_3 = 0. \tag{30}$$

We move on to the curl equations:

$$\begin{aligned}
&\frac{\partial E^3}{\partial t} = \frac{\partial B^2}{\partial x} - \frac{\partial B^1}{\partial y} = \partial_\eta e_3 - 2\partial_\theta e_3 \\
&= -\left(\tfrac{2xy}{\rho^4}\right) b_1 + \left(\tfrac{xy}{\rho^3}\right)\partial_\rho b_1 - \left(\tfrac{y^2}{\rho^4}\right)\partial_\theta b_1 + \left(-\tfrac{1}{\rho^2} + \tfrac{2x^2}{\rho^4}\right) b_2 - \left(\tfrac{x^2}{\rho^3}\right)\partial_\rho b_2 + \left(\tfrac{xy}{\rho^4}\right)\partial_\theta b_2 \\
&\quad + \left(\tfrac{2xy}{\rho^4}\right) b_1 - \left(\tfrac{xy}{\rho^3}\right)\partial_\rho b_1 - \left(\tfrac{x^2}{\rho^4}\right)\partial_\theta b_1 - \left(\tfrac{1}{\rho^2} - \tfrac{2y^2}{\rho^4}\right) b_2 - \left(\tfrac{y^2}{\rho^3}\right)\partial_\rho b_2 - \left(\tfrac{xy}{\rho^4}\right)\partial_\theta b_2 \\
&= -\left(\tfrac{x^2}{\rho^4}\right)\partial_\theta b_1 - \left(\tfrac{y^2}{\rho^4}\right)\partial_\theta b_1 - \left(\tfrac{x^2}{\rho^3}\right)\partial_\rho b_2 - \left(\tfrac{y^2}{\rho^3}\right)\partial_\rho b_2
\end{aligned}$$

which simplifies to the equation

$$\partial_\eta e_3 - 2\partial_\theta e_3 = -\frac{1}{\rho^2}\partial_\theta b_1 - \frac{1}{\rho}\partial_\rho b_2. \tag{31}$$

Similarly, we obtain $\partial_\eta b_3 - 2\partial_\theta b_3 = \dfrac{1}{\rho^2}\partial_\theta e_1 + \dfrac{1}{\rho}\partial_\rho e_2.$ (32)

We find that we must manipulate the remaining curl equations in order to cancel out the original independent variables: we take pairs of curl equations, multiply one by x and one by y, and then take the sum or difference of the resulting equations. In this manner we successfully eliminate all x, y, z, t.

We first multiply equation (3) by x and equation (5) by y and take the sum of the resulting equations. After substitution, cancellation, and combination of terms, we obtain

$$\partial_\eta e_1 - 2\partial_\theta e_1 = -\partial_\eta b_2 - 2\partial_\theta b_2 + \partial_\theta b_3. \tag{33}$$

Similarly, we multiply equation (4) by x and equation (6) by y, and take the sum of the resulting equations. We obtain

$$\partial_\eta b_1 - 2\partial_\theta b_1 = \partial_\eta e_2 + 2\partial_\theta e_2 - \partial_\theta e_3. \tag{34}$$

Next, we add equations (3) and (6), giving

$$\left(\frac{2x}{\rho^2}\right)\partial_\eta e_1 + \left(\frac{2y}{\rho^2}\right)\partial_\eta e_2 + \left(\frac{x}{\rho}\right)\partial_\rho e_3 - \left(\frac{y}{\rho^2}\right)\partial_\theta e_3$$

$$= \left(\frac{2y}{\rho^2}\right)\partial_\eta b_1 - \left(\frac{2x}{\rho^2}\right)\partial_\eta b_2 + \left(\frac{y}{\rho}\right)\partial_\rho b_3 + \left(\frac{x}{\rho^2}\right)\partial_\theta b_3, \qquad (35)$$

and subtract (5) from (4), giving

$$-\left(\frac{2y}{\rho^2}\right)\partial_\eta e_1 + \left(\frac{2x}{\rho^2}\right)\partial_\eta e_2 - \left(\frac{y}{\rho}\right)\partial_\rho e_3 - \left(\frac{x}{\rho^2}\right)\partial_\theta e_3$$

$$= \left(\frac{2x}{\rho^2}\right)\partial_\eta b_1 + \left(\frac{2y}{\rho^2}\right)\partial_\eta b_2 + \left(\frac{x}{\rho}\right)\partial_\rho b_3 - \left(\frac{y}{\rho^2}\right)\partial_\theta b_3. \qquad (36)$$

We multiply (35) by y and (36) by x, and add the resulting equations, giving

$$2\partial_\eta e_2 - \partial_\theta e_3 = 2\partial_\eta b_1 + \rho\partial_\rho b_3, \qquad (37)$$

and multiply (35) by x and (36) by y, and take the difference of the resulting equations, giving

$$2\partial_\eta b_2 - \partial_\theta b_3 = -2\partial_\eta e_1 - \rho\partial_\rho e_3. \qquad (38)$$

We have derived a reduced system consisting of eight equations: (29), (30), (31), (32), (33), (34), (37) and (38). Unfortunately the system has resisted integration, so we were unable to derive any explicit solutions in this case. In our next example, we increase the degree of symmetry by one, thereby making it easier to construct solutions to the system.

4.7 Subgroup generated by u_1 and v_2

We construct a two-dimensional subgroup using the generators u_1 and v_2. We expect that a two-dimensional subgroup will eliminate two independent variables from our system. We choose our invariants as follows, simply eliminating the η invariant from our work on the previous subgroup.

$$\rho = \sqrt{x^2+y^2}, \theta = -2(t+z) + \arctan\frac{y}{x}$$

$$\begin{aligned} e_1 &= xE^1 + yE^2 & b_1 &= xB^1 + yB^2 \\ e_2 &= yE^1 - xE^2 & b_2 &= yB^1 - xB^2 \\ e_3 &= E^3 & b_3 &= B^3 \end{aligned}$$

It follows that our solutions for E^i and B^i are identical to those found for the previous subgroup. We can obtain our reduced system by simply setting all η-derivatives from the reduced system above equal to zero.

The divergence equations yield (from equations (29) and (30) above):

$$\frac{1}{\rho}\partial_\rho e_1 - \frac{1}{\rho^2}\partial_\theta e_2 - 2\partial_\theta e_3 = 0 \qquad \frac{1}{\rho}\partial_\rho b_1 - \frac{1}{\rho^2}\partial_\theta b_2 - 2\partial_\theta b_3 = 0 \tag{39}$$

The curl equations yield (from equations (31), (32), (33), (34), (37), and (38) above):

$$2\partial_\theta e_3 = \frac{1}{\rho^2}\partial_\theta b_1 + \frac{1}{\rho}\partial_\rho b_2 \qquad 2\partial_\theta b_3 = -\frac{1}{\rho^2}\partial_\theta e_1 - \frac{1}{\rho}\partial_\rho e_2 \tag{40}$$

$$2\partial_\theta e_1 = 2\partial_\theta b_2 - \partial_\theta b_3 \qquad 2\partial_\theta b_1 = -2\partial_\theta e_2 + \partial_\theta e_3 \tag{41}$$

$$\partial_\theta e_3 = -\rho\partial_\rho b_3 \qquad \partial_\theta b_3 = \rho\partial_\rho e_3 \tag{42}$$

It is now our task to solve this system of eight equations. We do so by recognizing that several of our reduced equations are Cauchy-Riemann equations, and thus can be solved by choosing some of our variables to be the components of complex analytic functions. The following are several sample solutions. Here we choose $f(z) = e^z$ to be our analytic function. We omit the details of finding these solutions. Assume l, k to be constant.

$$\begin{aligned} E^1 &= \frac{1}{2}\left(\frac{kx+ly}{x^2+y^2}\right) & B^1 &= \frac{1}{2}\left(\frac{lx-ky}{x^2+y^2}\right) \\ E^2 &= \frac{1}{2}\left(\frac{ky-lx}{x^2+y^2}\right) & B^2 &= \frac{1}{2}\left(\frac{ly+kx}{x^2+y^2}\right) \\ E^3 &= 0 & B^3 &= 0 \end{aligned}$$

$$\begin{aligned} E^1 &= \left(\frac{\rho}{2x^2+2y^2}\right)(x\sin\theta - y\cos\theta) & B^1 &= \left(\frac{\rho}{2x^2+2y^2}\right)(x\cos\theta + y\sin\theta) \\ E^2 &= \left(\frac{\rho}{2x^2+2y^2}\right)(y\sin\theta + x\cos\theta) & B^2 &= \left(\frac{\rho}{2x^2+2y^2}\right)(y\cos\theta - x\sin\theta) \\ E^3 &= 0 & B^3 &= 0 \end{aligned}$$

$$
\begin{aligned}
E^1 &= \frac{x(k+\rho\sin\theta)+y(l-\rho\cos\theta)}{2x^2+2y^2} & B^1 &= \frac{y(-k+\rho\sin\theta)+x(l+\rho\cos\theta)}{2x^2+2y^2} \\
E^2 &= \frac{y(k+\rho\sin\theta)-x(l-\rho\cos\theta)}{2x^2+2y^2} & B^2 &= \frac{-x(-k+\rho\sin\theta)+y(l+\rho\cos\theta)}{2x^2+2y^2} \\
E^3 &= 0 & B^3 &= 0
\end{aligned}
$$

Note that all of these solutions have singularities at $x^2 + y^2 = 0$, which suggests that the electromagnetic field might be created by a charge at the origin or along the z-axis.

4.8 Subgroup generated by u_1, v_1 and v_3

We construct a three-dimensional subgroup using the generators u_1, v_1 and v_3. By the algorithm, we expect that a three dimensional subgroup will eliminate three variables from our system of equations, thus resulting in a system of ordinary differential equations.

We choose our invariants as follows:

$$\eta = t - z$$

$$
\begin{aligned}
\rho &= \sqrt{(E^1)^2+(E^2)^2} & \xi &= \sqrt{(B^1)^2+(B^2)^2} \\
\theta &= -2(t+z)+\arctan(\frac{E^2}{E^1}) & \phi &= -2(t+z)+\arctan(\frac{B^2}{B^1}) \\
e_3 &= E^3 & b_3 &= B^3
\end{aligned}
$$

Solving for our original dependent variables, we obtain:

$$
\begin{aligned}
E^1 &= \rho\cos(\theta+2(t+z)) & B^1 &= \xi\cos(\varphi+2(t+z)) \\
E^2 &= \rho\sin(\theta+2(t+z)) & B^2 &= \xi\cos(\varphi+2(t+z)) \\
E^3 &= e_3 & B^3 &= b_3
\end{aligned}
$$

Assuming that $\rho, \theta, \xi, \varphi$ are new variables dependent on η, we compute the partial derivatives and substitute them into Maxwell's Equations.

The divergence equations yield $\dfrac{\partial e_3}{\partial \eta} = 0$ and $\dfrac{\partial b_3}{\partial \eta} = 0.$

After some algebraic manipulation, the curl equations yield

$$\rho\cos\alpha = -\xi\sin\beta \qquad \rho\sin\alpha = \xi\cos\beta \tag{43}$$

and

$$0 = \cos\alpha\frac{\partial\rho}{\partial\eta} - \rho\sin\alpha\frac{\partial\theta}{\partial\eta} - \sin\beta\frac{\partial\xi}{\partial\eta} - \xi\cos\beta\frac{\partial\varphi}{\partial\eta} \tag{44}$$

$$0 = \sin\alpha\frac{\partial\rho}{\partial\eta} + \rho\cos\alpha\frac{\partial\theta}{\partial\eta} + \cos\beta\frac{\partial\xi}{\partial\eta} - \xi\sin\beta\frac{\partial\varphi}{\partial\eta} \tag{45}$$

$$0 = \sin\alpha\frac{\partial\rho}{\partial\eta} + \rho\cos\alpha\frac{\partial\theta}{\partial\eta} + \cos\beta\frac{\partial\xi}{\partial\eta} - \xi\sin\beta\frac{\partial\varphi}{\partial\eta} \tag{46}$$

$$0 = -\cos\alpha\frac{\partial\rho}{\partial\eta} + \rho\sin\alpha\frac{\partial\theta}{\partial\eta} + \sin\beta\frac{\partial\xi}{\partial\eta} + \xi\cos\beta\frac{\partial\varphi}{\partial\eta} \tag{47}$$

where $\alpha = \theta + 2(t+z)$, $\beta = \varphi + 2(t+z)$.

Squaring (43) and summing the results gives $\rho = \pm\xi$, so we get

$$\cos\alpha = -\sin\beta \tag{48}$$

$$\sin\alpha = \cos\beta \tag{49}$$

from which we get

$$\alpha = \beta + \tfrac{\pi}{2} \qquad \text{and} \qquad \theta = \varphi + \tfrac{\pi}{2}.$$

From (44) through (49), we get $\begin{array}{r} \sin\alpha\partial\rho + \rho\cos\alpha\partial\theta = 0 \\ -\cos\alpha\partial\rho + \rho\sin\alpha\partial\theta = 0 \end{array} \implies \partial\rho = \partial\theta = 0$

Thus, we obtain the following solutions, with ρ, θ constant:

$$\begin{array}{rclrcl} E^1 &=& \rho\cos(\theta + 2(t+z)) & B^1 &=& \rho\sin(\theta + 2(t+z)) \\ E^2 &=& \rho\sin(\theta + 2(t+z)) & B^2 &=& -\rho\cos(\theta + 2(t+z)) \\ E^3 &=& \text{constant} & B^3 &=& \text{constant} \end{array}$$

We observe that this solution describes a circularly polarized plane wave propagated in the negative z direction. This solution is invariant under the symmetries described by the generators of the subgroup: rotation about the z-axis, spatial translation along x, y, z, and time translation.

5 Conclusion

We have performed symmetry reduction on Maxwell's equations under various Poincaré subgroups of dimensions one, two, and three. We have constructed several physically relevant and interesting solutions, including potential applications to the Goursat problem. We also observe that the type of a system of partial differential equations can change under symmetry reduction.

It can be noted that after performing symmetry reduction, the resulting reduced equations might have new symmetries not present in the original system. In this case, it might be instructive to perform further symmetry reduction, especially if the reduced system cannot be solved using standard methods. Such multiple applications of the symmetry reduction process has not been well studied up to the present, and might produce interesting results.

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