Tomography of Non-convex Polygons from a Single Point X-ray

Mike Lauzon
Harvey Mudd College

Ben Schmidt
Emory University

Advisor
Donald Solmon
Oregon State University

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1 Introduction

When the x-ray of an object is taken, an x-ray of known intensity is emitted from an x-ray source and then the intensity is measured at a sensor. The decrease in intensity is related to how much "stuff" is between the source and the sensor. Mathematically, "how much stuff" is along a line is the line integral of a function describing where "stuff" is along that line.

Tomography is the study of what one can tell about a function just by knowing the line integrals of that function. Fan beam tomography is tomography where integrals along the rays emanating from a point are known. Lam and Solmon [3] have reconstructed convex polygons away from the source of the x-ray. We seek to expand this technique to nonconvex polygons.

1.1 Some simplifying assumptions

Throughout the paper there are several assumptions that we will make to simplify our problem. First we will always assume that the x-ray source is at the origin. This allows us to define an x-ray of a function $f(r, \theta)$ by

$$X[f](\theta) = \int_{r=0}^{\infty} f(r, \theta) \mathrm{d}\theta,$$

where (r, θ) are polar coordinates in the plane.

We also note that we may rotate coordinates however we like about the fixed origin. A proof may be found in [3]. If we can determine the shape of a polygon in a rotated set of coordinates, rotating coordinates back gives a solution in the original coordinates. That is with α being an arbitrary angle of rotation, let

$$f_\alpha(r, \theta) = f(r, \theta + \alpha),$$

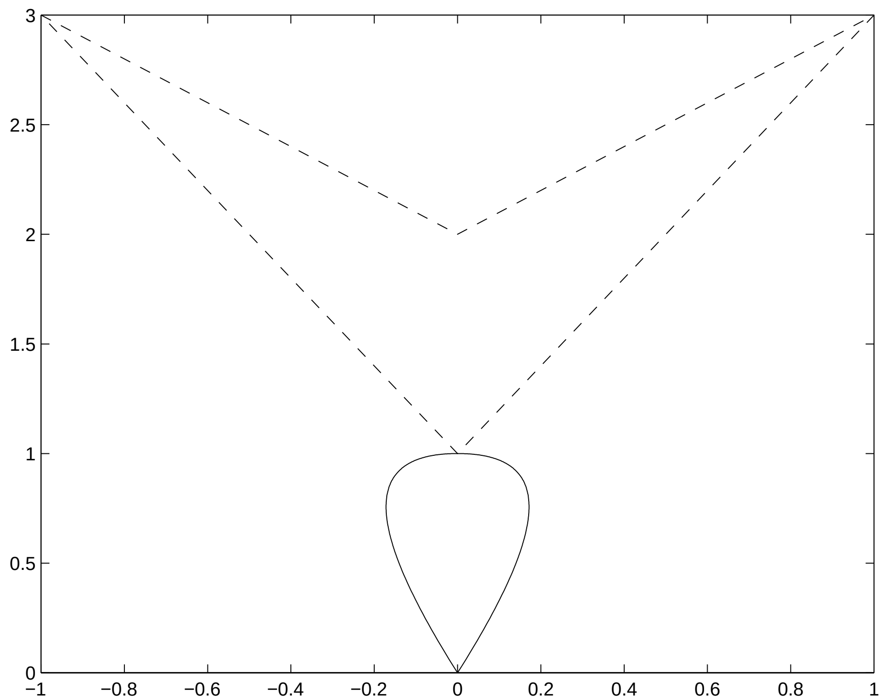

Figure 1: A sample polygon and its X-ray.

and note that

$$X[f](\theta + \alpha) = X[f_\alpha](\theta).$$

1.2 Definition of the x-ray of a polygon

We should also carefully define what we mean by "the x-ray of a polygon." We will assume that all polygons are compact subsets of the plane with nonempty interior. Place a polygon P anywhere in the plane away form the origin. Then consider the characteristic function of the polygon, so

$$P(r,\theta) = \begin{matrix} 1 & (r,\theta) \in P \\ 0 & (r,\theta) \notin P. \end{matrix} \tag{1}$$

We will also assume that no polygon will have the origin in its interior. In other words,

$$P(0,\theta) = 0 \tag{2}$$

for all polygons considered.

Now we define "the x-ray of P" to be

$$X[P](\theta) = \int_{r=0}^{\infty} P(r,\theta)\mathrm{d}\theta \tag{3}$$

When it is clear what polygon we are talking about we will surpress the P and write only $X(\theta)$. More conceptually, the $X[P](\theta)$ can be thought of as the length of the intersection of a ray emanating from the origin at angle θ with the polygon. See Figure 1 for an example of a polygon and its x-ray.

2 The Problem of Reconstruction

The problem of reconstructing polygons can be divided into two subproblems: Finding at what angles the corners of a polygon lie and reconstructing the pieces of a polygon once cones containing no corners have been found.

2.1 The problem of finding corners

We define corners and cones.

Definition 1. *A point $\boldsymbol{x}$ is called a corner of a polygon $\boldsymbol{P}$ if $\boldsymbol{x} \in \partial\boldsymbol{P}$ and the tangent line to $\boldsymbol{P}$ is not well defined at $\boldsymbol{x}$.*

Definition 2. *A **cone** is the part of the plane between two rays emanating from the origin.*

Often we will use the term cone to refer to a piece of the plane between consecutive corners of a polygon. Our first goal is to take an x-ray and seperate the plane into cones at angles $\phi_1, \phi_2 \ldots$, where corners of a the x-rayed polygon lay along the rays at angles ϕ_i.

2.2 The problem of reconstructing wedges

After a polygon has been divided into cones which have no corners in them, we would like to demonstrate uniqueness, and also a reconstruction, from the x-ray of the polygon. We now define a wedge.

Definition 3. *A* ***wedge*** *is a closed, connected component of a polygon both containing no corners and lying inside a cone. Two wedges in the same cone may not intersect in more than one point.*

Definition 4. *The defining edges of a wedge are the two edges that form the boundry of a wedge but are not subsets of the rays defining the cone the wedge is inside.*

We also define a special kind of wedge, a parallel wedge, and show that the location of a parallel wedge cannot be determined from its x-ray.

Definition 5. *A* ***parallel wedge*** *is a wedge with parallel defining edges.*

Theorem 2.1. *Two parallel wedges in the same cone cannot be distinguished by a single point x-ray if the slopes of the defining edges are equal and the distances between the lines that the defining edges are subsets of is equal.*

Proof. Consider the x-ray of two such parallel wedges. Each will have the same x-ray. The graph will be a line segment inside the cone such that the line segment will have the same slope, m as the defining edges of the wedges, and the line it is a subset of will be the distance d from the origin. Since the two wedges have the same x-ray, there is no way to distinguish them using x-ray data. □

The same problem can occur if there are multiple wedges in the same cone and two of the wedges have parallel defining edges. Take one of the wedges bounded by one of the parallel edges and slice a small parallel wedge off the side containing one of the parallel edges. This parallel wedge can be attatched to the second parallel edge to create a second polygon with the same x-ray. Therefore, there is no way to distinguish between the two polygons using the x-ray data.

Definition 6. *A non-parallel n-wedge cone is a cone containing n wedges with no two of the defining edges of the wedges parallel.*

In [3] uniqueness of a non-parallel wedge from x-rays in four different directions is shown. We generalize the result to show that for a non-parallel n-wedge cone, $4n$ x-rays will uniquely determine the wedges in the cone. The proof given here is simpler than the proof of uniqueness in [3].

3 The Problem of Finding Corners

The first step in determining polygons is locating the corners of the polygon. For the special case of convex polygons, write $X(\theta) = R(\theta) \Leftrightarrow r(\theta)$, where $R(\theta)$

parameterizes the sides of the polygon further from the origin and $r(\theta)$ parameterizes the sides of the polygon nearer to the origin. In [3], it is shown that corners of convex polygons will always appear as discontinuities in the first derivative of directed X-ray data. The general idea is that $R(\theta)$ is convex, $r(\theta)$ is concave, but since $\Leftrightarrow r(\theta)$ is considered when calculating the x-ray, discontinuities in R' and r' cannot cancel each other out. This means that when there is a corner in R or r, $X'(\theta)$ is discontinuous.

3.1 The More Complicated Star-Shaped Case

The above derivation does not work for star-shaped polygons since $R(\theta)$ may not be convex and $r(\theta)$ may not be concave. We would still like a local condition for finding corners. One's first instinct is to look at a corner, then take right and left of derivatives at the corner until one finds a discrepency between the two. This method is unnecessarily complicated. An easier question to answer is,"What are local conditions for determining the two lines defining a wedge?" Once we know the answer to this question, we will be able to tell if all left hand neighborhoods and right hand neighborhoods of a point belong to the same wedge. If we find a point where the left and right neighborhoods of the point belong to different wedges, we have found a corner. Now we state the conditions that locally determine the top and bottom edges of a non-parallel wedge from local x-ray data.

Lemma 3.1. *Three derivatives of x-ray data of a wedge W at a point uniquely determine the edges that bound W near and away form the origin.*

Proof. First we choose coordinates such that a ray coming from the origin at angle $\frac{\pi}{2}$ intersects the wedge W. Let $R(\theta)$ and $r(\theta)$ parameterize the inner and outer edge, respectively.

Let the top edge have slope B and y-intercept A and the bottom edge have slope b and y-intercept a in rectangular coordinates, and parameterize $R(\theta)$ and $r(\theta)$ as follows:

$$R(\theta) = \frac{A}{\sin(\theta) + B\cos(\theta)} \tag{4}$$

$$r(\theta) = \frac{a}{\sin(\theta) + b\cos(\theta)}. \tag{5}$$

By setting $X = R \Leftrightarrow r$, we can calculate the general form of the first three derivatives of the directed x-ray.

$$X(\theta) = \frac{A}{\sin(\theta) + B\cos(\theta)} \Leftrightarrow \frac{a}{\sin(\theta) + b\cos(\theta)} \tag{6}$$

$$X'(\theta) = \Leftrightarrow \frac{A(\cos(\theta) \Leftrightarrow B\sin(\theta))}{(\sin(\theta) + B\cos(\theta))^2} + \frac{a(\cos(\theta) \Leftrightarrow b\sin(\theta))}{(\sin(\theta) \Leftrightarrow b\cos(\theta))^2} \tag{7}$$

$$X''(\theta) = \frac{2A(cos(\theta) \Leftrightarrow B\sin(\theta))^2}{(\sin(\theta) + B\cos(\theta))^3} + \frac{A}{\sin(\theta) + B\cos(\theta)} \Leftrightarrow \frac{2a(cos(\theta) \Leftrightarrow b\sin(\theta))^2}{(\sin(\theta) + b\cos(\theta))^3} \Leftrightarrow \frac{a}{\sin(\theta) + b\cos(\theta)} \quad (8)$$

$$X^{(3)}(\theta) = \frac{\Leftrightarrow 6A(cos(\theta) \Leftrightarrow B\sin(\theta))^3}{(\sin(\theta) + B\cos(\theta))^4} \Leftrightarrow \frac{A(\cos(\theta) \Leftrightarrow B\sin(\theta))}{(\sin(\theta) + B\cos(\theta))^2} + \frac{6a(cos(\theta) \Leftrightarrow b\sin(\theta))^3}{(\sin(\theta) + b\cos(\theta))^4} + \frac{a(\cos(\theta) \Leftrightarrow b\sin(\theta))}{(\sin(\theta) + b\cos(\theta))^2} \quad (9)$$

Evaluating the derivatives at $\frac{\pi}{2}$ yields

$$X(\pi/2) = A \Leftrightarrow a \quad (10)$$
$$X'(\pi/2) = AB \Leftrightarrow ab \quad (11)$$
$$X''(\pi/2) = 2(AB^2 \Leftrightarrow ab^2) + A \Leftrightarrow a \quad (12)$$
$$X^{(3)}(\pi/2) = 6(AB^3 \Leftrightarrow ab^3) + AB \Leftrightarrow ab. \quad (13)$$

By taking linear combinations of these to cancel out all but the highest order term from each equation we obtain constants such that:

$$X(\frac{\pi}{2}) = C_0 = A \Leftrightarrow a \quad (14)$$
$$X'(\frac{\pi}{2}) = C_1 = AB \Leftrightarrow ab \quad (15)$$
$$X''(\frac{\pi}{2}) \Leftrightarrow X(\frac{\pi}{2}) = C_2 = AB^2 \Leftrightarrow ab^2 \quad (16)$$
$$\frac{1}{6}\left[X^{(3)}(\frac{\pi}{2}) \Leftrightarrow X'(\frac{\pi}{2})\right] = C_3 = AB^3 \Leftrightarrow ab^3. \quad (17)$$

Now, multiply (15) by B^2 and substitute for AB^3 in (17). Similarly we can multiply (16) by b and substitute for ab^3 yielding,

$$C_3 = C_2 B \Leftrightarrow Bab^2 + C_2 b \Leftrightarrow AB^2 b. \quad (18)$$

Substituting $ab^2 = ABb \Leftrightarrow C_2 b$ we calculate:

$$C_3 = C_2 B \Leftrightarrow C_1 Bb + AB^2 b + C_2 b \Leftrightarrow Ab^2 b = \Leftrightarrow C_1 Bb + C_2 B + C_2 b. \quad (19)$$

By making similar substitutions of BC_1 and bC_1 into (16), and also using $A = C_0 + a$ we get:

$$C_2 = C_1(B + b) \Leftrightarrow C_0 Bb. \quad (20)$$

Multiply(19) by C_0 and (20) by $\Leftrightarrow C_1$ and add.

$$C_0 C_3 \Leftrightarrow C_1 C_2 = (C_0 C_2 \Leftrightarrow C_1^2)(B + b) \quad (21)$$

(20) and (21) are linear in (B+b) and (Bb). The equations are independent because if the determinant of the system, $\Leftrightarrow C_0(C_0C_2 \Leftrightarrow C_1^2)$ were 0 we would be forced to conclude that either $A = a$ or $B = b$, both of which are not allowed by the assumption that the wedge is not degenerate and the wedge is not parallel. Since we can solve for $B + b$ and Bb, we have at most two choices for (B, b). There is at least one solution, since the data came from a wedge. We show that if there are two solutions, only one of them will be acceptable as wedge data.

Proof. Assume that $(A = A_0, a = a_0, B = B_0, b = b_0)$ solves the system. We note that $(A = \Leftrightarrow a_0, a = \Leftrightarrow A_0, B = b_0, b = B_0)$ also solves the system presented in equations 14-17. These are the only two solutions, and only one of these, the correct one, will yield $A > 0$. Therefore, there is a unique wedge satisfying the x-ray data. □

Corollary 3.1. *Let P be a polygon with at most one wedge in every cornerless cone. Let ψ be the polar angle of a ray emanating from the origin intersecting at least one corner of P. Then one of $X[P]$, $X[P]'$, $X[P]''$, and $X[P]^{(3)}$ will be discontinuous at ψ.*

An example of this can be found in Figure 2.

3.2 Generalization to a Generic Non-Convex Polygon

The above theorem allows us to locate corners on star-shaped polygons, but is not helpful in finding corners of more generic non-convex polygons. We prove the following theorem about finding corners of a generic nonconvex polygon.

Theorem 3.1. *If a cone contains at most n wedges in it, and all cornerless cones are non-parallel n-wedge cones then all corners of a polygon will appear as discontinuities in the x-ray or one of its first $4n \Leftrightarrow 1$ derivatives.*

Theorem 3.1 is the case where $n = 1$. We will follow the same general strategy used to prove Theorem 3.1. First we will show that $4n \Leftrightarrow 1$ derivatives uniquely determine n-wedges in a cone. Then, since a corner is at the boundary of two wedges, it must have different right and left hand limits in the x-ray or in one of the first $4n \Leftrightarrow 1$ derivatives. Therefore, there will be a discontinuity in the x-ray or one of its first $4n \Leftrightarrow 1$ derivatives at a corner.

Now we show that $4n \Leftrightarrow 1$ derivatives determine a wedge. To do this we prove a lemma about derivatives of a line which will carry over into derivatives of x-rays.

Lemma 3.2. *The mth polar derivative of a line not passing through the origin is of the form*

$$R^{(m)}(\theta) = a_{(m,m)}A(\cos(\theta) \Leftrightarrow B\sin(\theta))^m(\sin(\theta) + B(\cos(\theta))^{-(m+1)} + \sum_{i=1}^{m-1} a_{(m,i)}R^{(i)}(\theta)$$

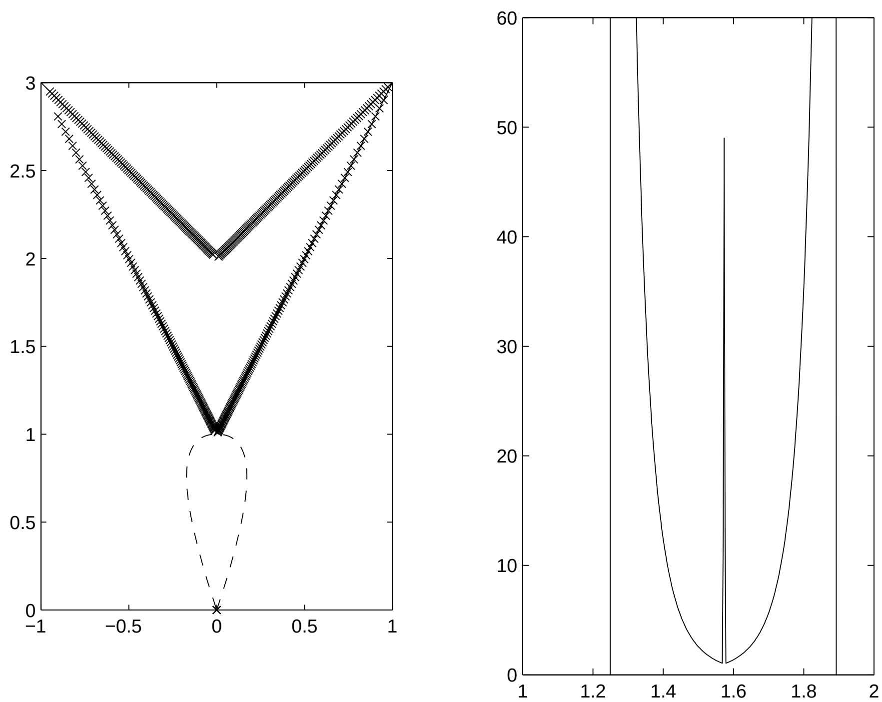

Figure 2: This polygon's (left) corners do not show up earlier than the third derivative of the x-ray data. The consecutive difference of numerical estimates of the third derivative are shown on the right. Notice the spike at $\frac{\pi}{2}$.

where

$$R(\theta) = A(\sin(\theta) + B\cos(\theta))^{-1}$$

and the $a_{(m,i)}$ are constants.

Proof. The proof uses induction. For base cases we have the parameterization of the line itself and its first derivatives. Inductively we differentiate

$$R^{(m)}(\theta) = a_{(m,m)} A(\cos(\theta) \Leftrightarrow B\sin(\theta))^m (\sin(\theta) + B\cos(\theta))^{-(m+1)} + \sum_{i=1}^{m-1} a_{(m,i)} R^{(i)}(\theta)$$

to get

$$\begin{aligned} R^{(m+1)}(\theta) &= \Leftrightarrow (m+1) a_{(m,m)} A(\cos(\theta) \Leftrightarrow B\sin(\theta))^{m+1} (\sin(\theta) + B(\cos(\theta))^{-(m+2)} \Leftrightarrow \\ &\quad m a_{(m,m)} A(\cos(\theta) \Leftrightarrow B\sin(\theta))^{m-1} (\sin(\theta) + B(\cos(\theta))^{-m} + \\ &\quad \sum_{i=1}^{m-1} a_{(m,i)} R^{(i+1)}(\theta) \\ &= a_{(m+1,m+1)} A(\cos(\theta) \Leftrightarrow B\sin(\theta))^{m+1} (\sin(\theta) + B(\cos(\theta))^{-(m+2)} + \\ &\quad \sum_{i=1}^{m} a_{(m+1,i)} R^{(i)}(\theta). \end{aligned}$$

The last step is taken by substituting for $A(\cos(\theta) \Leftrightarrow mB\sin(\theta))^{m-1}(\sin(\theta) + B)^{-m}(\cos(\theta))^{-(m)}$ using the formula for the $(m \Leftrightarrow 1)$th derivative, then combining like terms and calling the new constants $a_{(m+1,i)}$.

This lemma tells us that by taking linear combinations of the derivatives and evaluating at $\frac{\pi}{2}$ we can find constants $K_0, K_1, \ldots K_{4n-1}$ such that $K_i = AB^i$.

We may write the x-ray of n wedges in a cone as a sum of the $2n$ defining edges of the wedge, using the convention that a negative value for some A_i means edge i is the bottom edge of some wedge. Given an n-wedge cone, we can derive constants C_i such that

$$\begin{array}{ccccccccc} C_0 & = & A_0 & + & A_1 & + & \ldots & + & A_{2n} \\ C_1 & = & A_0 B_0 & + & A_1 B_1 & + & \ldots & + & A_{2n} B_{2n} \\ C_2 & = & A_0 B_0^2 & + & A_1 B_1^2 & + & \ldots & + & A_{2n} B_{2n}^2 \\ \vdots & & \vdots & & \vdots & & & & \vdots \\ C_{4n-1} & = & A_0 B_0^{4n-1} & + & A_1 B_1^{4n-1} & + & \ldots & + & A_{2n} B_{2n}^{4n-1}, \end{array}$$

where (A_i, B_i) together determine some edge bounding a wedge in the cone.

To prove uniqueness of the n-wedges, assume that two n-wedge cones, W and W', agree at the x-ray and the first $4n \Leftrightarrow 1$ derivatives. We characterize W

as above, and for W' write

$$\begin{array}{rcccccccc} C_0 & = & A'_0 & + & A'_1 & + & \dots & + & A'_{2n} \\ C_1 & = & A'_0B'_0 & + & A'_1B'_1 & + & \dots & + & A'_{2n}B'_{2n} \\ C_2 & = & A'_0(B'_0)^2 & + & A'_1(B'_1)^2 & + & \dots & + & A'_{2n}(B'_{2n})^2 \\ \vdots & & \vdots & & \vdots & & & & \vdots \\ C_{4n-1} & = & A'_0(B'_0)^{4n-1} & + & A'_1(B'_1)^{4n-1} & + & \dots & + & A'_{2n}(B'_{2n})^{4n-1}. \end{array}$$

Since we are assuming $X[W](\frac{\pi}{2}) = X[W'](\frac{\pi}{2})$ we can set up the matrix equation:

$$B\vec{A} = \vec{0}, \tag{22}$$

where

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ B_1 & B_2 & \cdots & B_{2n} & B'_1 & B'_2 & \cdots & B'_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ B_1^{4n-1} & B_2^{4n-1} & \cdots & B_{2n}^{4n-1} & (B'_1)^{4n-1} & (B'_2)^{4n-1} & \cdots & (B'_{2n})^{4n-1} \end{bmatrix}$$

and

$$\vec{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_{2n} & \Leftrightarrow A'_1 & \Leftrightarrow A`_2 & \cdots & \Leftrightarrow A'_{2n} \end{bmatrix}^{\mathrm{T}}.$$

B is both singular, since $\vec{A} \neq \vec{0}$, and a Vandermonde matrix. This means that either $B_i = B_j$ $(i \neq j)$ or $B'_i = B'_j$ $(i \neq j)$ or $B_i = B'_j$ (for any i and j). However, we are assuming that no two edges are parallel, so $B_i = B_j$ and $B'_i = B'_j$ are both impossible.[1] Therefore, for some (i, j) $B_i = B'_j$. The indicies of the A's and B's have no purpose other than differentiating the A's and B's. Without loss of generality we can re-index so that we reach the conclusion $B_{2n} = B'_{2n}$.

Setting $B_{2n} = B'_{2n}$ we can write down a new system of equations. The subscripts of the matrices will be in parenthesis to distinguish them from entries in the matrix.

$$B_{(1)} = \vec{A}_1 \tag{23}$$

$$B_{(1)} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ B_1 & B_2 & \cdots & B_{2n} & B'_1 & B'_2 & \cdots & B'_{2n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ B_1^{4n-2} & B_2^{4n-2} & \cdots & B_{2n}^{4n-2} & (B'_1)^{4n-2} & (B'_2)^{4n-2} & \cdots & (B'_{2n-1})^{4n-2} \end{bmatrix}$$

[1]After following the argument made below, it will become clear that if $B'_i = B'_j$ we will eventualy have to conclude that $B_k = B_l$ for some k, l, reaching a contradiction.

$$\vec{A}_1 = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{2n} \Leftrightarrow A'_{2n} \\ \Leftrightarrow A'_1 \\ \Leftrightarrow A‘_2 \\ \vdots \\ \Leftrightarrow A'_{2n-1} \end{bmatrix}$$

We ignore the $(4n \Leftrightarrow 1)$th equation for the purpose of maintaining a square matrix. Now we iterate this argument. $B_{(1)}$ is a Vandermonde matrix. $\vec{A}_1$ is non-zero, but $B_{(1)}\vec{A}_1 = \vec{0}$. Therefore $B_{(1)}$ is also singular. This means that for some (i, j), $B_i = B'_j$, $i \neq 2n$, since this would imply that either some $B_i = B_{2n}$ or $B'_i = B'_{2n}$. Without loss of generality we assume that $B_{2n-1} = B'_{2n-1}$.

We continue to iterate this argument until we conclude that $B_1 = B'_1$, $B_2 = B'_2$, $\ldots B_{2n} = B'_{2n}$. When all necessary substitutions are made the system in matrix form is

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ B_1 & B_2 & \cdots & B_{2n} \\ \vdots & \vdots & & \vdots \\ B_1^{2n-1} & B_2^{2n-1} & \cdots & B_{2n}^{2n-1} \end{bmatrix} \begin{bmatrix} A_1 \Leftrightarrow A'_1 \\ A_2 \Leftrightarrow A'_2 \\ \vdots \\ A_{2n} \Leftrightarrow A'_{2n} \end{bmatrix} = \vec{0}.$$

The matrix is Vandermonde and must be non-singular since we assumed $B_i \neq B_j$ $(i \neq j)$. We conclude $\begin{bmatrix} A_1 \Leftrightarrow A'_1 & A_2 \Leftrightarrow A'_2 & \cdots & A_{2n} \Leftrightarrow A'_{2n} \end{bmatrix}^T = \vec{0}$.

That is, $A_1 = A'_1$, $A_2 = A'_2 \ldots A_{2n} = A'_{2n}$. Since all the slopes and corresponding intercepts of the defining edges are equal, we conclude that $W = W'$.

Since an n-wedges cone is uniquely determined by the x-ray and $4n \Leftrightarrow 1$ derviatives, it follows that every corner in a polygon with at most n-wedges in a cone will appear as a discontinuity in the x-ray or one of the first $4n \Leftrightarrow 1$ derivatives.

4 Uniqueness of Non-parallel Wedges

In this section of the paper we show that x-rays in $4n$ directions uniquely determine a non-parallel n-wedge. Using this result we suggest an algebraic reconstruction for a non-parallel 2-wedge from x-ray data from eight directions. There is already a known algebraic reconstruction for a single non-parallel wedge from four directions. Here we will use a similar technique for reconstructing a non-parallel 2-wedge.

4.1 Uniqueness of an n-wedge from $4n$ directions

In [3], it is shown that x-rays in 4 directions uniquely determine a non-parallel 1-wedge. In this section of the paper, it will be shown that a non-parallel n-wedge is uniquely determined by x-rays in $4n$ directions. As mentioned before, it is important to assume that the n-wedge has no two defining edges parallel. To prove uniqueness, we use theorem 5.3.13 of [2] which first appeared in [1]. The proof works by showing that these conditions are met in each cone drawn such that no corners of the polygon lie in the interior of any cone. Thus when two n-wedges have equal x-rays everywhere in a cone, it is possible to use theorem 5.3.13 to impose conditions on the defining edges of the wedges even though it is stated for polygons. First, notation is developed for the statement of the theorem.

Let W and W' be two n-wedges, each with $2n$ defining edges. Let $C(\alpha, \beta) = \{(r, \theta) : \alpha \leq \theta \leq \beta\}$ be a cone with $0 < \beta \Leftrightarrow \alpha \leq \pi$. Let ϕ be an angle such that $0 \leq \phi < \pi$. Label the set of defining edges of W and W' by $e_i, 1 \leq i \leq 2n$, and denote $\{e_i, i \in N(\phi)\}$ those e_i parallel to the direction ϕ. Suppose e_i intersects the ray $\{\theta = \alpha\}$ at (r_i, α). Define $a_i = +1 (or \Leftrightarrow 1)$ if a moving point on the ray $\{\theta = \gamma, \alpha < \gamma < \beta\}$ leaves W (or enters W, respectively) as its distance from the origin increases across e_i; and visa versa for defining edges of W'. The theorem may now be properly stated. A proof is in [2].

Theorem 4.1. *With this notation, two n-wedges, W and W' have the same x-rays in every direction in the cone $C(\alpha, \beta) = \{(r, \theta) : \alpha \leq \theta \leq \beta\} \Leftrightarrow \forall\ \phi$ such that $0 \leq \phi < \pi$, the family $\{e_i, i \in N(\phi)\}$ of edges satisfies $\sum\{a_i r_i : n \in N(\phi)\} = 0$.*

Before using this theorem to show uniqueness of a non-parallel n-wedge from $4n$ x-rays, we show that if x-rays of two non-parallel n-wedges agree in $4n$ directions, then the x-rays agree in every direction in the cone.

Lemma 4.1. *If two nonparallel n-wedges have x-rays equal in $4n$ directions, then the x-rays agree in the entire cone enclosing the n-wedges.*

Proof. Let W and W' be two n-wedges with equal x-rays in $4n$ directions. W and W' are composed of 1-wedges $(W_1, W_2, \ldots, W_n)$ and $(W'_1, W'_2, \ldots, W'_n)$ where the 1-wedges move away from the origin as their index increases. For a general 1-wedge W_k of W, let (r_k, α) and (R_k, α) denote the points of intersection of the wedge's lower defining edge and upper defining edge with the ray $\{\theta = \alpha\}$, respectively. Then by requiring the n-wedge to be in the half plane $\{(x, y) | x > 0\}$, we have $0 < r_1 \leq R_1 \leq r_2 \leq R_2 \leq \ldots \leq r_n \leq R_n$. For W', the corresponding 1-wedge W'_k has points of intersection $(r_k + d_k, \alpha)$ and $(R_k + D_k, \alpha)$, with, necessarily, $d_k > \Leftrightarrow r_k$ and $D_k > \Leftrightarrow R_k$. Thus we have $0 < r_1 + d_1 \leq R_1 + D_1 \leq r_2 + d_2 \leq R_2 + D_2 \leq \ldots \leq r_n + d_n \leq R_n + D_n$ and $\sum_{i=1}^{n} (D_i \Leftrightarrow d_i) = 0$. Denote the points of W_k and W'_k intersecting the ray $\{\theta = \beta\}$ in the analagous way by (g_k, β), (G_k, β), $(g_k + c_k, \beta)$, and $(G_k + C_k, \beta)$. Here, we get $\sum_{i=1}^{n} (C_i \Leftrightarrow c_i) = 0$. Finally, note that if two consecutive edges intersect in one point on the ray $\{\theta = \alpha\}$, then these edges must intersect the ray $\{\theta = \beta\}$ in separate points. This requirement is to prevent consecutive defining edges to be the same and, consequently, having fewer than n-wedges.

For a general direction ϕ, the x-ray of a 1-wedge W_k from W is given by

$$\frac{G_k R_k}{G_k \sin(\phi \Leftrightarrow \alpha) + R_k \sin(\beta \Leftrightarrow \phi)} \Leftrightarrow \frac{g_k r_k}{g_k \sin(\phi \Leftrightarrow \alpha) + r_k \sin(\beta \Leftrightarrow \phi)}.$$

Likewise, the x-ray of a 1-wedge W'_k from W' is given by

$$\frac{(G_k + C_k)(R_k + D_k)}{(G_k + C_k)\sin(\phi \Leftrightarrow \alpha) + (R_k + D_k)\sin(\beta \Leftrightarrow \phi)}$$

$$\Leftrightarrow \frac{(g_k + c_k)(r_k + d_k)}{(g_k + c_k)\sin(\phi \Leftrightarrow \alpha) + (r_k + d_k)\sin(\beta \Leftrightarrow \phi)}.$$

These equations are found by parameterizing the equation of the defining edges of the 1-wedges and finding the points of intersection with the ray $\{\theta = \phi\}$. Let, for convenience, $L_\phi = \sin(\phi \Leftrightarrow \alpha)$ and $M_\phi = \sin(\beta \Leftrightarrow \phi)$. In this notation, an x-ray of W in the direction ϕ, the sum of its 1-wedge x-rays, is given by

$$\sum_{i=1}^{n} \frac{G_i R_i}{G_i L_\phi + R_i M_\phi} \Leftrightarrow \frac{g_i r_i}{g_i L_\phi + r_i M_\phi}$$

$=$

$$\sum_{i=1}^{n} \frac{G_i g_i (R_i \Leftrightarrow r_i) L_\phi + R_i r_i (G_i \Leftrightarrow g_i) M_\phi}{G_i g_i L_\phi^2 + (G_i r_i + R_i g_i) L_\phi M_\phi + R_i r_i M_\phi^2}.$$

With the equation for the x-ray of W' in the direction ϕ found analagously, the x-rays of W and W' are equal in the directon ϕ when

$$\sum_{i=1}^{n} [\frac{t_{1_i} L_\phi + t_{2_i} M_\phi}{t_{3_i} L_\phi^2 + t_{4_i} L_\phi M_\phi + t_{5_i} M_\phi^2}$$

$$\Leftrightarrow \frac{t_{6_i} L_\phi + t_{7_i} M_\phi}{t_{8_i} L_\phi^2 + t_{9_i} L_\phi M_\phi + t_{10_i} M_\phi^2}] = 0,$$

where

$$t_{1_i} = (G_i + C_i)(g_i + d_i)(R_i + D_i \Leftrightarrow r_i \Leftrightarrow d_i),$$

$$t_{2_i} = (R_i + D_i)(r_i + d_i)(G_i + C_i \Leftrightarrow g_i \Leftrightarrow c_i),$$

$$t_{3_i} = (G_i + C_i)(g_i + c_i),$$

$$t_{4_i} = (G_i + C_i)(r_i + d_i) + (R_i + D_i)(g_i + c_i),$$

$$t_{5_i} = (R_i + D_i)(r_i + d_i),$$

$$t_{6_i} = G_i g_i (R_i \Leftrightarrow r_i),$$

$$t_{7_i} = R_i r_i (G_i \Leftrightarrow g_i),$$

$$t_{8_i} = G_i g_i,$$

$$t_{9_i} = G_i r_i + R_i g_i, and$$

$$t_{10_i} = R_i r_i.$$

After simplifying, $\sum_{i=1}^{n} \frac{A_i}{B_i} = 0$, where

$$A_i = [G_i g_i (G_i + C_i)(g_i + c_i)(D_i \Leftrightarrow d_i)]L_\phi^3 + [k_1]L_\phi^2 M_\phi$$

$$+[k_2]M_\phi L_\phi^2 + [R_i r_i (R_i + D_i)(r_i + d_i)(C_i \Leftrightarrow c_i)]M_\phi^3,$$

and

$$B_i = [G_i g_i (G_i + C_i)(g_i + c_i)]L_\phi^4 + [k_3]L_\phi^3 M_\phi + [k_4]L_\phi^2 M_\phi^2$$

$$+[k_5]L_\phi M_\phi^3 + [R_i r_i (R_i + D_i)(r_i + d_i)]M^4,$$

where the k_i are coefficients depending on the points of intersection of W_i and W_i' with the defining rays of the enclosing cone.

Now expand the sum, take a common denominator, clear it, and notice the result is homogenous of order 3+4(n-1)=4n-1. This equation has the form $kL_\phi^{4n-1} + (f_2)L_\phi^{4n-2}M_\phi + \cdots + (f_{4n-1})L_\phi M_\phi^{4n-2} + qM_\phi^{4n-1} = 0$, where the f_i are coefficients depending on points of intersection of wedges with the enclosing cone, and

$$k = \sum_{i=1}^{n}(D_i \Leftrightarrow d_i)\prod_{i=1}^{n}(G_i g_i)(G_i + C_i)(g_i + c_i) = 0$$

and

$$q = \sum_{i=1}^{n}(C_i \Leftrightarrow c_i)\prod_{i=1}^{n}(R_i r_i)(R_i + D_i)(r_i + d_i) = 0$$

since

$$\sum_{i=1}^{n}(C_i \Leftrightarrow c_i) = \sum_{i=1}^{n}(D_i \Leftrightarrow d_i) = 0.$$

Thus, two n-wedge x-rays agree in the direction ϕ when $(f_2)L_\phi^{4n-2}M_\phi + \cdots + (f_{4n-1})L_\phi M_\phi^{4n-2} = 0$. First note that $\sin(\phi \Leftrightarrow \alpha) = 0 \Longrightarrow \phi = \alpha$, or $\phi \Leftrightarrow \alpha = \pi$, neither of which is possible with $0 < \alpha < \phi < \beta < \pi$. Thus, $L_\phi \neq 0$. Similarly, $M_\phi \neq 0$. Divide by $L_\phi M_\phi$ to get $(f_2)L_\phi^{4n-3} + \cdots + (f_{4n-1})M_\phi^{4n-3} = 0$. Again divide by L_ϕ^{4n-3} to get $f_2 + f_3(\frac{M_\phi}{L_\phi}) + f_4(\frac{M_\phi}{L_\phi})^2 + \cdots + f_{4n-1}(\frac{M_\phi}{L_\phi})^{4n-3} = 0$.

For simplicity, let $Y_\phi = \frac{M_\phi}{L_\phi}$. Then we have the polynomial

$$f_2 + f_3(Y_\phi) + \cdots + f_{4n-1}(Y_\phi)^{4n-3} = 0.$$

Here there can be no more than $4n \Leftrightarrow 3$ distinct values of Y_ϕ that are roots unless the polynomial is identically zero. Suppose $Y_{\phi_1} = Y_{\phi_2}$. Then

$$\frac{\sin(\beta \Leftrightarrow \phi_1)}{\sin(\phi_1 \Leftrightarrow \alpha)} = \frac{\sin(\beta \Leftrightarrow \phi_2)}{\sin(\phi_2 \Leftrightarrow \alpha)}.$$

Expanding and cross multiplying gives

$$[\sin(\beta)\cos(\phi_1) \Leftrightarrow \cos(\beta)\sin(\phi_1)][\sin(\phi_2)\cos(\alpha) \Leftrightarrow \cos(\phi_2)\sin(\alpha)]$$

$$= [\sin(\beta)\cos(\phi_2) \Leftrightarrow \cos(\beta)\sin(\phi_2)][\sin(\phi_1)\cos(\alpha) \Leftrightarrow \cos(\phi_1)\sin(\alpha)].$$

After simplifying, one obtains $\tan(\alpha)\sin(\phi_1 \Leftrightarrow \phi_2) = \tan(\beta)\sin(\phi_1 \Leftrightarrow \phi_2)$. Since $\sin(\phi_1 \Leftrightarrow \phi_2) \neq 0$ in the acceptable range of angles, $\tan(\alpha) = \tan(\beta)$ which is impossible for $0 < \alpha < \beta < \pi$. Then there are $4n \Leftrightarrow 2$ distinct values of Y_ϕ which satisfy the equation. Hence the polynomial must be identically zero. Thus x-rays of W and W' agree in every direction in the cone $C(\alpha, \beta)$. □

From the above lemma we know that when two non-parallel n-wedges have equal x-rays in $4n$ directions, they have equal x-rays in the cone enclosing them. Now theorem 4.1 is used to demonstrate uniqueness.

Theorem 4.2. *$4n$ x-rays uniquely determine a non-parallel n-wedge.*

Proof. Consider an identical setup as in the above lemma. From the above lemma, x-rays of the two n-wedges must agree in every direction in the cone $C(\alpha, \beta)$. First note that each defining edge of W must be parallel to a unique defining edge of W'. Suppose a defining edge ℓ of W were not parallel to any defining edge of W'. Then since, by hypothesis, ℓ is not parallel to any of the defining edges of W, it is not parallel to any other defining edge. By theorem 4.1, the point of intersection of ℓ with the ray $\{\theta = \alpha\}$ is $(0, \alpha)$. This contradicts the original assumption that the n-wedge lies entirely in the upper half plane. Now suppose that ℓ is parallel to two edges, ℓ_1 and ℓ_2 of W'. No other of the remaining $2n$-1 defining edges of W may be parallel to ℓ_1 or ℓ_2 since that would imply the defining edge is parallel to ℓ. Thus one of the remaining $2n$-1 defining edges of W is not parallel to any of the remaining $2n$-2 defining edges of W'. Hence there exists a defining edge of W parallel to no other defining edges, contradicting what has been said above. It is assumed hereafter that each defining edge of W is parallel to a unique defining edge of W'. The proof proceeds by induction on the number of wedges.

Case, n=1:

By the above argument, it is true that either

(1) the bottom defining edge of W is parallel to the top defining edge of W' and the top defining edge of W is parallel to the bottom defining edge of W', or (2) the bottom defining edge of W is parallel to the bottom defining edge of W' and the top defining edge of W is parallel to the top defining edge of W'.

If (1) occurs, $R_1 + r_1 + d_1 = 0$ by theorem 4.1 and the fact that the top defining edge of W and bottom defining edge of W' are parallel. This is impossible since both $R_1 > 0$ and $r_1 + d_1 > 0$. When (2) occurs, note that $r_1 = r_1 + d_1$ and $R_1 = R_1 + D_1$, by theorem 4.1. Then $d_1 = 0 = D_1$. Thus, the points of intersection of the defining edges of W and W' with the ray $\{\theta = \alpha\}$ are the same. Now the same argument may be used to show that the points of intersection are the same on the line $\{\theta = \beta\}$, or simply note that it is forced by the lines being parallel. In either case, W and W' are the same wedge. This completes the proof for n=1.

Now suppose that for $n = P \Leftrightarrow 1$, $4(P \Leftrightarrow 1)$ x-rays uniquely determine a non-parallel $P \Leftrightarrow 1$ wedge. Let W and W' be non-parallel P-wedges equal in

$4(P)$ directions. By the above lemma, W and W' have the same x-rays in every direction. Still, each defining edge of W is parallel to a unique defining edge of W'. Suppose that the bottom defining edge of W_1 is not parallel to the bottom defining edge of W_1' and that the top defining edge of W_1 is not parallel to the top defining edge of W_1'. It will be shown that in either case, a contradiction is reached.

The bottom edges of W_1 and of W_1' not being parallel implies the bottom edge of W_1 is parallel to some defining edge from a 1-wedge $W_k', k \geq 2$ or that it is parallel to the top defining edge of W_1'. In other words,

(1) $\Leftrightarrow r_1 \Leftrightarrow R_k \Leftrightarrow D_k = 0$, some $k > 1$, or

(2) $\Leftrightarrow r_1 + r_k + d_k = 0$, some $k > 1$ hold.

(1) is impossible since $r_1 > 0$ and $R_k + D_k > 0$.

From the top lines and an analagous argument, either

(3) $R_1 + r_g + d_g = 0$, some $g \geq 1$, or

(4) $R_1 \Leftrightarrow R_g \Leftrightarrow D_g = 0$, some $g > 1$.

Likewise, (3) is impossible.

Thus, when the bottom lines of W_1 and W_1' are not parallel and the top lines of W_1 and W_1' are not parallel, $r_1 = r_k + d_k$, some $k > 0$, and $R_1 = R_g + D_g$, some $g > 0$.

Identically, for the edges of W_1', $r_1 + d_1 = r_q$, for some $q > 0$ and $R_1 + D_1 = R_f$, for some $f > 0$. Then $r_1 < R_1 \leq r_q \Longrightarrow r_k + d_k = r_1 < r_q = r_1 + n_1$, contradicting $r_1 + n_1 \leq r_k + n_k$. Although this part of the proof relies on the inequality $r_1 < R_1$, if in fact $r_1 = R_1$, then $g_1 < G_1$ and the argument may be reformulated accordingly.

Now we conclude that the bottom edge of W_1 is parallel to the bottom edge of W_1' and the top edge of W_1 is parallel to the top edge of W_1'. From the case n=1, we have that W_1 and W_1' are the same wedge. This implies that the $P \Leftrightarrow 1$ other wedges of W and W' have the same x-ray in every direction in the cone. By the inductive hypothesis, these $P \Leftrightarrow 1$ wedges are uniquely determined. Thus, a non-parallel P-wedge is uniquely determined by $4P$ x-rays. The general result follows. □

5 Non-parallel 2-wedge Reconstruction

Using the fact that a non-parallel n-wedge is uniquely determined by its integrals along $4n$ rays, a technique for the algebraic reconstruction of a non-parallel 2-wedge from x-ray data along eight rays is suggested.

Let $W = (W_1, W_2)$ be a non-parallel 2-wedge in the upper-half plane where W_1 and W_2 are the non-parallel 1-wedges with W_1 closer to the origin, and W_1 and W_2 sharing no more than one point in common. Using standard polar coordinates let $\phi_1, \phi_2, \ldots, \phi_8$, angles measured from the positive x-axis, denote the eight directions from which data is taken. Moreover, require that ϕ_1 and ϕ_8 define the cone $C = \{(r, \theta) | \phi_1 \leq \theta \leq \phi_8\}$ enclosing $W = (W_1, W_2)$. For each ray $\{\theta = \phi_j\}$, denote the points of intersection of the ray with W_1 by (r_j, ϕ_j) and (R_j, ϕ_j), with (r_j, ϕ_j) closer to the origin. Similarly, denote the points of intersection of the ray with W_2 by (q_j, ϕ_j) and (Q_j, ϕ_j). Let λ_1 and λ_2, with respective slopes m_1 and m_2, be the defining edges of W_1. For W_2, let the defining edges be called λ_3 and λ_4 with respective slopes m_3 and m_4. Since $W = (W_1, W_2)$ is a non-parallel 2-wedge, $m_1 \neq m_2 \neq m_3 \neq m_4$. We also require that none of the defining edges is vertical to prevent some m_i being infinite. Nevertheless, results found here will hold for vertical edges by continuity.

5.1 Generating equations

By writing the points of intersection, (r_j, ϕ_j) in rectangular coordinates as $(r_j \cos(\phi_j), r_j \sin(\phi_j))$ and substituting them into the equation for the slope of a line, we see

$$m_1 = \frac{r_j \sin(\phi_j) \Leftrightarrow r_1 \sin(\phi_1)}{r_j \cos(\phi_j) \Leftrightarrow r_1 \cos(\phi_1)}.$$

Similarly, the equations

$$m_2 = \frac{R_j \sin(\phi_j) \Leftrightarrow R_1 \sin(\phi_1)}{R_j \cos(\phi_j) \Leftrightarrow R_1 \cos(\phi_1)} \tag{24}$$

$$m_3 = \frac{q_j \sin(\phi_j) \Leftrightarrow q_1 \sin(\phi_1)}{q_j \cos(\phi_j) \Leftrightarrow q_1 \cos(\phi_1)} \tag{25}$$

$$m_4 = \frac{Q_j \sin(\phi_j) \Leftrightarrow Q_1 \sin(\phi_1)}{Q_j \cos(\phi_j) \Leftrightarrow Q_1 \cos(\phi_1)} \tag{26}$$

$$2 \leq j \leq 8$$

hold.

After some elementary algebra,

$$r_j = \frac{\sin(\phi_1) \Leftrightarrow m_1 \cos(\phi_1)}{\sin(\phi_j) \Leftrightarrow m_1 \cos(\phi_j)} r_1 \tag{27}$$

$$R_j = \frac{\sin(\phi_1) \Leftrightarrow m_2 \cos(\phi_1)}{\sin(\phi_j) \Leftrightarrow m_2 \cos(\phi_j)} R_1 \tag{28}$$

$$q_j = \frac{\sin(\phi_1) \Leftrightarrow m_3 \cos(\phi_1)}{\sin(\phi_j) \Leftrightarrow m_3 \cos(\phi_j)} q_1 \tag{29}$$

$$Q_j = \frac{\sin(\phi_1) \Leftrightarrow m_4 \cos(\phi_1)}{\sin(\phi_j) \Leftrightarrow m_4 \cos(\phi_j)} Q_1 \tag{30}$$

$$2 \leq j \leq 8.$$

Now define

$$b_j = \frac{\sin(\phi_1) \Leftrightarrow m_1 \cos(\phi_1)}{\sin(\phi_j) \Leftrightarrow m_1 \cos(\phi_j)} \tag{31}$$

$$a_j = \frac{\sin(\phi_1) \Leftrightarrow m_2 \cos(\phi_1)}{\sin(\phi_j) \Leftrightarrow m_2 \cos(\phi_j)} \tag{32}$$

$$d_j = \frac{\sin(\phi_1) \Leftrightarrow m_3 \cos(\phi_1)}{\sin(\phi_j) \Leftrightarrow m_3 \cos(\phi_j)} \tag{33}$$

$$c_j = \frac{\sin(\phi_1) \Leftrightarrow m_4 \cos(\phi_1)}{\sin(\phi_j) \Leftrightarrow m_4 \cos(\phi_j)} \tag{34}$$

$$2 \leq j \leq 8.$$

By construction, the linear system of equations

$$R_1 \Leftrightarrow r_1 + Q_1 \Leftrightarrow q_1 = \chi_1 \tag{35}$$

$$a_j R_1 \Leftrightarrow b_j r_1 + c_j Q_1 \Leftrightarrow d_j q_1 = \chi_j \tag{36}$$

$$2 \leq j \leq 8$$

with coefficients nonlinear in m_1, m_2, m_3, and m_4 has a solution $0 < r_1 \leq q_1$. By theorem 4.2 , the solution is unique.

Moreover, the matrix

$$\begin{bmatrix} 1 & \Leftrightarrow 1 & 1 & \Leftrightarrow 1 \\ a_g & \Leftrightarrow b_g & c_g & \Leftrightarrow d_g \\ a_j & \Leftrightarrow b_j & c_j & \Leftrightarrow d_j \\ a_k & \Leftrightarrow b_k & c_k & \Leftrightarrow d_k \end{bmatrix}$$

has rank four when $2 \leq g < j < k$.

Proof. Suppose not. Then $\exists\ \gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{R}$, not all zero, such that

$$\gamma_1(1, \Leftrightarrow 1, 1, \Leftrightarrow 1) + \gamma_2(a_j, \Leftrightarrow b_j, c_j, \Leftrightarrow d_j) + \gamma_3(a_k, \Leftrightarrow b_k, c_k, \Leftrightarrow d_k) = \gamma_4(a_g, \Leftrightarrow b_g, c_g, \Leftrightarrow d_g).$$

Suppose that $\gamma_4 \neq 0$. By dividing through by γ_4,

$$\alpha_1(1, \Leftrightarrow 1, 1, \Leftrightarrow 1) + \alpha_2(a_j, \Leftrightarrow b_j, c_j, \Leftrightarrow d_j) + \alpha_3(a_k, \Leftrightarrow b_k, c_k, \Leftrightarrow d_k) = (a_g, \Leftrightarrow b_g, c_g, \Leftrightarrow d_g),$$

with the α_d not all zero.

Then

$$\frac{\alpha_2 \sin(\phi_1) \Leftrightarrow \alpha_2 m_2 \cos(\phi_1)}{\sin(\phi_j) \Leftrightarrow m_2 \cos(\phi_j)} + \frac{\alpha_3 \sin(\phi_1) \Leftrightarrow \alpha_3 m_2 \cos(\phi_1)}{\sin(\phi_k) \Leftrightarrow m_2 \cos(\phi_k)}$$
$$= \frac{\sin(\phi_1) \Leftrightarrow m_2 \cos(\phi_1)}{\sin(\phi_g) \Leftrightarrow m_2 \cos(\phi_g)} \Leftrightarrow \frac{\alpha_1 \sin(\phi_g) \Leftrightarrow \alpha_1 m_2 \cos(\phi_g)}{\sin(\phi_g) \Leftrightarrow m_2 \cos(\phi_g)}, \tag{37}$$

follows from equality in the first coordinate.

After simplification, we can rewrite this equation as

$$Am_2^3 + Bm_2^2 + Cm_2 + D = 0, \tag{38}$$

where

$$\begin{aligned}
A = &\Leftrightarrow \alpha_1 \cos(\phi_g)\cos(\phi_k)\cos(\phi_j) \\
&\Leftrightarrow \alpha_2 \cos(\phi_1)\cos(\phi_j)\cos(\phi_g) \\
&\Leftrightarrow \alpha_3 \cos(\phi_1)\cos(\phi_j)\cos(\phi_g) \\
&+ \cos(\phi_1)\cos(\phi_k)\cos(\phi_j), \\
B = &\ \alpha_1(\sin(\phi_j + \phi_k)\cos(\phi_g) + \cos(\phi_k)\cos(\phi_j)\sin(\phi_g)) \\
&+\alpha_2(\sin(\phi_1 + \phi_k)\cos(\phi_g) + \cos(\phi_1)\cos(\phi_k)\sin(\phi_g)) \\
&+\alpha_3(\sin(\phi_1 + \phi_j)\cos(\phi_g) + \cos(\phi_1)\cos(\phi_j)\sin(\phi_g)) \\
&\Leftrightarrow(\sin(\phi_k + \phi_j)\cos(\phi_1) + \cos(\phi_k)\cos(\phi_j)\sin(\phi_1)), \\
C = &\Leftrightarrow \alpha_1(\sin(\phi_j + \phi_k)\sin(\phi_g) + \cos(\phi_g)\sin(\phi_j)\sin(\phi_k)) \\
&\Leftrightarrow \alpha_2(\sin(\phi_1 + \phi_k)\sin(\phi_g) + \sin(\phi_1)\sin(\phi_k)\cos(\phi_g)) \\
&\Leftrightarrow \alpha_3(\sin(\phi_1 + \phi_j)\sin(\phi_g) + \cos(\phi_g)\sin(\phi_1)\sin(\phi_j)) \\
&+(\sin(\phi_k + \phi_j)\sin(\phi_1) + \cos(\phi_1)\sin(\phi_j)\sin(\phi_k)), \text{and} \\
D = &\ \alpha_1 \sin(\phi_g)\sin(\phi_j)\sin(\phi_k) \\
&+\alpha_2 \sin(\phi_1)\sin(\phi_k)\sin(\phi_g) \\
&+\alpha_3 \sin(\phi_1)\sin(\phi_j)\sin(\phi_g) \\
&\Leftrightarrow \sin(\phi_1)\sin(\phi_j)\sin(\phi_k).
\end{aligned}$$

Similarly, from the second, third, and fourth coordinates, we get the system

$$Am_1^3 + Bm_1^2 + Cm_1 + D = 0 \tag{39}$$

$$Am_4^3 + Bm_4^2 + Cm_4 + D = 0 \tag{40}$$

$$Am_3^3 + Bm_3^2 + Cm_3 + D = 0. \tag{41}$$

These four cubics, 38-41 have the same coefficients, forcing at least two of the m_d to be equal unless $A = B = C = D = 0$. By assumption, the 2-wedge is non-parallel, so it must be that $A = B = C = D = 0$. From setting $A = B = C = D = 0$, and using the rotational invariance to let $\phi_1 = \frac{\pi}{2}$, the following equations in the α_i are satisfied:

$\alpha_1 \cos(\phi_g)\cos(\phi_k)\cos(\phi_j) = 0$

$\alpha_1(t_1) + \alpha_2(\cos(\phi_k)\cos(\phi_g)) + \alpha_3(\cos(\phi_j)\cos(\phi_g)) = \cos(\phi_k)\cos(\phi_j)$

$\alpha_1(t_2) + \alpha_2 \sin(\phi_k + \phi_g) + \alpha_3 \sin(\phi_j + \phi_g) = \sin(\phi_k + \phi_j)$, and

$\alpha_1(t_3) + \alpha_2 \sin(\phi_k)\sin(\phi_j) + \alpha_3 \sin(\phi_j)\sin(\phi_g) = \sin(\phi_j)\sin(\phi_k)$, where the t_i are coeficients determined by cosines of the x-ray directions (angles).

These equations hold for $(\alpha_1, \alpha_2, \alpha_3) \neq (0,0,0)$ if and only if the matrix

$$\begin{bmatrix} \cos(\phi_g)\cos(\phi_k)\cos(\phi_j) & 0 & 0 & 0 \\ t_1 & \cos(\phi_k)\cos(\phi_g) & \cos(\phi_j)\cos(\phi_g) & \Leftrightarrow\cos(\phi_k)\cos(\phi_j) \\ t_2 & \sin(\phi_k+\phi_g) & \sin(\phi_j+\phi_g) & \Leftrightarrow\sin(\phi_k+\phi_j) \\ t_3 & \sin(\phi_k)\sin(\phi_j) & \sin(\phi_j)\sin(\phi_g) & \Leftrightarrow\sin(\phi_j)\sin(\phi_k) \end{bmatrix}$$

has zero determinant.

First note that $\cos(\phi_g)\cos(\phi_k)\cos(\phi_j)$ is nonzero following the rotation because for each $d \neq 1, \phi_d$ lies in the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$. By setting the determinant equal to zero, we find that $\sin(\phi_j+\phi_k)\sin(\phi_g)\cos(\phi_g)\sin(\phi_j \Leftrightarrow \phi_k) + \sin(\phi_g + \phi_j)\sin(\phi_k)\cos(\phi_k)\sin(\phi_g \Leftrightarrow \phi_j) + \sin(\phi_k+\phi_g)\sin(\phi_j)\cos(\phi_j)\sin(\phi_k \Leftrightarrow \phi_g) = 0$. After more simplification, $\sin(2(\phi_k \Leftrightarrow \phi_j)) + \sin(2(\phi_j \Leftrightarrow \phi_g)) = \sin(2(\phi_k \Leftrightarrow \phi_g))$.

To see if there are any such ϕ_g, ϕ_j, and ϕ_k in the allowed range of angles satisfying this equation, first fix ϕ_k and ϕ_g. Now $f(\phi_j) = \sin(2(\phi_k \Leftrightarrow \phi_j)) + \sin(2(\phi_j \Leftrightarrow \phi_g)) \Leftrightarrow \sin(2(\phi_k \Leftrightarrow \phi_g)$ is a function in the variable ϕ_j which vanishes at $\phi_j = \phi_k$ and $\phi_j = \phi_g$. First note that $f(\phi_j)$ is an analytic function. Then the derivative may be used to find how many maximum and minimum values occur in the interval (ϕ_g, ϕ_k), and thus how many times the graph will cross the x-axis. We will find that either there is only one extremum, and hence no roots of $f(\phi_j)$ in the proper range of angles, or that $f(\phi_j)$ is identically zero. To see that the latter case is ruled out, f is evaluated at an angle in the allowed range and seen not to be zero. By taking the derivative, and setting it equal to zero, $2\cos(2(\phi_k \Leftrightarrow \phi_j)) = 2\cos(2(\phi_j \Leftrightarrow \phi_g)$. But $0 < \phi_j \Leftrightarrow \phi_g < \pi$ and $0 < \phi_k \Leftrightarrow \phi_j < \pi \Longrightarrow 0 < 2(\phi_g \Leftrightarrow \phi_g) < 2\pi$ and $0 < 2(\phi_k \Leftrightarrow \phi_j) < 2\pi$. So, either (1) $2(\phi_j \Leftrightarrow \phi_g) = 2(\phi_k \Leftrightarrow \phi_j)$, or (2) $2(\phi_j \Leftrightarrow \phi_g) + 2(\phi_k \Leftrightarrow \phi_j) = 2\pi$. (1) implies that $\phi_j = \frac{\phi_k+\phi_g}{2}$ and (2) implies that $\phi_k \Leftrightarrow \phi_g = \pi$ which is not possible in the allowed range of angles. Thus there exists only one maximum or minimum in the interval (ϕ_g, ϕ_k), unless $f(\phi_j)$ vanishes everywhere in the interval (ϕ_g, ϕ_k). Consider $f(\frac{\phi_g+\phi_k}{2}) = 2\sin(\phi_k \Leftrightarrow \phi_g) \Leftrightarrow \sin(2(\phi_k \Leftrightarrow \phi_g))$. Since $2\sin(\phi_k \Leftrightarrow \phi_g) \Leftrightarrow \sin(2(\phi_k \Leftrightarrow \phi_g)) = 2\sin(\phi_k \Leftrightarrow \phi_g)(1 \Leftrightarrow \cos(\phi_k \Leftrightarrow \phi_g))$ is equal to zero only when $\cos(\phi_k \Leftrightarrow \phi_g) = 1$, we have that $\phi_k \Leftrightarrow \phi_g = 0$, a contradiction since $\phi_g < \phi_k$. Thus we conclude that the determinant is not equal to zero for all ϕ_j in (ϕ_g, ϕ_k), reaching a contradiction when $\gamma_4 \neq 0$. Now suppose that $\gamma_4 = 0$.

Then

$$\gamma_1 + \frac{\gamma_2\sin(\phi_1) \Leftrightarrow \gamma_2 m_2\cos(\phi_1)}{\sin(\phi_j) \Leftrightarrow m_2\cos(\phi_j)} + \frac{\gamma_3\sin(\phi_1) \Leftrightarrow \gamma_3 m_2\cos(\phi_1)}{\sin(\phi_k) \Leftrightarrow m_2\cos(\phi_k)} = 0$$

follows from equality in the first coordinate.

After rotating to $\phi_1 = \frac{\pi}{2}$ and further simplification we rewrite this equation as $Am_2^2 + Bm_2 + C = 0$, where

$$A = \gamma_1\cos(\phi_j)\cos(\phi_k),$$

$$B = \Leftrightarrow \gamma_1\sin(\phi_j+\phi_k) \Leftrightarrow \gamma_2\cos(\phi_k) \Leftrightarrow \gamma_3\cos(\phi_j),$$

and

$$C = \gamma_1\sin(\phi_j)\sin(\phi_k) + \gamma_2\sin(\phi_k) + \gamma_3\sin(\phi_j).$$

Similarly, from the second, third, and fourth coordinates,

$$Am_1^2 + Bm_1 + C = 0 \tag{42}$$
$$Am_4^2 + Bm_4 + C = 0 \tag{43}$$
$$Am_3^2 + Bm_3 + C = 0. \tag{44}$$

Since $m_1 \neq m_2 \neq m_3 \neq m_4$, $A = B = C = 0$.

A=0 $\Longrightarrow \gamma_1 = 0$ since $\cos(\phi_j)\cos(\phi_k) \neq 0$. Then B=0 and C=0 $\Longrightarrow$ (1) $\gamma_2 \sin(\phi_k) + \gamma_3 \sin(\phi_j) = 0$, and (2) $\gamma_2 \cos(\phi_k) + \gamma_3 \cos(\phi_j) = 0$.

Since $(\gamma_2, \gamma_3) \neq (0,0)$, (1) and (2) imply that

$$\sin(\phi_k)\cos(\phi_j) \Leftrightarrow \sin(\phi_j)\cos(\phi_k) = \sin(\phi_k \Leftrightarrow \phi_j) = 0.$$

However, this is impossible in the permitted range of angles. Hence a contradiction is also derived when $\gamma_4 = 0$.

Thus,

$$\begin{bmatrix} 1 & \Leftrightarrow 1 & 1 & \Leftrightarrow 1 \\ a_g & \Leftrightarrow b_g & c_g & \Leftrightarrow d_g \\ a_j & \Leftrightarrow b_j & c_j & \Leftrightarrow d_j \\ a_k & \Leftrightarrow b_k & c_k & \Leftrightarrow d_k \end{bmatrix}$$

has rank four when $2 \leq g < j < k$. □

From this last result Cramer's rule may be used to solve for R_1. Using Cramer's rule to write R_1 in terms of determinants yields

$$R_1 = \frac{\begin{vmatrix} \chi_1 & \Leftrightarrow 1 & 1 & \Leftrightarrow 1 \\ \chi_2 & \Leftrightarrow b_2 & c_2 & \Leftrightarrow d_2 \\ \chi_3 & \Leftrightarrow b_3 & c_3 & \Leftrightarrow d_3 \\ \chi_w & \Leftrightarrow b_w & c_w & \Leftrightarrow d_w \end{vmatrix}}{\begin{vmatrix} 1 & \Leftrightarrow 1 & 1 & \Leftrightarrow 1 \\ a_2 & \Leftrightarrow b_2 & c_2 & \Leftrightarrow d_2 \\ a_3 & \Leftrightarrow b_3 & c_3 & \Leftrightarrow d_3 \\ a_w & \Leftrightarrow b_w & c_w & \Leftrightarrow d_w \end{vmatrix}},$$

for each $w \in \{4, 5, 6, 7, 8\}$.

Next, set the solutions for R_1 from consecutive w's equal and cross multiply to obtain

$$\begin{vmatrix} \chi_1 & \Leftrightarrow 1 & 1 & \Leftrightarrow 1 \\ \chi_2 & \Leftrightarrow b_2 & c_2 & \Leftrightarrow d_2 \\ \chi_3 & \Leftrightarrow b_3 & c_3 & \Leftrightarrow d_3 \\ \chi_w & \Leftrightarrow b_w & c_w & \Leftrightarrow d_w \end{vmatrix} \begin{vmatrix} 1 & \Leftrightarrow 1 & 1 & \Leftrightarrow 1 \\ a_2 & \Leftrightarrow b_2 & c_2 & \Leftrightarrow d_2 \\ a_3 & \Leftrightarrow b_3 & c_3 & \Leftrightarrow d_3 \\ a_{w+1} & \Leftrightarrow b_{w+1} & c_{w+1} & \Leftrightarrow d_{w+1} \end{vmatrix}$$

$$\Leftrightarrow \begin{vmatrix} \chi_1 & \Leftrightarrow 1 & 1 & \Leftrightarrow 1 \\ \chi_2 & \Leftrightarrow b_2 & c_2 & \Leftrightarrow d_2 \\ \chi_3 & \Leftrightarrow b_3 & c_3 & \Leftrightarrow d_3 \\ \chi_{w+1} & \Leftrightarrow b_{w+1} & c_{w+1} & \Leftrightarrow d_{w+1} \end{vmatrix} \begin{vmatrix} 1 & \Leftrightarrow 1 & 1 & \Leftrightarrow 1 \\ a_2 & \Leftrightarrow b_2 & c_2 & \Leftrightarrow d_2 \\ a_3 & \Leftrightarrow b_3 & c_3 & \Leftrightarrow d_3 \\ a_w & \Leftrightarrow b_w & c_w & \Leftrightarrow d_w \end{vmatrix} = 0,$$

for each $w \in \{4, 5, 6, 7\}$.

This scheme yields four equations, one for each possible w. Using Maple, these determinants are taken, and the following equations in the m_i emerge:

$$A(j,j+1)(m_1m_2m_3m_4)+B(j,j+1)(m_1m_2m_3+m_1m_2m_4+m_1m_3m_4+m_2m_3m_4)$$
$$+C(j,j+1)(m_1m_2+m_1m_3+m_1m_4+m_2m_3+m_2m_4+m_3m_4)$$
$$+D(j,j+1)(m_1+m_2+m_3+m_4)+E(j,j+1)=0,$$

for $j \in \{4, 5, 6, 7\}$, where

$$A(j,j+1) = \cos(\phi_2)\cos(\phi_3)\cos(\phi_j)\cos(\phi_{j+1})$$
$$[\chi_2\cos(\phi_2)^3\sin(\phi_{j+1}\Leftrightarrow\phi_j)\sin(\phi_j\Leftrightarrow\phi_3)\sin(\phi_{j+1}\Leftrightarrow\phi_3)$$
$$+\chi_3\cos(\phi_3)^3\sin(\phi_{j+1}\Leftrightarrow\phi_j)\sin(\phi_j\Leftrightarrow\phi_2)\sin(\phi_{j+1}\Leftrightarrow\phi_2)$$
$$+\chi_j\cos(\phi_j)^3\sin(\phi_{j+1}\Leftrightarrow\phi_3)\sin(\phi_3\Leftrightarrow\phi_2)\sin(\phi_{j+1}\Leftrightarrow\phi_2)$$
$$+\chi_{j+1}\cos(\phi_{j+1})^3\sin(\phi_j\Leftrightarrow\phi_3)\sin(\phi_3\Leftrightarrow\phi_2)\sin(\phi_j\Leftrightarrow\phi_2)],$$
$$B(j,j+1) = \cos(\phi_2)\cos(\phi_3)\cos(\phi_j)\cos(\phi_{j+1})$$
$$[\chi_2\cos(\phi_2)^2\sin(\phi_2)\sin(\phi_{j+1}\Leftrightarrow\phi_j)\sin(\phi_j\Leftrightarrow\phi_3)\sin(\phi_{j+1}\Leftrightarrow\phi_3)$$
$$\Leftrightarrow\chi_3\cos(\phi_3)^2\sin(\phi_3)\sin(\phi_{j+1}\Leftrightarrow\phi_j)\sin(\phi_j\Leftrightarrow\phi_2)\sin(\phi_{j+1}\Leftrightarrow\phi_2)$$
$$+\chi_j\cos(\phi_j)^2\sin(\phi_j)\sin(\phi_{j+1}\Leftrightarrow\phi_3)\sin(\phi_3\Leftrightarrow\phi_2)\sin(\phi_{j+1}\Leftrightarrow\phi_2)$$
$$\Leftrightarrow\chi_{j+1}\cos(\phi_{j+1})^2\sin(\phi_{j+1})\sin(\phi_j\Leftrightarrow\phi_3)\sin(\phi_3\Leftrightarrow\phi_2)\sin(\phi_j\Leftrightarrow\phi_2)],$$
$$C(j,j+1) = \cos(\phi_2)\cos(\phi_3)\cos(\phi_j)\cos(\phi_{j+1})$$
$$[\Leftrightarrow\chi_2\cos(\phi_2)\sin(\phi_2)^2\sin(\phi_{j+1}\Leftrightarrow\phi_j)\sin(\phi_{j+1}\Leftrightarrow\phi_3)\sin(\phi_j\Leftrightarrow\phi_3)$$
$$+\chi_3\cos(\phi_3)\sin(\phi_3)^2\sin(\phi_{j+1}\Leftrightarrow\phi_j)\sin(\phi_j\Leftrightarrow\phi_2)\sin(\phi_{j+1}\Leftrightarrow\phi_2)$$
$$\Leftrightarrow\chi_j\cos(\phi_j)\sin(\phi_j)^2\sin(\phi_{j+1}\Leftrightarrow\phi_3)\sin(\phi_3\Leftrightarrow\phi_2)\sin(\phi_{j+1}\Leftrightarrow\phi_2)$$
$$+\chi_{j+1}\cos(\phi_{j+1})\sin(\phi_{j+1})^2\sin(\phi_j\Leftrightarrow\phi_3)\sin(\phi_3\Leftrightarrow\phi_2)\sin(\phi_j\Leftrightarrow\phi_2)],$$
$$D(j,j+1) = \cos(\phi_2)\cos(\phi_3)\cos(\phi_j)\cos(\phi_{j+1})$$
$$[\chi_2\sin(\phi_2)^3\sin(\phi_{j+1}\Leftrightarrow\phi_j)\sin(\phi_{j+1}\Leftrightarrow\phi_3)\sin(\phi_j\Leftrightarrow\phi_3)$$
$$+\chi_3\sin(\phi_3)^3\sin(\phi_{j+1}\Leftrightarrow\phi_j)\sin(\phi_j\Leftrightarrow\phi_2)\sin(\phi_{j+1}\Leftrightarrow\phi_2)$$
$$+\chi_j\sin(\phi_j)^3\sin(\phi_{j+1}\Leftrightarrow\phi_3)\sin(\phi_3\Leftrightarrow\phi_2)\sin(\phi_{j+1}\Leftrightarrow\phi_2)$$
$$+\chi_{j+1}\sin(\phi_{j+1})^3\sin(\phi_j\Leftrightarrow\phi_3)\sin(\phi_3\Leftrightarrow\phi_2)\sin(\phi_j\Leftrightarrow\phi_2)],$$

and

$$E(j,j+1) = \chi_1\sin(\phi_{j+1}\Leftrightarrow\phi_j)\sin(\phi_{j+1}\Leftrightarrow\phi_3)\sin(\phi_j\Leftrightarrow\phi_3)\sin(\phi_{j+1}\Leftrightarrow\phi_2)$$
$$\sin(\phi_j\Leftrightarrow\phi_2)\sin(\phi_3\Leftrightarrow\phi_2)$$

$$
\begin{aligned}
&\Leftrightarrow \chi_2 \sin(\phi_2)^4 \cos(\phi_3) \cos(\phi_j) \cos(\phi_{j+1}) \sin(\phi_{j+1} \Leftrightarrow \phi_3) \sin(\phi_j \Leftrightarrow \phi_3) \sin(\phi_{j+1} \Leftrightarrow \phi_j) \\
&+\chi_3 \sin(\phi_3)^4 \cos(\phi_2) \cos(\phi_j) \cos(\phi_{j+1}) \sin(\phi_{j+1} \Leftrightarrow \phi_j) \sin(\phi_j \Leftrightarrow \phi_2) \sin(\phi_{j+1} \Leftrightarrow \phi_2) \\
&\Leftrightarrow \chi_j \sin(\phi_4)^4 \cos(\phi_2) \cos(\phi_3) \cos(\phi_{j+1}) \sin(\phi_{j+1} \Leftrightarrow \phi_3) \sin(\phi_{j+1} \Leftrightarrow \phi_2) \sin(\phi_3 \Leftrightarrow \phi_2) \\
&+\chi_{j+1} \sin(\phi_{j+1})^4 \cos(\phi_2) \cos(\phi_3) \cos(\phi_j) \sin(\phi_j \Leftrightarrow \phi_3) \sin(\phi_j \Leftrightarrow \phi_2) \sin(\phi_3 \Leftrightarrow \phi_2).
\end{aligned}
$$

We think, but do not know whether Cramer's rule may be used again to solve for the symmetric polynomials in the m_i. However, assuming it could, finishing the reconstruction would amount to solving a quartic polynomial in one of the m_i and ordering the m_i. This polynomial will yield four distinct roots since, by 4.2, the wedge is uniquely determined. Furthermore, no attempts have been made at implementing this reconstruction, but it seems as if one sought a generalization, this technique may prove to be computationally difficult. Likely, a least-squares method of reconstruction, or using Newton's method to solve for the roots would be more efficient.

6 Conclusion

We have proven that a generic polygon with no parallel wedges is uniquely determined by its x-ray from a point source. We can uniquely determine the arrangement of a nonparallel n wedge cone from either $4n$ x-rays or $4n \Leftrightarrow 1$ derivatives at a point and the x-ray at the same point. An area for further study would be whether $4n$ total x-rays and derivatives from points in the wedge are enough to uniquely determine a nonparallel n-wedge cone in general.

The results proved here generalize to a collection of polygons in the plane, as we never needed the assumption that the polygon was connected. The shortcoming of these results is that we require *apriori* knowledge of how many wedges will appear in a cone. Determination of how many wedges are in a cornerless cone from x-ray data is an open problem.

References

[1] R. J. Gardner. X-rays of polygons. *Discret and Computational Geometry*, pages 281–293, 1992.

[2] Richard J. Gardner. *Geometric Tomography*. Cambridge University Press, New York, 1995.

[3] Dorothy Lam and Donald Solmon. Reconstructing polygons in the plane from one directed x-ray. Submitted to Journal of Discrete and Computational Geometry., 1999.