Properties of Planar Convex Bodies from One Directed X-ray

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1 Introduction

The motivation for this paper was to make progress in the field of geometric tomography toward solving Hammer's x-ray problem. Geometric tomography is the study of convex bodies and their idealized x-rays. Thus, factors like density are ignored when taking an x-ray of a convex body. This paper focuses only on directed x-rays. For more information on geometric tomography including both parallel and directed x-rays, consult Gardner [1].

In this paper, we analyze and characterize convex bodies with the same x-ray data from a single point source. First, we derive various families of convex bodies through parameterizations and by adding x-ray data to a line. A second goal is to improve the method for determining the convexity of a body by extending the quadratic form introduced by Lam and Solmon [2]. Finally, we find conditions that determine which x-rays can be associated with a convex body away from the source and what restrictions the x-ray data puts on the convex body.

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2 Notation and Definitions

Definition 2.1 *A* **convex body** *is a compact, convex subset of the plane with non-empty interior.*

Definition 2.2 *The* **characteristic function** *of a convex body K is*

$$\chi_K(p) = \begin{cases} 0 \text{ if } p \notin K \\ 1 \text{ if } p \in K \end{cases} \text{ for } p \in \mathbf{E}^2$$

Throughout the paper, K is assumed to be a convex body. Unless otherwise specified, points will be in polar coordinates (r, φ) where $r \geq 0$ and φ is measured counterclockwise from the positive x-axis.

Definition 2.3 *A* **directed x-ray transform**, *also known as a point source x-ray or a fan-beam x-ray, gives the chord length of the convex body along a particular direction θ from a given point O, which is called the* **point source**. *Thus, the directed x-ray of K, $\mathcal{D}_{K_O}$ is*

$$\mathcal{D}_{K_O}(\varphi) = \int_0^\infty \chi_K(O + t\theta)dt, \;\; \theta =< \cos\varphi, \sin\varphi >$$

Since we are only considering a single source for the directed x-rays, we assume, without loss of generality, that the source, O, is at the origin.

From a single point source, O, we can define the convex body K by a unique pair of functions $r(\varphi), R(\varphi)$ satisfying

$$K = \{(s, \varphi) : r(\varphi) \leq s \leq R(\varphi)\}$$

We refer to the points with polar coordinates $(r(\varphi), \varphi)$ as points on the near boundary of K and the function $r(\varphi)$ as the near boundary function of K. In a similar manner, the terms far boundary and far boundary function refer to the points $(R(\varphi), \varphi)$ and the function $R(\varphi)$. As a convention throughout this paper, we use lowercase letters to refer to the near boundary and uppercase letters to refer to the far boundary. With this notation, the directed x-ray of K in the direction $\theta =< \cos\varphi, \sin\varphi >$ becomes $X(\varphi) = R(\varphi) - r(\varphi)$. Throughout the paper, we refer to these functions simply as X,R, and r.

Definition 2.4 *Let K be a convex body with an associated directed x-ray from point O. Then the* **supporting cone** *$\mathcal{C}[\alpha, \beta]$ of K is the cone with vertex at O such that for all φ, $\alpha < \varphi < \beta$, $\mathcal{D}_{K_O}(\varphi) > 0$. The* **supporting rays** *of K are $\varphi = \alpha$ and $\varphi = \beta$.*

Throughout the paper, the supporting rays are labeled α and β where $\alpha < \beta$. Since K is bounded away from O, $\beta - \alpha < \pi$, and we can always choose coordinates such that K lies above the y-axis.

We would to be able to determine whether a near or far side function is convex the right way. One method that is extremely useful in determining this is the quadratic form defined by Lam and Solmon [2].

Definition 2.5 *Given three points on the curve x, $\{x_1, x_2, x_3\}$, with associated angles $\varphi_1, \varphi_2, \varphi_3$ such that $\varphi_1 < \varphi_2 < \varphi_3 < \varphi_1 + \pi$, we define the quadratic form, $\mathcal{Q}(x)$, as*

$$\mathcal{Q}(x) = \mathcal{Q}(x_1, x_2, x_3) = x_1 x_2 \sin(\varphi_2 - \varphi_1) + x_2 x_3 \sin(\varphi_3 - \varphi_2) - x_1 x_3 \sin(\varphi_3 - \varphi_1)$$

where x_i is a function of φ_i.

This form can be written $\mathcal{Q}(x) = xAx^T$, where

$$A = \frac{1}{2}\begin{pmatrix} 0 & \sin(\varphi_2 - \varphi_1) & -\sin(\varphi_3 - \varphi_1) \\ \sin(\varphi_2 - \varphi_1) & 0 & \sin(\varphi_3 - \varphi_2) \\ -\sin(\varphi_3 - \varphi_1) & \sin(\varphi_3 - \varphi_2) & 0 \end{pmatrix}$$

and x is the vector $< x_1, x_2, x_3 >$. Both forms appear in the paper, but the second expression is useful in expressing the middle term when expanding $\mathcal{Q}(r+s) = \mathcal{Q}(r) + 2rAs^T + \mathcal{Q}(s)$.

From Lam and Solmon [2], we find some important properties of $\mathcal{Q}$. A bounded function r on $[\alpha, \beta]$, $0 < \beta - \alpha < \pi$, is the near side of a convex body K if and only if for all $\alpha \leq \varphi_1 < \varphi_2 < \varphi_3 \leq \beta$, $\mathcal{Q}(r) \leq 0$. Similarly, a bounded function R on $[\alpha, \beta]$ is the far side of K if and only if $\mathcal{Q}(R) \geq 0$. This allows us to uniquely identify any convex body bounded away from the source O with supporting cone $\mathcal{C}[\alpha, \beta]$ with a pair of bounded functions (r, R) satisfying

i) $0 < r(\varphi) \leq R(\varphi)$ on $[\alpha, \beta]$ with $r < R$ on (α, β)

ii) $\mathcal{Q}(r) \leq 0$ and $\mathcal{Q}(R) \geq 0$ on $[\alpha, \beta]$

Having made this identification, given the source O, we may refer to K as the pair of functions (r, R) satisfying i and ii above.

3 Properties of $\mathcal{Q}$

In this section, we introduce some useful properties associated with $\mathcal{Q}$.

Theorem 3.1 *The quadratic form $\mathcal{Q}$ satisfies the following*

i) *$\mathcal{Q}(tx) = t^2\mathcal{Q}(x)$ for all functions x and scalars t.*

ii) *If $\mathcal{Q}(R) \geq 0$, $\mathcal{Q}(s) \leq 0$, and $R \geq s$, then $\mathcal{Q}(R - s) \geq 0$.*

iii) *If $\mathcal{Q}(r) \leq 0$ and $\mathcal{Q}(s) \leq 0$, then $\mathcal{Q}(r + s) \leq 0$.*

Proof. i) is trivial. ii) was proved by Lam and Solmon [2]. To prove iii), we first expand

$$\mathcal{Q}(r + s) = \mathcal{Q}(r) + \mathcal{Q}(s) + 2rAs^T$$

where

$$2rAs^T = (r_1 s_2 + r_2 s_1)\sin(\varphi_2 - \varphi_1) + (r_2 s_3 + r_3 s_2)\sin(\varphi_3 - \varphi_2) - (r_1 s_3 + r_3 s_1)\sin(\varphi_3 - \varphi_1). \quad (1)$$

Since $\mathcal{Q}(r) \leq 0$, we know that

$$r_2 \leq \frac{r_1 r_3 \sin(\varphi_3 - \varphi_1)}{r_1 \sin(\varphi_2 - \varphi_1) + r_3 \sin(\varphi_3 - \varphi_2)}$$

and similarly because $\mathcal{Q}(s) \leq 0$,

$$s_2 \leq \frac{s_1 s_3 \sin(\varphi_3 - \varphi_1)}{s_1 \sin(\varphi_2 - \varphi_1) + s_3 \sin(\varphi_3 - \varphi_2)}.$$

Substituting into (1), we find that $2rAs^T \leq$

$$-\frac{\sin(\varphi_2 - \varphi_1)\sin(\varphi_3 - \varphi_2)\sin(\varphi_3 - \varphi_1)(r_1 s_3 - s_3 r_1)^2}{(r_1 \sin(\varphi_2 - \varphi_1) + r_3 \sin(\varphi_3 - \varphi_2))(s_1 \sin(\varphi_2 - \varphi_1) + s_3 \sin(\varphi_3 - \varphi_2))} \leq 0$$

Since this expression is always less than or equal to 0 and $\mathcal{Q}(r)$, $\mathcal{Q}(s) \leq 0$ as given, $\mathcal{Q}(r+s) \leq 0$. □

The first result of this theorem is useful in computations of $\mathcal{Q}$. The second result implies that the directed x-ray of a convex body is itself the far side of a convex body with source O, a result of Longinetti [3]. The last result of this theorem provides a property that allows the comparison of combinations of the near sides of two different convex bodies.

4 Some convex bodies with a common directed x-ray

We begin with a convex body K bounded away from O and its associated x-ray data X from a given source O. Then $K = (R, r)$ where $\mathcal{Q}(r) \leq 0$ and $\mathcal{Q}(R) \geq 0$. In addition, we can associate a convex body M with the x-ray data of K as $(0, X)$ since $\mathcal{Q}(0) = 0$ and $\mathcal{Q}(X) \geq 0$.

4.1 Linear Combinations of $(0, X)$ and (r, R)

Given a source O and convex bodies $K = (r, R)$ and $M = (0, X)$, we define a linear function L as

$$L(t) = (1-t)(0, X) + t(r, R) = (tr, (1-t)X + tR)$$

$L(t)$ is a body with near side tr, the result of scaling r, but the far edge $(1-t)X + tR$ is not necessarily convex and may look different from R, especially as t approaches 0.

Theorem 4.1 *For all $t \in [0, 1]$, $L(t)$ is always a convex body with x-ray data $X = R - r$.*

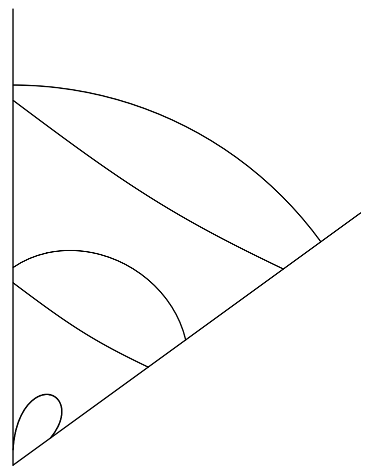

Figure 1: The convex bodies $L(1)$ and $L(0.5)$ and their x-ray data

PROOF. First, we show that the x-ray data is preserved for all t. Since (r, R) has directed x-ray data X, we can write $R = r + X$. Substituting into $L(t)$, we find

$$L(t) = (tr, (1-t)X + tR) = (tr, (1-t)X + t(r+X)) = (tr, tr + X)$$

which certainly has x-ray data X.

Second, we must show that the new body $L(t)$ is convex for $t \in [0,1]$. Hence, we must show that $\mathcal{Q}(tr) \leq 0$ and $\mathcal{Q}(tr + X) \geq 0$. The first condition is trivial since $\mathcal{Q}(tr) = t^2 \mathcal{Q}(r)$, but the second condition is more difficult. We know that at $t = 0$, $\mathcal{Q}(tr + X) = \mathcal{Q}(X) \geq 0$ and at $t = 1$, $\mathcal{Q}(tr + X) = \mathcal{Q}(R) \geq 0$. Expanding gives

$$\mathcal{Q}(tr + X) = t^2 \mathcal{Q}(r) + 2trAX^T + Q(X)$$

which is a quadratic with leading coefficient $\mathcal{Q}(r)$. By hypothesis, $\mathcal{Q}(r) \leq 0$, so the parabola must open down, and since at $t = 0$ and $t = 1$, $\mathcal{Q}(tr + X) \geq 0$, $\mathcal{Q}(tr + X) \geq 0$ for all $t \in [0,1]$. □

It should be noted that of the three necessary conditions for the body $(tr, tr + X)$ to satisfy the above theorem, two hold for all t while the other always holds for $t \in [0,1]$ but not necessarily for other t. Both the preservation of x-ray data and convexity of the near side tr hold for all t. The convexity of $tr + X$ restricts the interval for t in most cases. We can, in fact, show that this interval is bounded, provided $\mathcal{Q}(r) \neq 0$. In some cases, the interval is bounded when $\mathcal{Q}(r) = 0$.

Theorem 4.2 *There exists some $t_0 \geq 1$ such that $\mathcal{Q}(t_0 r + X) < 0$, provided that either*

i) $\mathcal{Q}(r) < 0$ *for* $\varphi_1 < \varphi_2 < \varphi_3 < \varphi_1 + \pi$*, or*

ii) $\mathcal{Q}(r) = 0$ *and* $\mathcal{Q}(X) > \mathcal{Q}(R)$.

PROOF. First, we consider the condition in part i. Since $\mathcal{Q}(tr+X)$ is quadratic and the coefficient of t^2, $\mathcal{Q}(r)$, is always less than 0, $\mathcal{Q}(tr+X)$ is a parabola that opens down. In addition, $\mathcal{Q}(X)$,$\mathcal{Q}(R) \geq 0$, so because $\mathcal{Q}(r) \neq 0$, the quadratic must have two roots, one of which is greater than 1. Call this root t^*. Then for $t_0 > t^*$,

$$\mathcal{Q}(t_0 r + X) < \mathcal{Q}(t^* r + X) = 0$$

Now, we consider the second case. Here the quadratic reduces to a linear expression so

$$\mathcal{Q}(tr + X) = 2trAX^T + \mathcal{Q}(X).$$

At $t = 0$, $\mathcal{Q}(tr + X) = \mathcal{Q}(X)$, and at $t = 1$, $\mathcal{Q}(tr + X) = \mathcal{Q}(R)$. Since $\mathcal{Q}(X) > \mathcal{Q}(R)$ by hypothesis, this line must have negative slope and therefore must intersect the x-axis at some T^*. So for $t_0 > T^*$,

$$\mathcal{Q}(t_0 r + X) < \mathcal{Q}(T^* r + X) = 0$$

Therefore, there exists some $t_0 \geq 1$ such that $\mathcal{Q}(t_0 r + X) < 0$ and $L(t)$ is not convex for $t \geq t_0$. □

4.2 Extending Convex Bodies from (r, R)

In the previous section, we showed that it is possible to construct convex bodies that have the same x-rays by taking a certain linear combination of $(0, X)$ and (r, R). This involved scaling the near side and constructing the far side by adding the x-ray data to the scaled near side. In this section, we examine what happens when we scale the far side and construct the near side by subtracting the x-ray data.

Given a source O and a convex body $K = (r, R)$ with x-ray data $X = R - r$, we define a function G so that

$$G(s) = (s-1)(X, 0) + s(r, R) = (sR - X, sR)$$

Theorem 4.3 *If* $\mathcal{Q}(R) = 0$*, then* $G(s)$ *is a convex body with x-ray data* $X = R - r$ *for all* $s \geq 1$.

PROOF. It is obvious that the x-ray data is preserved in this parameterization, so we address convexity. Again, in order for $G(s)$ to be convex, $\mathcal{Q}(sR) \geq 0$ and $\mathcal{Q}(sR - X) \leq 0$. Since $\mathcal{Q}(sR) = s^2 Q(R) = 0$ holds for all s, we are left with only the last condition. Rewriting,

$$\mathcal{Q}(sR - X) = \mathcal{Q}(sR - (R - r)) = \mathcal{Q}((s-1)R + r)$$

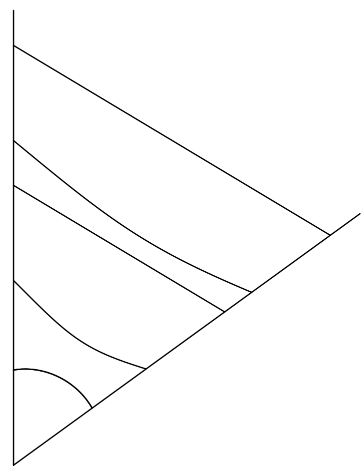

Figure 2: The convex bodies $G(1)$ and $G(1.5)$ and their x-ray data

Because $s \geq 1 \implies s-1 \geq 0$, let $s^* = s-1 \geq 0$. Then, $\mathcal{Q}((s-1)R+r) = \mathcal{Q}(s^*R+r)$, and because $\mathcal{Q}(s^*R) = 0$ and $\mathcal{Q}(r) \leq 0$, by part iii of Theorem 3.1, $\mathcal{Q}(s^*R+r) \leq 0$. □

Notice that we must restrict R to be a line in order to guarantee convexity for $G(s)$ for all $s \geq 1$. If we do not, we are left with a quadratic with leading coefficient $\mathcal{Q}(R) \geq 0$. That implies that graph of $\mathcal{Q}(sR-X)$ is a parabola that opens up. Because $\mathcal{Q}(sR-X) = \mathcal{Q}(r) \leq 0$ at $s=1$, we know that there is some $s' \geq 1$ such that $\mathcal{Q}(s'R-X) = 0$, and therefore $G(s)$ will not be convex for all $s > s'$.

4.3 Combining the two parameterizations

Using both of the parameterizations in the above sections, we can construct a two-parameter family of convex bodies with the same directed x-ray from source O. We begin with the family $(sR-X, sR)$ and apply the first parameterization $(tr, tr+X)$ by substituting $r = sR-X$ to produce the family

$$H(s,t) = (t(sR-X), t(sR-X)+X) = (tsR-tX, tsR+(1-t)X)$$

Again, the x-ray data is preserved, while convexity is limited by the s and t parameters.

5 The Differential Operator $\mathcal{K}$

One of the problems with the quadratic form is that it requires three points to compute the concavity of a curve. It is also limited when analyzing behavior

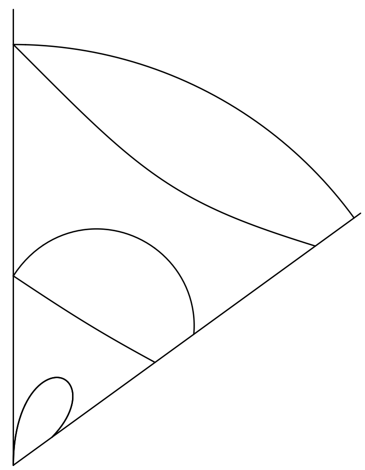

Figure 3: The convex bodies $H(1,1)$ and $H(1.5,0.3)$ and their x-ray data

if $\varphi_i = \varphi_j$ for $i < j$ since it is always 0. Thus, it becomes apparent that a differential operator that requires only information from one point is superior for many applications. While such an operator requires the curve be twice differentiable, the curves we will be examining have only a small number of points where this is an issue.

We derive the differential operator $\mathcal{K}$ from the quadratic form $\mathcal{Q}$ by taking limits as $\varphi_1, \varphi_3 \to \varphi_2$. To accomplish this, we first compute $\mathcal{Q}^*(x) = \lim_{\varphi_1 \to \varphi_2} \frac{\mathcal{Q}(x)}{\varphi_2 - \varphi_1}$. By L'hospital's Rule, $\mathcal{Q}^*(x)$

$$
\begin{aligned}
&= \lim_{\varphi_1 \to \varphi_2} \frac{x_1 x_2 \sin(\varphi_2 - \varphi_1) + x_2 x_3 \sin(\varphi_3 - \varphi_2) - x_1 x_3 \sin(\varphi_3 - \varphi_1)}{\varphi_2 - \varphi_1} \\
&= \lim_{\varphi_1 \to \varphi_2} x_1 x_2 \frac{\sin(\varphi_2 - \varphi_1)}{\varphi_2 - \varphi_1} + \frac{\frac{d}{d\varphi_1}(x_2 x_3 \sin(\varphi_3 - \varphi_2) - x_1 x_3 \sin(\varphi_3 - \varphi_1))}{\frac{d}{d\varphi_1}(\varphi_2 - \varphi_1)} \\
&= \lim_{\varphi_1 \to \varphi_2} x_1 x_2 \frac{\sin(\varphi_2 - \varphi_1)}{\varphi_2 - \varphi_1} + \lim_{\varphi_1 \to \varphi_2} (x_1' x_3 \sin(\varphi_3 - \varphi_1) - x_1 x_3 \cos(\varphi_3 - \varphi_1)) \\
&= x_2^2 + x_2' x_3 \sin(\varphi_3 - \varphi_2) - x_2 x_3 \cos(\varphi_3 - \varphi_2).
\end{aligned}
$$

After this we take the $\lim_{\varphi_3\to\varphi_2} \frac{\mathcal{Q}^*(x)}{(\varphi_3-\varphi_2)^2}$

$$
\begin{aligned}
&= \lim_{\varphi_3\to\varphi_2} \frac{x_2^2 + x_2' x_3 \sin(\varphi_3-\varphi_2) - x_2 x_3 \cos(\varphi_3-\varphi_2)}{(\varphi_3-\varphi_2)^2} \\
&= \lim_{\varphi_3\to\varphi_2} \frac{(x_2' x_3' + x_2 x_3)\sin(\varphi_3-\varphi_2) + (x_2' x_3 - x_2 x_3')\cos(\varphi_3-\varphi_2)}{2(\varphi_3-\varphi_2)} \\
&= \frac{(x_2')^2 + x_2^2}{2} + 1/2 \lim_{\varphi_3\to\varphi_2} (x_3' x_2' - x_2 x_3'')\cos(\varphi_3-\varphi_2) \\
&= \frac{(x_2')^2 + x_2^2}{2} + \frac{(x_2')^2 - x_2 x_2''}{2} \\
&= (x_2^2 + 2(x_2')^2 - x_2 x_2'')/2
\end{aligned}
$$

Since the constant is unimportant when considering whether the operator is positive or negative, we define $\mathcal{K}(x) = x^2 + 2(x')^2 - x''x$.

Remark This derivation of $\mathcal{K}$ assumes that the curve x is $\mathcal{C}^2$, but the limit formula will hold except on a set of zero Lebesgue measure for curves satisfying $\mathcal{Q}(r) \leq 0$ or $\mathcal{Q}(R) \geq 0$.

Remark Since we apply the differential operator to sums, we derive $\mathcal{K}(x+y)$ now.

$$
\begin{aligned}
\mathcal{K}(x+y) &= (x+y)^2 + 2((x+y)')^2 - (x+y)(x+y)'' \\
&= \mathcal{K}(x) + (2xy + 4x'y' - xy'' - x''y) + \mathcal{K}(y)
\end{aligned}
$$

Next, we establish conditions for concavity similar to those $\mathcal{Q}$ produces by showing that for φ in some interval (α, β), when $\mathcal{K}(x) \geq 0$, the graph of the polar function $x(\varphi)$ is concave toward the source and when $\mathcal{K}(x) \leq 0$, the graph of x is concave away from the source.

Theorem 5.1 *Let $g(\varphi)$ be a C^2 polar function on (α,β) and continuous on $[\alpha,\beta]$, $0 < \beta-\alpha < \pi$. Define the parametric curve $\Gamma =< g(\varphi)\cos\varphi, g(\varphi)\sin\varphi >$. Then,*

i) *Γ is concave toward the origin if and only if $\mathcal{K}(g)(\varphi) \geq 0$ for all $\varphi \in (\alpha,\beta)$*

ii) *Γ is concave away from the origin if and only if $\mathcal{K}(g)(\varphi) \leq 0$ for all $\varphi \in (\alpha,\beta)$.*

PROOF. First, let $\theta =< \cos\varphi, \sin\varphi >$ and $\theta^\perp =< -\sin\varphi, \cos\varphi >$. Consider the polar parametric curve

$$\Gamma =< g(\varphi)\cos\varphi, g(\varphi)\sin\varphi >= g(\varphi)\theta.$$

Then,

$$\begin{aligned}
\Gamma' &= < g'(\varphi)\cos\varphi - g(\varphi)\sin\varphi, g'(\varphi)\sin\varphi + g(\varphi)\cos\varphi > \\
&= g'(\varphi) < \cos\varphi, \sin\varphi > + g(\varphi) < -\sin\varphi, \cos\varphi > \\
&= g'(\varphi)\theta + g(\varphi)\theta^{\perp} \\
\Gamma'' &= g''(\varphi)\theta + g'(\varphi)\theta^{\perp} + g'(\varphi)\theta^{\perp} - g(\varphi)\theta \\
&= \Gamma'' = (g''(\varphi) - g(\varphi))\theta + 2g'(\varphi)\theta^{\perp}
\end{aligned}$$

A simple cross product yields

$$\begin{aligned}
\Gamma' \times \Gamma'' &= (g'(\varphi)\theta + g(\varphi)\theta^{\perp}) \times (g''(\varphi) - g(\varphi))\theta + 2g'(\varphi)\theta^{\perp} \\
&= 2[g'(\varphi)]^2(\theta \times \theta^{\perp}) + g(\varphi)(g''(\varphi) - g(\varphi)(\theta^{\perp} \times \theta) \\
&= 2[g'(\varphi)]^2 k + (g^2(\varphi) - g(\varphi)g''(\varphi))k \\
&= \mathcal{K}(g)(\varphi)k
\end{aligned}$$

where k is the unit vector in the positive z direction. Recall from vector calculus that the second derivative of a parametric curve Γ is called the acceleration vector and can be written as

$$\Gamma'' = \frac{dv}{d\varphi}\vec{T} + \kappa v^2 \vec{n} \tag{2}$$

where $v = \|\Gamma'(\varphi)\|$,κ is the curvature of Γ, $\vec{T} = \frac{\Gamma'}{\|\Gamma'\|}$ is the unit tangent vector, and $\vec{n}$ is the principal unit vector ($\vec{n}$ always points to the side of the curve which is concave). Observe that $\vec{T} = \frac{\Gamma'}{\|\Gamma'\|} \implies$

$$\Gamma'(\varphi) = \|\Gamma'\|\vec{T} = v\vec{T} \tag{3}$$

From equations (2) and (3), we can write

$$\begin{aligned}
\Gamma' \times \Gamma'' &= v\vec{T} \times (\frac{dv}{d\varphi}\vec{T} + \kappa v^2 \vec{n}) \\
&= \kappa v^3(\vec{T} \times \vec{n})
\end{aligned}$$

By definition $\kappa \geq 0$ and since $v = \|\Gamma'(\varphi)\| \geq 0$, $v^3 \geq 0 \implies \kappa v^3 \geq 0$. Furthermore, since $\vec{T}$ is perpendicular to $\vec{n}$ and they both are unit vectors, we know $\vec{T} \times \vec{n} = \pm k$. Thus, $\Gamma' \times \Gamma'' = \pm\kappa v^3 k = \mathcal{K}(g)(\varphi)k \implies$

$$\mathcal{K}(g)(\varphi) \geq 0 \iff \vec{T} \times \vec{n} = k$$

and

$$\mathcal{K}(g)(\varphi) \leq 0 \iff \vec{T} \times \vec{n} = -k$$

By the right hand rule $\vec{T} \times \vec{n} = k$ iff $\vec{n}$ points toward the origin and the curve Γ is concave toward the origin, and $\vec{T} \times \vec{n} = -k$ iff $\vec{n}$ points away from the origin and the curve Γ is concave away from the origin. □

Remark One interesting and logical property of the differential operator $\mathcal{K}$ is that it is related to the formula for curvature κ. Deriving this formula in polar coordinates yields

$$\kappa(x) = \frac{x^2 + 2(x')^2 - x''x}{(x^2 + (x')^2)^{3/2}}$$

Since the denominator is always positive, the numerator of the formula determines whether the curve has positive or negative curvature. But this numerator is exactly the differential operator $\mathcal{K}$.

6 Creating convex bodies by adding lines to x-ray data

Lemma 6.1 *Let $g(\varphi)$ and $h(\varphi)$ be C^1 functions on (α, β). Then the tangent lines of g and h are parallel at φ_0 if and only if $\frac{g'}{g} = \frac{h'}{h}$ for some $\varphi_0 \in (\alpha, \beta)$.*

PROOF. Parameterize $g(\varphi)$ as $y = g(\varphi)\sin(\varphi)$,$x = g(\varphi)\cos(\varphi)$. Then by the chain rule the slope of the tangent line of g at φ, $m_g(\varphi)$ is

$$\begin{aligned} m_g(\varphi) &= \frac{dy}{dx} = \frac{\frac{dy}{d\varphi}}{\frac{dx}{d\varphi}} \\ &= \frac{g'(\varphi)\sin\varphi + g(\varphi)\cos\varphi}{g'(\varphi)\cos\varphi - g(\varphi)\sin\varphi} \\ &= \frac{\frac{g'}{g}\sin\varphi + \cos\varphi}{\frac{g'}{g}\cos\varphi - \sin\varphi} \end{aligned}$$

Similarly the slope of the tangent line of h at φ is

$$m_h(\varphi) = \frac{\frac{h'}{h}\sin\varphi + \cos\varphi}{\frac{h'}{h}\cos\varphi - \sin\varphi}$$

If $\frac{g'}{g} = \frac{h'}{h}$ at some φ_0, then clearly $m_g(\varphi_0) = m_h(\varphi_0)$. Conversely, if the slopes of the tangent lines are parallel at φ_0, then

$$\begin{aligned} m_g(\varphi_0) = m_h(\varphi_0) &\iff \frac{\frac{g'}{g}\sin\varphi_0 + \cos\varphi_0}{\frac{g'}{g}\cos\varphi_0 - \sin\varphi_0} = \frac{\frac{h'}{h}\sin\varphi_0 + \cos\varphi_0}{\frac{h'}{h}\cos\varphi_0 - \sin\varphi_0} \\ &\iff -\frac{g'}{g}\sin^2\varphi_0 + \frac{h'}{h}\cos^2\varphi_0 = -\frac{h'}{h}\sin^2\varphi_0 + \frac{g'}{g}\cos^2\varphi_0 \\ &\iff \frac{h'}{h}(\sin^2\varphi_0 + \cos^2\varphi_0) = \frac{g'}{g}(\sin^2\varphi_0 + \cos^2\varphi_0) \\ &\iff \frac{h'}{h} = \frac{g'}{g} \text{ at } \varphi = \varphi_0 \end{aligned}$$

□

Lemma 6.2 *If X is the x-ray data from a non-parallel wedge $W = (r, R)$, then there exists a $\delta > 0$ such that $\mathcal{K}(x) > \delta$ for all $\varphi \in [\alpha, \beta]$.*

PROOF. Since $W = (r, R)$ is a non-parallel wedge, $\mathcal{K}(r) = 0$ and $\mathcal{K}(R) = 0$ for all $\varphi \in [\alpha, \beta]$. Hence

$$r'' = r + 2\frac{(r')^2}{r} \tag{4}$$

and

$$R'' = R + 2\frac{(R')^2}{R} \tag{5}$$

for all $\varphi \in [\alpha, \beta]$. Because $X = R - r$, we can expand $\mathcal{K}(X)$ as

$$\begin{aligned}\mathcal{K}(R - r) &= \mathcal{K}(R) - (2rR + 4r'R' - rR'' - r''R) + \mathcal{K}(r)\\ &= -(2rR + 4r'R' - rR'' - r''R)\end{aligned}$$

Substituting (4) and (5), we find

$$\begin{aligned}\mathcal{K}(X) &= -(2rR + 4r'R' - r\left(R + 2\frac{(R')^2}{R}\right) - R\left(r + 2\frac{(r')^2}{r}\right)\\ &= \frac{1}{rR}\left(-2r^2R^2 - 4rr'RR' + r^2R^2 + 2r^2(R')^2 + R^2r^2 + 2R^2(r')^2\right)\\ &= \frac{1}{rR}(rR' - r'R)^2 \geq 0\end{aligned}$$

Equality holds if and only if $rR' = r'R$ in which case R is parallel to r and the wedge is a parallel wedge by Lemma 6.1. Contradiction. The existence of δ follows since $\mathcal{K}(X)$ is continuous on $[\alpha, \beta]$ by the Extreme Value Theorem. □

Remark To ensure that $\mathcal{K}$ is defined on the closed interval $[\alpha, \beta]$, we make the following definitions. We define $\mathcal{K}(x)(\alpha) = x^2 + 2(x'_-)^2 - xx''_-$ and $\mathcal{K}(x)(\beta) = x^2 + 2(x'_+)^2 - xx''_+$ where x'_- and x''_- denote the left hand first and second derivatives and x'_+ and x''_+ the right hand derivatives.

Theorem 6.3 *Given a non-parallel wedge $W = (r, R)$ with x-ray data $X = R - r$, for any line l there exists a $t_0 > 0$ such that $W' = (t_0 l, t_0 l + X)$ is a convex body provided l intersects both α and β away from the source.*

PROOF. Since W is a non-parallel wedge, we know that $\mathcal{K}(r) = \mathcal{K}(R) = 0$ so (4) and (5) from the previous lemma hold. In order to show $W' = (t_0 l, t_0 l + X)$ is convex we must show there exists a $t_0 > 0$ such that $\mathcal{K}(t_0 l) \leq 0$ and $\mathcal{K}(t_0 l + X) \geq 0$. Since l is a line, $\mathcal{K}(tl) = t^2\mathcal{K}(l) = 0$ for all $t \in \mathbb{R}$. So

$$\begin{aligned}\mathcal{K}(tl + X) &= \mathcal{K}(tl) + t(2lX + 4l'X' - l''X - lX'') + \mathcal{K}(X)\\ &= t(2lX + 4l'X' - l''X - lX'') + \mathcal{K}(X)\end{aligned}$$

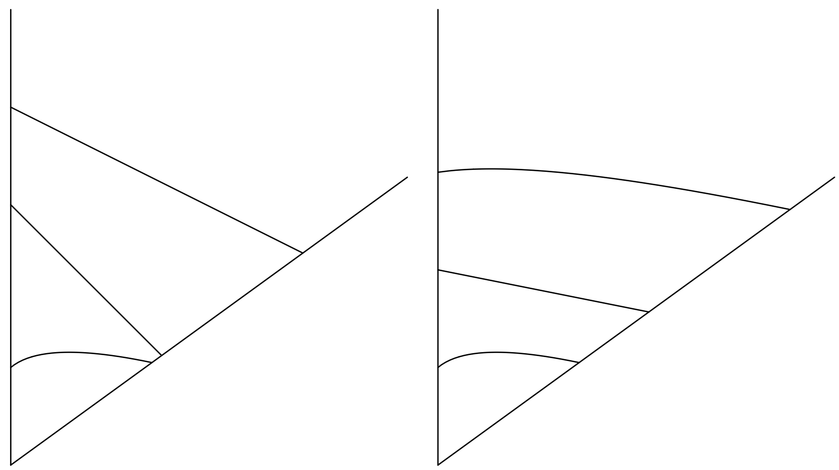

Figure 4: A wedge W and the result of adding its x-ray data to a line

Substituting (4) and (5), we find

$$\mathcal{K}(tl + X) = -\frac{t}{lX}(l'X - lX')^2 + \mathcal{K}(X)$$

We now have a linear expression in t with a y-intercept greater than $\delta > 0$. By continuity there exists a $t_0 > 0$ such that $\mathcal{K}(t_0 l + X) > 0$. □

Theorem 6.4 *Given a non-parallel wedge $W = (r, R)$ with x-ray data $X = R - r$, then for each line L whose slope is sufficiently close to R, there exists an s_0 such that $W' = (s_0 L - X, s_0 L)$ is a convex body with the same directed x-ray as W.*

PROOF. To show $W' = (s_0 L - X, s_0 L)$ is a convex body, we need to show there exists an $s_0 > 0$ such that $\mathcal{K}(s_0 L - X) \leq 0$ and $\mathcal{K}(s_0 L) \geq 0$.

Since W is a non-parallel wedge, (4) and (5) from Lemma 6.2 still hold. We also know from hypothesis that $\mathcal{K}(sL) = s^2\mathcal{K}(L) = 0$. Next, we show $\mathcal{K}(sL - X) \leq 0$ for some $s > 0$.

$$\begin{aligned}
\mathcal{K}(sL - X) &= s^2\mathcal{K}(L) - s(2LX + 4L'X' - LX'' - L''X) + \mathcal{K}(X) \\
&= \mathcal{K}(X) - s(2LX + 4L'X' - L''X - LX'') \qquad (6) \\
&= \mathcal{K}(X) - s(2LX + 4L'X' - L''X - L(R - r)'')
\end{aligned}$$

This is a linear expression in s. Since $\mathcal{K}(X) > 0$, it has a positive y-intercept. Thus, expression (6) can only become negative for some $s > 0$ if the expression $2LX + 4L'X' - L''X - LX''$ is positive. We are only concerned with the slope and therefore can drop $\mathcal{K}(X)$. After substituting (4) and (5) from Lemma 6.2,

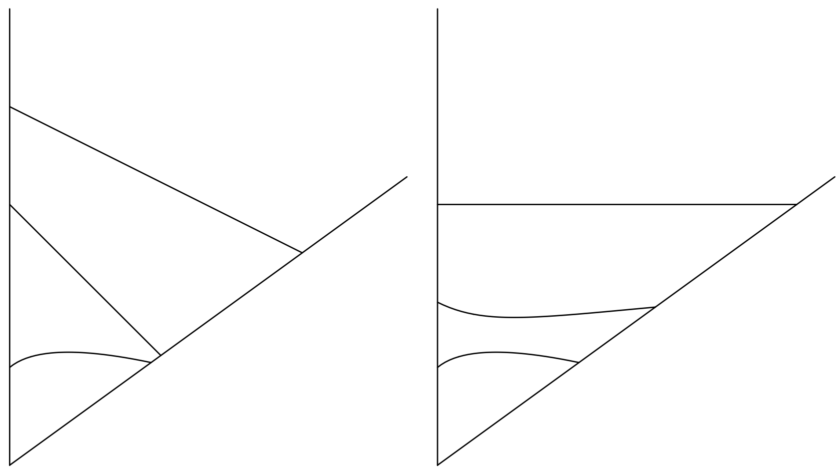

Figure 5: A wedge W and the result of subtracting its x-ray data from a line

we find $2LX + 4L'X' - L''X - LX''$

$$\begin{aligned}
&= 2LX + 4L'X' - L''X - LR'' + Lr'' \\
&= 2LX + 4L'X' - L''X - L\left(R + 2\frac{(R')^2}{R}\right)) + L\left(r + 2\frac{(r')^2}{r}\right) \\
&= L(2R - 2r - R + r) + 4L'X' - L''X - 2L\left(\frac{(R')^2}{R} - \frac{(r')^2}{r}\right)
\end{aligned} \quad (7)$$

Since L is a straight line, $\mathcal{K}(L) = 0$ and

$$L'' = L + 2\frac{(L')^2}{L} \quad (8)$$

Substituting (8) into (7), we find

$$\begin{aligned}
(7) &= LX + 4L'X' - \left(L + 2\frac{(L')^2}{L}\right) X - 2L\left(\frac{(R')^2}{R} - \frac{(r')^2}{r}\right) \\
&= 4L'X' - \frac{2R(L')^2}{L} + \frac{2r(L')^2}{L} - \frac{2L(R')^2}{R} + \frac{2L(r')^2}{r} \\
&= \frac{2}{LRr}(2RrLL'(R-r)' - R^2r(L')^2 + Rr^2(L')^2 - rL^2(R')^2 + RL^2(r')^2 \\
&= \frac{2}{LRr}(R((rL')^2 - 2rr'LL' + (r'L)^2) - r((RL')^2 - 2RR'LL' + (R'L)^2) \\
&= -\frac{2}{LRr}(r(RL' - R'L)^2 - R(rL' - r'L)^2)
\end{aligned} \quad (9)$$

Studying expression (9), we can see the slope of the linear equation is negative when $r(RL' - LR')^2 - R(rL' - r'L)^2 < 0$ since L, R, and r are positive.

Rewriting,

$$R\left(\frac{L'}{L}-\frac{R'}{R}\right)^2 - r\left(\frac{L'}{L}-\frac{r'}{r}\right)^2 < 0 \tag{10}$$

From Lemma 6.1, we know

$$m_p = \frac{\frac{p'}{p}\sin\varphi + \cos\varphi}{\frac{p'}{p}\cos\varphi - \sin\varphi}$$

where m_p is the slope of a polar function $p(\varphi)$. Then this expression

$$\begin{aligned}
&\implies m_p(\frac{p'}{p}\cos\varphi - \sin\varphi) = \frac{p'}{p}\sin\varphi + \cos\varphi \\
&\implies \frac{p'}{p}(m_p\cos\varphi - \sin\varphi) = m_p\sin\varphi + \cos\varphi \\
&\implies \frac{p'}{p} = \frac{m_p\sin\varphi + \cos\varphi}{m_p\cos\varphi - \sin\varphi}
\end{aligned}$$

The slope of p can be expressed in terms of the angle of inclination, ψ_p in the following way

$$\begin{aligned}
m_p = \tan\psi_p &= \frac{\sin\psi_p}{\cos\psi_p} \\
\implies \frac{p'}{p} &= \frac{\frac{\sin\psi_p}{\cos\psi_p}\sin\varphi + \cos\varphi}{\frac{\sin\psi_p}{\cos\psi_p}\cos\varphi - \sin\varphi}
\end{aligned}$$

Multiplying by $\frac{\cos\psi_p}{\cos\psi_p}$ yields

$$\frac{p'}{p} = \frac{\sin\psi_p\sin\varphi + \cos\varphi\cos\psi_p}{\sin\psi_p\cos\varphi - \sin\varphi\cos\psi_p} \tag{11}$$

Using the trigonometric identities $\cos(x-y) = \sin x\sin y + \cos x\cos y$ and $\sin(x-y) = \sin x\cos y - \sin y\cos x$ we can write (11) as

$$\begin{aligned}
\frac{p'}{p} &= \frac{\cos(\psi_p-\varphi)}{\sin(\psi_p-\varphi)} \\
&= \cot(\psi_p-\varphi)
\end{aligned}$$

Using this relation, we can rewrite (10) as

$$\begin{aligned}
&R(\cot(\psi_R-\varphi)-\cot(\psi_L-\varphi))^2 - r(\cot(\psi_r-\varphi)-\cot(\psi_L-\varphi))^2 < 0 \\
&\implies \frac{R}{r} < \frac{(\cot(\psi_r-\varphi)-\cot(\psi_L-\varphi))^2}{(\cot(\psi_R-\varphi)-\cot(\psi_L-\varphi))^2}
\end{aligned} \tag{12}$$

and the slope is negative when (12) holds.

Observe that $\frac{R}{r}$ is bounded as R and r are bounded and $0 < r < R$. Therefore as $\psi_L \to \psi_R$ the right hand side goes to infinity. This means that if the angle of inclination of L is sufficiently close to the angle of inclination of R (i.e. the slope of L is sufficiently close to the slope of R) then the slope of (10) is negative. Hence there exists a $s_0 > 0$ such that $\mathcal{K}(s_0 L - X) \leq 0$. Also notice that the only time the right hand side of this inequality is undefined is when $\varphi = \psi_r$ or $\varphi = \psi_L$ or $\varphi = \psi_R$. If $\varphi = \psi_r$ or $\varphi = \psi_R$ then $\tan\varphi = \tan\psi_R$ or $\tan\varphi = \tan\psi_r$. This implies that the slope of R or r is parallel to the slope of the ray φ. If that were the case we would have a parallel wedge. Suppose $\varphi = \psi_L \implies \tan\varphi = \tan\psi_L = m_L$. Then $\tan\alpha < m_L$ or $m_L < \tan\beta$. Then L would intersect α and β only at the origin. Contradiction. □

Remark Also notice that when $\psi_L \to \psi_r$ the right hand side of 12 goes to zero. This means that when the slope of L is sufficiently close to the slope of r the inequality does not hold as $\frac{R}{r}$ is bounded below by 1. Hence for lines L who have slopes close to that of r there does not exist an $s_0 > 0$ such that $\mathcal{K}(s_0 L - X) \leq 0$.

Corollary 6.5 *The set of all convex bodies with same directed x-ray is not convex.*

Proof. From the theorem, we know that we can find two convex bodies, (r, R) and (s, S), with the same x-ray data that both have a line for their far side, and different slopes. Examine the linear combination of these bodies: $L(s,t) = (1-t)(r,R) + t(s,S)$ at $t = s = 0.5$. Then $L(0.5, 0.5) = (0.5(r+s), 0.5(R+S))$. So for this body to be convex, $\mathcal{Q}(0.5(R+S)) \geq 0$. But, $\mathcal{Q}(0.5(R+S)) = (0.5)^2\mathcal{Q}(R+S)$, and we know that because R and S are both lines, $\mathcal{Q}(R) = 0$ and $\mathcal{Q}(S) = 0$. Then, by part iii of Theorem 3.1, $\mathcal{Q}(R+S) \leq 0$ This means that $\mathcal{Q}(R+S) = 0$, but this is impossible because that implies that R and S are parallel. □

7 Determining the existence of a convex body from the x-ray data

We have shown that we can construct families of convex bodies that have the same x-ray by using the s and t parameterizations (§4) and by adding and subtracting the x-ray data from a line (§6). In this section, we focus our attention on when a function $X(\varphi)$ can be the x-ray data of a convex body. We assume that X' and X'' are bounded throughout this section.

First, we examine x-ray data that has one point where $\mathcal{K}(X) = 0$, and show that if a convex body bounded away from the origin exists, there must be points of zero curvature at the same angle on r and R. Second, we show that if there exist two different points of zero curvature on the x-ray and a point in between that has curvature greater than zero, then there can be no convex body bounded away from the origin that has that x-ray data.

Lemma 7.1 *If $K = (r, R)$ is a convex body bounded away from the origin in $C[\alpha, \beta]$ and its directed x-ray X satisfies $\mathcal{K}(X)(\varphi_0) = 0$ for some $\varphi_0 \in (\alpha, \beta)$, then $\mathcal{K}(r)(\varphi_0) = \mathcal{K}(R)(\varphi_0) = 0$ and the tangent lines to the functions $X(\varphi)$,$r(\varphi)$, and $R(\varphi)$ are parallel at $\varphi = \varphi_0$.*

PROOF.

$$\mathcal{K}(R) = \mathcal{K}(X + r) = \mathcal{K}(X) + (2Xr + 4X'r' - Xr'' - X''r) + \mathcal{K}(r)$$

Since R is the far side of the convex body K, by Lemma 5.1 we know that $\mathcal{K}(R) \geq 0$. So, $\mathcal{K}(X)+(2Xr+4X'r'-Xr''-X''r)+\mathcal{K}(r) \geq 0$. Since $\mathcal{K}(X)(\varphi_0) = 0$, we know that

$$\mathcal{K}(R) = (2Xr + 4X'r' - Xr'' - X''r) + \mathcal{K}(r) \geq 0 \text{ at } \varphi = \varphi_0.$$

By Lemma 5.1, we know

$$\mathcal{K}(r) \leq 0 \implies r'' \geq r - 2\frac{(r')^2}{r} \tag{13}$$

and by hypothesis, we know

$$\mathcal{K}(X)(\varphi_0) = 0 \implies X'' = X - 2\frac{(X')^2}{X} \text{ at } \varphi = \varphi_0. \tag{14}$$

Substituting (13) and (14) into $\mathcal{K}(R)$, we find

$$\begin{aligned}
\mathcal{K}(R) &\leq (2Xr + 4X'r' - Xr - 2X\frac{(r')^2}{r} - Xr - 2r\frac{(X')^2}{X}) + \mathcal{K}(r) \\
&= (4X'r - 2X\frac{(r')^2}{r} - 2r\frac{(X')^2}{X}) + \mathcal{K}(r) \\
&= -\frac{2}{Xr}(X^2(r')^2 - 2XX'rr' + (X')^2r^2) + \mathcal{K}(r) \\
&= -\frac{2}{Xr}(X'r - Xr')^2 + \mathcal{K}(r)
\end{aligned}$$

$$0 \leq \mathcal{K}(R) \leq -\frac{2}{Xr}(X'r - Xr')^2 + \mathcal{K}(r) \text{ at } \varphi = \varphi_0. \tag{15}$$

Since $-\frac{2}{Xr}(X'r - Xr')^2$ is always less than or equal to zero and $\mathcal{K}(r) \leq 0$, inequality (15) only holds when $-\frac{2}{Xr}(X'r - Xr')^2 = 0$ and $\mathcal{K}(r) = 0$. So $\mathcal{K}(r)(\varphi_0) = 0$ and $\frac{X'}{X} = \frac{r'}{r}$ which implies the tangent line of X is parallel to the tangent line of r at $\varphi = \varphi_0$.

A similar argument yields the inequality

$$0 \geq \mathcal{K}(r) \geq \frac{2}{XR}(X'R - XR')^2 + \mathcal{K}(R) \text{ at } \varphi = \varphi_0 \tag{16}$$

This inequality holds when $\mathcal{K}(R) = 0$ and $\frac{2}{XR}(X'R - XR')^2 = 0$. So $\mathcal{K}(R)(\varphi_0) = 0$, and the tangent line of X is parallel to the tangent line of R at $\varphi = \varphi_0$. □

Theorem 7.2 *Let $X(\varphi)$ be a function on $[\alpha, \beta]$ that is C^2 on (α, β) and satisfies $\mathcal{K}(X)(\varphi) \geq 0$ for $\varphi \in (\alpha, \beta)$. If there exist angles φ_0 and φ_1, $\alpha < \varphi_0 < \varphi_1 < \beta$ such that*

i) $\mathcal{K}(X)(\varphi_0) = \mathcal{K}(X(\varphi_1)) = 0$ *and*

ii) $\mathcal{K}(X)(\varphi) > 0$ *for some* $\varphi \in (\varphi_0, \varphi_1)$,

then the only convex body with directed x-ray data X is $K = (0, X)$.

PROOF. Suppose there exists a convex body $K' = (r, R)$, bounded away from the origin with directed x-ray data X. Then by Lemma 7.1 and condition i), we know that

$$\mathcal{K}(r)(\varphi_0) = \mathcal{K}(r)(\varphi_1) = \mathcal{K}(R)(\varphi_0) = \mathcal{K}(R)(\varphi_1) = 0$$

and the slopes of the tangent line of X,R, and r are equal at φ_0.

Choose coordinates so that the y-axis is parallel to the tangent lines of X,R, and r at φ_0 and $0 < r(\varphi_0) < R(\varphi_0)$. Since the ray going through $X(\varphi_0)$ and $R(\varphi_0)$ intersects the ray going through $X(\varphi_1)$ and $R(\varphi_1)$ at O and nowhere else, we can conclude that both $r(\varphi_1)$ and $R(\varphi_1)$ are on the same side of the line connecting $r(\varphi_0)$ and $R(\varphi_0)$. Observe that since K is a convex body, as we travel around ∂K from $r(\varphi_0)$ clockwise, the slope of the tangent line at each point on ∂K must be monotone decreasing from ∞ to 0 and then to $-\infty$ when we get to $R(\varphi_0)$. Likewise if we travel counterclockwise from $r(\varphi_0)$ around ∂K, the slopes of the tangent lines must be monotone increasing from $-\infty$ to 0 and then to ∞ at $R(\varphi_0)$. (We denote a slope of $-\infty$ to be equal to the slope of ∞, both of which are vertical. The only reason we use the different signs is so we can describe the slope as increasing or decreasing accordingly.) If the slopes of the tangent lines at $r(\varphi_1)$ and $R(\varphi_1)$ are equal and the curvature is zero at these two points, then there are only two possibilities in order to preserve the convexity of K. The first possibility is that $r(\varphi_1)$ and $R(\varphi_1)$ lie on a line segment. The second possibility is that $r(\varphi_1)$ lies on a vertical line connected to $r(\varphi_0)$, and $R(\varphi_1)$ lies on a vertical line connected to $R(\varphi_0)$.

Possibility 1: $r(\varphi_1)$ and $R(\varphi_1)$ lie on a line segment.

If this is true, then in order to preserve the convexity of K, the line segment connecting $r(\varphi_1)$ and $R(\varphi_1)$ must contained in ∂K and thus must be contained in a supporting ray of K. This implies that $r(\varphi_1)$ and $R(\varphi_1)$ are endpoints of the $r(\varphi)$ and $R(\varphi)$, respectively. Then, either $\varphi_1 = \alpha$ or $\varphi_1 = \beta$, and this is a contradiction.

Possibility 2: $r(\varphi_1)$ lies on a vertical line connected to $r(\varphi_0)$ and $R(\varphi_1)$ lies on a vertical line connected to $R(\varphi_0)$.

If this is true, then K contains a parallel wedge and by [2, Theorem 3.1], the x-ray data contains an interval of zero curvature connecting $X(\varphi_0)$ and $X(\varphi_1)$. Thus, $\mathcal{K}(X(\varphi)) = 0$ for all $\varphi \in (\varphi_0, \varphi_1)$,and again this is a contradiction. □

Theorem 7.3 *If $X \in \mathcal{C}[\alpha, \beta]$, $\mathcal{K}(X) > 0$ on $[\alpha, \beta]$, then there exists a convex body K_X bounded away from the origin such that X is the directed x-ray of K.*

PROOF. Let l be a line. Show $\mathcal{K}(tl + X) \geq 0$ for some $t > 0$. Since

$$\mathcal{K}(tl + X) = t(2lX + 4l'X' - l''X - lX'') + \mathcal{K}(X)$$

By the Extreme Value Theorem, there exists a δ such that $\mathcal{K}(X) > \delta > 0$ on $[\alpha, \beta]$. By hypothesis, $2lX + 4l'X' - l''X - LX''$ is bounded. Hence for some $t > 0$, $\mathcal{K}(tl + X) > 0$. □

Remark This result is not as strong as we would like since to show that as long as $\{\varphi \in (\alpha, \beta) : \mathcal{K}(x)(\varphi) = 0\}$ is connected, then there exists a convex body bounded away from the origin with directed x-ray X.

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