Perfect Domination in Kneser Graphs

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Abstract

Given a graph, a perfect dominating set, D, is a subset of the vertices such that no two vertices in D are adjacent, and every other vertex is adjacent to exactly one vertex in D. We examine the family of graphs called Kneser graphs to determine which ones have a perfect dominating set. A Kneser graph $K\binom{n}{k}$ has as its vertices the k element subsets of an n element set. Two vertices are adjacent exactly when their corresponding sets are disjoint. We show that a Kneser graph has a perfect dominating set only if it is of the form $\binom{2k+1}{k}$ where $k+2$ is prime. We define a "perfect set" as a subset of vertices from a Kneser graph such that all $k-m$ tuples appear the same number of times and show that any perfect dominating set on a Kneser graph is also a perfect set, and vice versa. An algorithm to find a perfect dominating set, as well as different perfect dominating sets on the same graph, are discussed. Given any vertex, we give a method to find the nearest member of the dominating set.

1 Introduction

A graph $G = (V, E)$ is a set of vertices and edges such that if $a, b \in V$, $(a, b) \in E$ then the vertices a and b share an edge. The distance between two vertices is the fewest number of edges that can be traversed to move from one vertex to the other. A perfect dominating set on a graph is a subset $D \subseteq V$ such that, for every $v \in V$, either $v \in D$ or $\exists$ exactly one $w \in D$ such that $(v, w) \in E$. Note that this requires each vertex $v \in D$ be a distance of 3 or greater from each other [1]. Previous research on perfect dominating sets has shown that determining whether a tree has a perfect dominating set can be solved in $O(\log |V|)$ time with $O(|V|)$ processors in the CREW PRAM model of parallel computation and in $O(|V|)$ time sequentially[5].

Perfect dominating sets can also be called perfect one-error correcting codes. Both sets have the same properties; perfect one-error correcting codes are used in message transmission. If an error in transmitting a code vertex occurs, an adjacent vertex is recieved. Since it is adjacent to exactly one code vertex, it can be corrected back to the desired code vertex.

A dominating set of a graph $G = (V, E)$ is a subset of the vertices such that each vertex in the graph is either in the subset or adjacent to a vertex in the subset. Note that, every perfect dominating set is a dominating set, but not every dominating set is a perfect dominating set. In particular, dominating vertices can be adjacent, and a vertex not in the dominating set can be adjacent to more than one vertex in the dominating set. Domination numbers of q-analogues of Kneser graphs have been previously studied[3]. Also, it has been found that the domination number of Kneser graphs of the form $\binom{n}{2}$, $n \geq 3$, is equal to 3[8].

We study here a family of graphs called Kneser graphs. The vertices of a $\binom{n}{k}$ Kneser graph correspond to the sets obtained by choosing k of n elements. Two vertices are adjacent if their corresponding sets are disjoint. The most well known example is the Petersen graph, which is the $\binom{5}{2}$ Kneser graph. The vertices are the following: {1,2}, {1,3}, {1,4}, {1,5}, {2,3}, {2,4}, {2,5}, {3,4}, {3,5}, {4,5}. This graph is well known for presenting counterexamples to many conjectures about graphs[7].

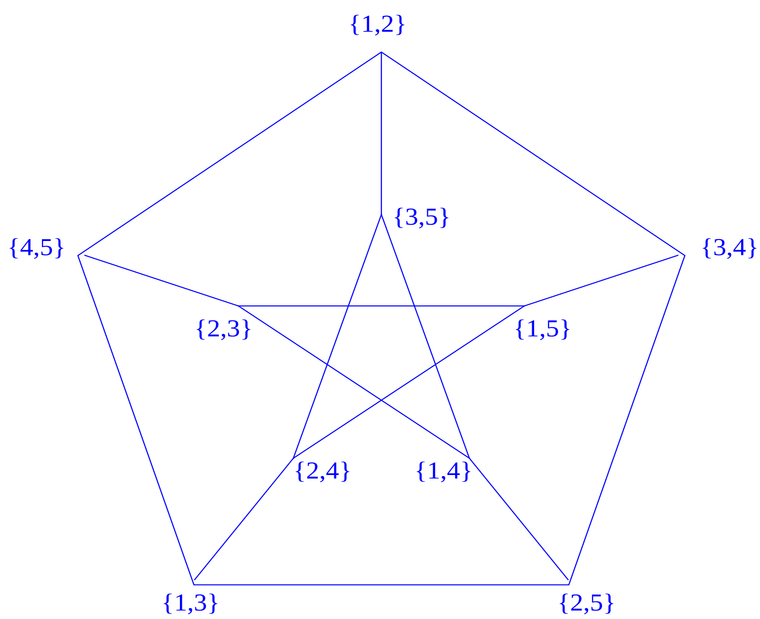


Figure 1: Petersen Graph

We determine which Kneser graphs can have a perfect dominating set. We also discuss an algorithm to find a perfect dominating set, the possibility of more than one perfect dominating set for a graph, and techniques to decode a vertex to the vertex which dominates it. In general, deciding whether or not a graph has a perfect one-error correcting code [and hence a perfect dominating set] is NP-complete[4]. Other families of graphs have been discovered to have perfect one-error correcting codes on them, such as the graph formed from the Towers of Hanoi puzzle[4]. One example of a Kneser graph that we know has a perfect dominating set is the $\binom{7}{3}$ graph. The $\binom{7}{3}$ Kneser graph can be found in Appendix A. The following table has a list of the elements in the dominating set, D, as well as the vertices which are adjacent to each element of D. We also found a perfect dominating set on the $\binom{11}{5}$ Kneser graph, which is in Appendix B.

vertex in D	adjacent vertices			
123	456	457	467	567
145	236	237	267	367
167	234	235	245	345
246	135	137	157	357
257	134	136	146	346
347	125	126	156	256
356	124	127	147	247

2 Preliminaries

Kneser graphs have some special properties. First of all, an $\binom{n}{k}$ Kneser graph is $\binom{n-k}{k}$ regular. Second, Kneser graphs are vertex transitive[9, 6]; each pair of vertices is equivalent under graph automorphism. We focus on Kneser graphs where n and k have certain relationships. We require that $n = 2k + r$ where $1 \leq r < k - 1$ because otherwise the graphs will be disconnected or have graph diameter of at most 2. (In requiring that our graph have a diameter of at least three, we also limit ourselves to graphs that have a perfect dominating set D where $|D| > 1$. Hence the $\binom{n}{1}$ graph, which has a perfect dominating set, is missed). Since each vertex has a neighborhood of size $\binom{n-k}{k}$, we know if D is our perfect dominating set,

$$\#\text{ of vertices in } D = \frac{\binom{n}{k}}{\binom{n-k}{k} + 1} \tag{1}$$

Suppose we wanted to calculate the number of vertices in D with the number 1 in its corresponding set. Let

$$\begin{aligned} a_1 &= \#\text{ of elements of } D \text{ with '1' in them} \\ b_1 &= \#\text{ of elements of } D \text{ without '1' in them} \end{aligned}$$

Then we know that the following equations are true.

$$a_1 + b_1 = \frac{\binom{n}{k}}{\binom{n-k}{k} + 1} \qquad (2)$$

$$a_1 + \binom{n-k-1}{k-1} b_1 = \binom{n-1}{k-1} \qquad (3)$$

Let $A =$ the set of vertices in D with a '1' in them, and $B =$ the set of vertices in D without a '1' in them. Equation (2) simply says that the elements in A, added to the elements of in B, equals the total number of elements in D. We also know that vertices in B are adjacent to some vertices with a '1' in them. Specifically, a vertex in B is adjacent to $\binom{n-k-1}{k-1}$ vertices with a '1' in them. Hence, equation (3) states that the number of vertices in B, added to $\binom{n-k-1}{k-1}$ multiplied by the number of vertices in B gives the total number of vertices in the graph that have a '1' in them.

This is a very useful idea. First of all, if we had wanted to write equations concerning the number of elements of D with and without a '2' in them, then the same equations would have been formed. We know that we can solve for the two variables a_1 and b_1, since our two equations are independent.[1] Hence, we can conclude that the number of '1's in the elements of D is equal to the number of '2's, is equal to ... is equal to the number of 'n's.

Second, it may be noted that this idea can be extended. Let

$$a_{12\ldots s} = \#\text{ of elements of } D \text{ with '}12\ldots s\text{' in them}$$
$$b_{12\ldots s} = \#\text{ of elements of } D \text{ with neither '1', nor '2', nor}\ldots\text{, nor '}s\text{' in them}$$

Then we know that the following equations are true.

$$(-1)^2\binom{s}{1}a_1 + (-1)^3\binom{s}{2}a_{12} + \cdots + (-1)^{s+1}\binom{s}{s}a_{12\ldots s} + b_{12\ldots s} = \frac{\binom{n}{k}}{\binom{n-k}{k} + 1} \qquad (4)$$

$$a_{12\ldots s} + \binom{n-k-s}{k-s} b_{12\ldots s} = \binom{n-s}{k-s} \qquad (5)$$

Equation (4) is an extension of equation (2). It uses the fact that the number of '1's is equal to the number of '2's, the number of '12's is equal to the number of '13's, etc. Equation (4) uses inclusion/exclusion to obtain the right number of vertices in D that include any combination of the numbers 1 through s. Then it adds on all the vertices in D that do not contain any of the numbers 1 through s so that it equals the total number of vertices in D.

Equation (5) is an extension of equation (3). Like equation (3), it counts all the vertices with '$12\ldots s$' in them. It should be noted here that the above equations only make sense if $s < k$. If $s = k$ then $a_{12\ldots s}$ and $b_{12\ldots s}$ may (if s is odd) both have a coefficient of 1 in equations (4) and (5). Hence, the equations

[1] This is assuming $n \neq 2k$, which we already excluded from study, and that $k \neq 1$, which we have also excluded from study.

may not be able to be solved for either variable. Since k is the number of elements in each vertex, if $s > k$, then there can be no vertex with the numbers 1 through s in its set. But regardless of whether s is odd or even, the equations are valid and independent so long as $s < k$.

3 The Special Case of $n = 2k + 1$

Using these equations, we set up matrices so that we could solve for the various a_i and b_i in our equations. We then plugged in various combinations of n and k. We knew that having part of a vertex in a perfect dominating set or negative of a vertex in a perfect dominating set is impossible. Hence, if any of the a_i's or b_i's came out to be anything other than a natural number, then that particular $\binom{n}{k}$ Kneser graph could not have a perfect dominating set.

We noticed that the graphs that were not eliminated had the form $\binom{2k+1}{k}$. Hence, we set out to prove that only graphs of this form could have a perfect dominating set.

Theorem 1 *Suppose a Kneser graph of the form* $\binom{2k+r}{k}$ *where* $1 \leq r < k - 1$ *has a perfect dominating set. Then* r *must equal 1.*

proof 1: Suppose a $\binom{2k+r}{k}$ Kneser graph, where $1 \leq r < k - 1$, has a perfect dominating set, D. It is easy to see that there can be no more than 1 of each $k-r$ tuple among the elements of D, otherwise two of them would be a distance of 2 apart. Hence, the maximum number of vertices, such that no 2 vertices have a distance of less than 3, includes one of each $k - r$ tuple. Note that $\binom{k}{k-r}$ $k - r$ tuples are combined into each vertex. Hence, the maximum number of vertices that can be chosen such that there is only one of each $k - r$ tuple is $\frac{\binom{2k+r}{k-r}}{\binom{k}{k-r}}$. We need this to be equal to the required number of elements in D, if the $\binom{2k+r}{k}$ Kneser graph is to have a perfect domination set. Hence, we need:

$$\frac{\binom{2k+r}{k-r}}{\binom{k}{k-r}} = \frac{\binom{2k+r}{k}}{\binom{k+r}{k} + 1}$$

Note that this is true for $r = 1$. What we want to show is that, for $r > 1$,

$$\frac{\binom{2k+r}{k-r}}{\binom{k}{k-r}} < \frac{\binom{2k+r}{k}}{\binom{k+r}{k} + 1}$$

This would show that the maximum number of vertices that can be chosen such that there is only one of each $k - r$ tuple is not enough to make a perfect dominating set [if $1 < r < k-1$]. Hence, if $1 < r < k-1$ then the $\binom{2k+r}{k}$ Kneser graph cannot have a perfect dominating set. Note that the above equation is equivalent to saying

$$\frac{(2k+r)!r!}{(k+2r)!k!} < \frac{(2k+r)!r!}{(k+r)!((k+r)! + r!k!)}$$

which, when reduced some, reads

$$\frac{(k+2r)!k! - ((k+r)!)^2}{(k+r)!k!r!} > 1 \tag{6}$$

Suppose that, when r goes up by one, then the equation $\frac{(k+2r)!k!-((k+r)!)^2}{(k+r)!k!r!}$ also gets larger. Since $\frac{(k+2r)!k!-((k+r)!)^2}{(k+r)!k!r!} = 1$ for $r = 1$, this would show that equation (6) is true for $1 < r < k-1$. Hence, all we need to do is show that the following equation is true.

$$\frac{\frac{(k+2r)!k!-((k+r)!)^2}{(k+r)!k!r!}}{\frac{(k+2r+2)!k!-((k+r+1)!)^2}{(k+r+1)!k!(r+1)!}} < 1$$

The following series of equations shows exactly what we want to prove.

$$\frac{(k+2r)!k! - ((k+r)!)^2}{(k+r)!k!r!} \cdot \frac{(k+r+1)!k!(r+1)!}{(k+2r+2)!k! - ((k+r+1)!)^2} =$$
$$\frac{(k+2r)!k!(k+r+1)(r+1) - ((k+r)!)^2(k+r+1)(r+1)}{(k+2r+2)!k! - (k+r+1)!(k+r+1)!} <$$
$$\frac{(k+2r+2)!k! - (k+r+1)!(k+r+1)!}{(k+2r+2)!k! - (k+r+1)!(k+r+1)!} = 1$$

□

While that was a lengthy proof, it was also very straightforward and assumed very little. The following proof is much more concise and conceptual.

proof 2: Suppose a Kneser graph of the form $\binom{2k+r}{k}$ has a perfect dominating set, D. From equations (2) and (3), we know that the number of n tuples within the elements of D must all be the same, where $1 \leq n < k$. (The number of vertices in D with '$12\ldots n$' in them is equal to the number of vertices in D with, for example, '$23\ldots n+1$' in them). Hence, each $k-1$ tuple must appear the same number of times within the elements of D. However, as explained in proof 1 of Theorem (1), a perfect dominating set on a $\binom{2k+r}{k}$ Kneser graph cannot have more than one of each of the $k-r$ tuples, otherwise two vertices in D will have a distance of 2 between them. Therefore, if $r > 1$, then not every $k-1$ tuple can be used, since that would cause each $k-r$ tuple to appear more than once. However, we already stated that each $k-1$ tuple must appear the same number of times within the elements of D. Hence, $r = 1$ is the only option for which the graph can have a perfect dominating set. Therefore, if a Kneser graph is to be connected and have a perfect dominating set, then it must be of the form $\binom{2k+1}{k}$. □

4 Requiring that $k+2$ be Prime

Proof 2 of Theorem 1 shows us that we must have 1 of each $k-1$ tuple within the elements of a perfect dominating set. Hence, we need to combine all of the $k-1$ tuples into groups of k sets, such that the union of the sets in each group has k elements. In requiring this, we have the following theorem.

Theorem 2 *If Kneser graph of the form $\binom{2k+1}{k}$ where $k \geq 1$ has a perfect dominating set, then $k+2$ is a prime number.*

proof: Let p be a prime less than k. Consider any $k-p$ tuple in a $\binom{2k+1}{k}$ Kneser graph. We know that, within any particular vertex, there are $\binom{p}{p-1} = p$ $k-1$ tuples that contain a particular $k-p$ tuple. Therefore, the number of $k-1$ tuples that contain $k-p$ must be divisible by p in order to be able to place all of the $k-1$ tuples into vertices. In equation form, this says

$$\binom{2k+1-(k-p)}{p-1} = \frac{(k+p+1)!}{(p-1)!(k+2)!} =$$

$$\frac{(k+3)(k+4)\dots(k+(p+1))}{(1)(2)\dots(p-1)} = pt \text{ where } t \in \mathbf{Z} \tag{7}$$

In looking at equation (7), notice it is true as long as $k+3 \not\equiv 1(\text{mod}p)$. Therefore, we know that a $\binom{2k+1}{k}$ Kneser graph where $k \equiv (p-2)(\text{mod}p)$ where $p < k$, is unable to have a perfect dominating set. In other words, if $k+2$ has a prime factorization consisting of primes that are smaller than k, then the $\binom{2k+1}{k}$ Kneser graph is unable to have a perfect dominating set. Hence, every k is eliminated except for k such that $k+2$ is prime. □

5 "Perfect Sets" as Perfect Dominating Sets

From the equations above, we seem to have discovered a fundamental property of our dominating sets; they have $\frac{\binom{2k+1}{k}}{k+2}\frac{\binom{k}{m}}{\binom{2k+1}{m}}$ of each m tuple. These sets are a special case of a more general object.

Definition 1 *A **Perfect Set** is a subset of vertices from a $\binom{2k+1}{k}$ Kneser graph such that all $k-m$ tuples appear the same number of times, $0 < m < k$.*[2]

For a perfect set of size p, there are $p \cdot \frac{\binom{k}{m}}{\binom{2k+1}{m}}$ of each m tuple. The following are some laws for perfect sets on the same graph.

1. If A, B are perfect sets where $A \subset B$, then $B \setminus A$ is also a perfect set.
2. If $A \cap B = \emptyset$, then $A \cup B$ is a perfect set.

Examples of perfect sets include: the entire graph, the graph minus the dominating set, the union of two disjoint dominating sets. The smallest perfect set must have one of each $k-1$ tuple. Since perfect dominating sets have one of each $k-1$ tuple, they are the smallest perfect set. After defining a perfect set, we created equations that would help explain how to construct one using 1 of each $k-1$ tuple.

[2]Note that, from proof 2 of Theorem 1 we know that a perfect dominating set must be a perfect set using 1 of each $k-1$ tuple.

Suppose we want to create a perfect set [using 1 of each $k-1$ tuple] from the $\binom{2k+1}{k}$ Kneser graph where $k = 3$. Note that, as previously stated, a Kneser graph is vertex transitive. Hence, we know that we can pick any particular vertex to be the first vertex in our Perfect Set. Choose the vertex $\{1, 2, \ldots, k\}$ to be in the perfect set. [At this point, we will try to keep things in terms of k, as we would like to generalize the equation.] From this vertex, we can count the number of $k-1$ tuples that share a certain number of elements with it:

$$\binom{k}{k-1} = k = \#\text{ of } k-1 \text{ tuples using } k-1 \text{ elements from} \{1, 2, \ldots, k\}$$

$$\binom{k+1}{1}\binom{k}{k-2} = (k+1)\frac{k(k-1)}{2} =$$
$$\#\text{ of } k-1 \text{ tuples using } k-2 \text{ elements from} \{1, 2, \ldots, k\}$$

$$\binom{k+1}{2}\binom{k}{k-3} = \frac{(k+1)k}{2}\frac{k(k-1)(k-2)}{2 \cdot 3} =$$
$$\#\text{ of } k-1 \text{ tuples using } k-3 \text{ elements from} \{1, 2, \ldots, k\}$$

Suppose we want to start creating vertices using our left over $k-1$ tuples. Since the $k-1$ tuples using only elements from $\{1, 2, \ldots, k\}$ are all already used up in our first vertex we created, we know that we cannot use them again. Hence, each of the $k-1$ tuples that share $k-2$ elements with the vertex $\{1, 2, \ldots, k\}$ must be used with one other tuple from the same set, and $k-2$ of the $k-1$ tuples that share $k-3$ elements with the vertex$\{1, 2, \ldots, k\}$. This may sound very confusing, but it all results from two facts:

1. This perfect set contains exactly 1 of each $k-1$ tuple.
2. In order to combine k $k-1$ tuples into a vertex, they must each share exactly $k-2$ elements with each other $k-1$ tuple.

Hence, we can even expand the last comment in the preceeding paragraph. Suppose we have already used up all of the $k-1$ tuples containing $k-(n-1)$ elements from the set $\{1, 2, \ldots, k\}$. Then the remaining $k-1$ tuples containing $k-n$ elements from the set $\{1, 2, \ldots, k\}$ must be joined such that n of them, along with $k-n$ of the $k-1$ tuples containing $k-(n+1)$ elements from the set $\{1, 2, \ldots, k\}$, form one vertex.[3] Again, this comes from the two facts above.

Suppose we want to count the number of vertices that we are creating. We would then create the following equation when $k = 3$. [We will keep k as k, though, because we would still like to generalize this].

$$\frac{k}{k} + \frac{(k+1)k(k-1)}{2 \cdot 2} + \frac{\left[\frac{(k-1)k}{2}\frac{k(k-1)(k-2)}{2 \cdot 3} - \frac{(k+1)k(k-1)(k-2)}{2 \cdot 2}\right]}{k} =$$

[3]This is assuming, of course, that $k > n$.

$$1 + \frac{(k+1)k(k-1)}{2\cdot 2} + \frac{(k+1)(k-1)(k-2)}{2\cdot 2}\left[\frac{k}{3} - 1\right] \tag{8}$$

To make explanation of this equation less confusing, we will create the following labelings:

$$\begin{aligned}
\mathcal{A} &= \text{set of } k-1 \text{ tuples sharing } k-1 \text{ elements from the set}\{1,2,\ldots,k\}\\
\mathcal{B} &= \text{set of } k-1 \text{ tuples sharing } k-2 \text{ elements from the set}\{1,2,\ldots,k\}\\
\mathcal{C} &= \text{set of } k-1 \text{ tuples sharing } k-3 \text{ elements from the set}\{1,2,\ldots,k\}
\end{aligned}$$

The first term of equation (8) accounts for the first vertex we picked for our perfect set: $\{1,2,\ldots,k\}$. There were k elements in $\mathcal{A}$ and we used k of them in creating the first vertex. The next term accounts for the vertices we made using the elements from $\mathcal{B}$. We used 2 of elements from $\mathcal{B}$ per vertex, so we divided the total number of them by two to get the total number of vertices using those elements. The third term subtracts the number of elements of $\mathcal{C}$ that were used with elements from $\mathcal{B}$. The remaining elements of $\mathcal{C}$ are combined into vertices, and we can count the number of vertices that are made by dividing the number of elements remaining in $\mathcal{C}$ by k.

Note that in the last term, we have something multiplied by $\frac{k}{3} - 1$. When $k = 3$, we know that that term is equal to 0. Since that term corresponds to the number of vertices containing $k-1$ tuples that have no numbers in common with the first vertex, we then know that no vertices in a perfect set created with $k = 3$ are disjoint. In fact, we will soon show that, if we want to make a perfect set from a $\binom{2k+1}{k}$ Kneser graph, there will always be no disjoint vertices when k is odd.

Lemma 1 *Suppose we create the equation that counts the number of words in a perfect set [using 1 of each $k-1$ tuple] on the $\binom{2k+1}{k}$ Kneser graph as we did in equation (8). Assume $k \geq 3$, $k = n$. Then the last term will be of the following form:*[4]

$$\frac{(k+1)(k-1)!}{2\cdot 2\cdot 3\cdot\ldots(n-1)}\left[\frac{\overbrace{k(k-1)\ldots(3)}^{n-2\ terms}}{\underbrace{n(n-1)\ldots(3)}_{n-2\ terms}} - \left[\frac{\overbrace{k(k-1)\ldots(4)}^{n-3\ terms}}{\underbrace{(n-1)(n-2)\ldots(3)}_{n-3\ terms}} - \left(\cdots - \left(\frac{k}{3} - 1\right)\right)\cdots\right]\right]$$

proof: We will prove this using induction. From viewing equation (8), you can see that this is true when $k = 3$. Now assume that this is true when $k = n$. In the equation for $k = n$ the last term rounds up all the remaining $k-1$ tuples that shared no elements with the vertex $\{1,2,\ldots,k\}$ and makes vertices out of them. Therefore, if we multiply it by k, we will get the total number of $k-1$ tuples that were remaining. Therefore, there are a total of

[4] The reason this was written with both n's and k's is to show which is the variable. Since the construction for the equations is the same regardless of the k, k is the variable. However, the n's in that equation stay the same if we move to a different k.

$$\frac{(k+1)!}{2\cdot 2\cdot 3\cdot\ldots(n-1)}\left[\frac{\overbrace{k(k-1)\ldots(3)}^{n-2\text{ terms}}}{\underbrace{n(n-1)\ldots(3)}_{n-2\text{ terms}}}-\left[\frac{\overbrace{k(k-1)\ldots(4)}^{n-3\text{ terms}}}{\underbrace{(n-1)(n-2)\ldots(3)}_{n-3\text{ terms}}}-\left(\cdots-\left(\frac{k}{3}-1\right)\right)\cdots\right]\right]$$

of the $k-1$ tuples containing $k-n$ of the set $\{1,2,\ldots,k\}$ left. However, when $k=n+1$, we know that these must be grouped into groups of n, along with 1 of the $k-1$ tuples containing none of the set $\{1,2,\ldots,k\}$. Hence, if we divide that number by n, we will obtain the number of vertices including those $k-1$ tuples. Now we just need to figure out how many $k-1$ tuples we have left, and make them into vertices. [From this point on, we will use the term $n+1$ instead of k, since that is the particular case we are discussing].

We know that the number of $(n+1)-1$ tuples that contain none of the set $\{1,2,\ldots,n+1\}$ is $\binom{(n+1)+1}{n}\binom{n+1}{0}$. Therefore, in subtracting the number of them that we had to use with the $(n+1)-1$ tuples containing 1 element in the vertex $\{1,2,\ldots,n+1\}$, there are

$$\binom{(n+1)+1}{n}\binom{n+1}{0}-\frac{1}{n}\frac{((n+1)+1)!}{2\cdot 2\cdot 3\cdot\ldots(n-1)}\left[\frac{\overbrace{(n+1)((n+1)-1)\ldots(3)}^{n-2\text{ terms}}}{\underbrace{n(n-1)\ldots(3)}_{n-2\text{ terms}}}-\right.$$

$$\left.\left[\frac{\overbrace{(n+1)((n+1)-1)\ldots(4)}^{n-3\text{ terms}}}{\underbrace{(n-1)(n-2)\ldots(3)}_{n-3\text{ terms}}}-\left(\cdots-\left(\frac{(n+1)}{3}-1\right)\right)\cdots\right]\right]=$$

$$\frac{((n+1)+1)(n+1)n\ldots 4\cdot 3}{n(n-1)(n-2)\ldots 3\cdot 2}\frac{(n+1)n\ldots 3\cdot 2}{(n+1)n\ldots 3\cdot 2}-$$

$$\frac{((n+1)+1)!}{2\cdot 2\cdot 3\cdot\ldots(n-1)n}\left[\frac{\overbrace{(n+1)((n+1)-1)\ldots(3)}^{n-2\text{ terms}}}{\underbrace{n(n-1)\ldots(3)}_{n-2\text{ terms}}}-\right.$$

$$\left.\left[\frac{\overbrace{(n+1)((n+1)-1)\ldots(4)}^{n-3\text{ terms}}}{\underbrace{(n-1)(n-2)\ldots(3)}_{n-3\text{ terms}}}-\left(\cdots-\left(\frac{(n+1)}{3}-1\right)\right)\cdots\right]\right]=$$

$$\frac{((n+1)+1)(n+1)\dots 3\cdot 2}{2\cdot 2\cdot 3\cdot\ldots\cdot n}\left[\frac{\overbrace{(n+1)n\dots(3)}^{(n+1)-2\text{ terms}}}{\underbrace{(n+1)n\dots(3)}_{(n+1)-2\text{ terms}}}-\right.$$

$$\left.\left[\frac{\overbrace{(n+1)((n+1)-1)\dots(4)}^{(n+1)-3\text{ terms}}}{\underbrace{n(n-1)\dots(3)}_{(n+1)-3\text{ terms}}}--\left(\dots-\left(\frac{(n+1)}{3}-1\right)\right)\dots\right]\right]$$

$(n+1)-1$ tuples including none from the set $\{1,2,\dots,n+1\}$. Hence, that number divided by $n+1$ should be the last term in the case where $k=n+1$ (since $n+1$ of them go together to make a vertex). This is exactly what we set out to show. □

Lemma 2 *Suppose we can and do create a perfect set [using 1 of each $k-1$ tuple] on the $\binom{2k+1}{k}$ Kneser graph, where k is odd.*[5] *Then the set contains no disjoint vertices.*

proof1: Pick a vertex from the perfect set. In the equations from Lemma (1), the last term gives the number of vertices that are created using only $k-1$ tuples sharing nothing with the original picked vertex. In other words, the last term gives the number of vertices created that are disjoint from our original vertex. From Theorem 5, we know that, when $k=n$, the last term is something finite multiplied by

$$\left[\frac{\overbrace{k(k-1)\dots(3)}^{n-2\text{ terms}}}{\underbrace{n(n-1)\dots(3)}_{n-2\text{ terms}}}-\left[\frac{\overbrace{k(k-1)\dots(4)}^{n-3\text{ terms}}}{\underbrace{(n-1)(n-2)\dots(3)}_{n-3\text{ terms}}}-\left(\dots-\left(\frac{k}{3}-1\right)\right)\dots\right]\right]$$

When k is odd, this reduces to

$$1-\frac{k}{3}+\frac{k(k-1)}{3\cdot 4}-\dots-\frac{k(k-1)}{3\cdot 4}+\frac{k}{3}-1=0$$

Hence, the last term is equal to zero. Therefore, there are no vertices disjoint from the vertex in the perfect set that we picked. Since the particular vertex we picked had no bearing on the equations, we can conclude that there are no disjoint vertices in the perfect set. □

The following proof is less computational and does not require the use of Lemma 1.

[5] Requiring that k be odd is not actually a limitation, since we know that k will be odd already from Theorem 2.

proof2: Let A be all the vertices in a perfect set [using 1 of each $k-1$ tuple] from a $\binom{2k+1}{k}$ Kneser graph. Let B be all the vertices from the same graph that are not in A. We know that any vertex in A cannot have more than one vertex in A to which it is adjacent, otherwise two vertices would be of distance 2 apart. If every vertex in A is not adjacent to any other vertex in A [all vertices are disjoint], we are done.

Suppose a vertex in A is adjacent to another vertex in A. Since each $k-1$ tuple appears once within all the vertices in A, we know that each vertex in A must be adjacent to the same number of vertices in A. Therefore, if one vertex in A is adjacent to another vertex in A, every vertex in A must be adjacent to exactly 1 other vertex in A.

Now note that B is also a perfect set; each $k-1$ tuple appears k+1 times within the vertices of B. Therefore, as in A, each vertex in B must be adjacent to the same number of vertices within B.

From equation (1) we know that a perfect dominating set on this graph must have $\binom{2k+1}{k}\frac{1}{k+2}$ vertices. This is equal to $\binom{2k+1}{k-1}\frac{1}{k}$, which is the number of vertices in A. Hence, if all the vertices in A were disjoint, every vertex in B would be adjacent to a vertex in A. But since we're assuming that each vertex in A is adjacent to another vertex in A, not every vertex in B is adjacent to a vertex in A. We know that $A \cup B$ is the set of all vertices in the graph, and $A \cap B = \emptyset$. We also know that the graph is $k+1$ regular. This means that some vertices in B are adjacent to $k+1$ vertices in B, and some vertices in B are adjacent to less than $k+1$ vertices in B. This contradicts the fact that each vertex in B must be adjacent to the same number of vertices within B. Therefore, no vertices can be adjacent. Since no vertices can ever be disjoint, they certainly can never be disjoint when k is odd. □

Lemma 3 *If there exists a perfect set on a $\binom{2k+1}{k}$ Kneser graph (k odd), then that perfect set is a perfect dominating set.*

proof: In order for two vertices to be a distance of 2 apart in a $\binom{2k+1}{k}$ Kneser graph, they must share exactly $k-1$ elements. However, since each $k-1$ tuple is used exactly once in a perfect set, we know that no 2 vertices can share $k-1$ elements. Therefore we know that no two vertices in the perfect set are a distance of 2 apart.

In order for two vertices to be adjacent in a perfect set, they must be disjoint. However, in Lemma(2) we showed that no two elements in a perfect set are disjoint when k is odd.

Since no vertices in a perfect set with k odd are a distance of 1 or 2 apart, they all must be a distance of 3 or greater from each other. Hence, if we can show that there are $\frac{\binom{n}{k}}{\binom{n-k}{k}+1}$ vertices in the perfect set, then we know that it is a perfect dominating set on the $\binom{2k+1}{k}$ Kneser graph.

The perfect set is made by combining all the $k-1$ tuples into vertices. Therefore, the total number of vertices it creates is the number of $k-1$ tuples divided by k. (k is the number of $k-1$ tuples in each vertex). Therefore, the total number of vertices created in a perfect set is:

$$\frac{\binom{2k+1}{k-1}}{k} = \frac{(2k+1)!}{(k-1)!(k+2)!}\frac{1}{k} = \frac{(2k+1)!}{k!(k+2)!}$$

From above, we know that, for the perfect set to be a perfect dominating set, this number must equal $\frac{\binom{n}{k}}{\binom{n-k}{k}+1}$. When $n = 2k+1$, this number is equal to:

$$\frac{\binom{2k+1}{k}}{\binom{k+1}{k}+1} = \frac{(2k+1)!}{k!(k+1)!}\frac{1}{(k+1)+1} = \frac{(2k+1)!}{k!(k+2)!}$$

Hence, we have the right number of vertices and our perfect set is a perfect dominating set. □

6 Creating a Perfect Set With 1 of each $k-1$ tuple

We now know that we have a perfect dominating set on a $\binom{2k+1}{k}$ Kneser graph iff we have a perfect set on that graph. We also know that we can only have a perfect dominating set if $k+2$ is prime. So now we want to show that we can make a perfect set on a $\binom{2k+1}{k}$ Kneser graph where $k+2$ is prime. We have been unable to do this thus far, so we will state our prediction as a conjecture.

Conjecture 1 *Each Kneser graph of the form $\binom{2k+1}{k}$, $k+2$ prime, has a perfect set.*

If this conjecture is proven, then from Theorem 3 we know that any $\binom{2k+1}{k}$ Kneser graph, $k+2$ prime, has a perfect dominating set. The following describes an algorithm that can be used to create a perfect set, assuming that one exists.

Since our perfect set includes 1 of each $k-1$ tuple, we know that each $k-m$ tuple ($1 \le m < k$) appears $\binom{2k+1-(k-m)}{k-1-(k-m)}\frac{1}{m}$ times. We also know that there will be a total of $\binom{2k+1}{k-1}\frac{1}{k}$ vertices in the perfect set, each having k numbers.

Begin with an empty $\binom{2k+1}{k}\frac{1}{k+2} \times k$ matrix. We will contruct the perfect dominating set by filling the matrix. Consider the digit '1'. As stated above, in our perfect set, each $k-m$ tuple appears $\binom{2k+1-(k-m)}{k-1-(k-m)}\frac{1}{m}$ times. So we know that '1' will be in $\binom{2k}{k-2}\frac{1}{k-1}$ vertices. For ease, each row in the matrix will have the digits placed in ascending order. Hence, all $\binom{2k}{k-2}\frac{1}{k-1}$ 1's will be in the first column. Next, consider the digit '2'. Place $\binom{2k-1}{k-3}\frac{1}{k-2}$ of the 2's next to the 1's (in the second column), since that is how many times each 2 tuple will appear within the perfect dominating set. Then place $\binom{2k}{k-2}\frac{1}{k-1} - \binom{2k-1}{k-3}\frac{1}{k-2}$ 2's in the first column. (This is because a total of $\binom{2k}{k-2}\frac{1}{k-1}$ 2's must appear in the code.) This process can be generalized for the nth digit. In adding the nth digit to the matrix, every $k-m$ tuple that can be formed using only the digits

$1, 2, \ldots, n$ must be created $\binom{2k+1-(k-m)}{k-1-(k-m)}\frac{1}{m}$ times. If this can be done with each digit $1, 2, \ldots, 2k+1$, then a perfect dominating set will be the result.

It is easy to place the digits that never appear in the last column, which are the digits 1 through $k-1$. This is because, in placing those digits, we never add more than one $k-1$ tuple at one time; where these digits go is already defined. Digits in the last column are more difficult to place, because $k-1$ tuples are added k at a time. There are more options for places to put the digit, and more tuples are formed with each placing. The following is an example of a partially formed perfect dominating set using this algorithm for the $\binom{11}{5}$ Kneser graph. (The vertices are placed side by side to save space).

12345	1236	123	123	1246	124
124	1256	125	125	126	12
1346	134	134	1356	135	135
136	13	1456	145	145	146
14	156	15	16	16	1
2346	234	234	2356	235	235
236	23	2456	245	245	246
24	256	25	26	26	2
3456	345	345	346	34	356
35	36	36	3	456	45
46	46	4	56	56	5

Note that, when the digit 5 is placed in the 1st row, the rest of the 5's are easy to place, as are all the 6's. We believe this to be true for any kth and $k+1$th digits in a Kneser graph that has a perfect dominating set. It is also important to notice that, for the same reason that all the '1's go in the 1st column, all of the $2k+1$'s will go in the last column. The same remark can be made about the 2nd and the $2k$th digit. The number of times each digit is placed in each column is not so well defined for the middle three digits: k, $k+1$, and $k+2$. Note that this algorithm does contain some backtracking, but we hope that not much backtracking is necessary.

7 Permuting a Perfect Dominating Set

Once we have found a perfect dominating set, we want to know whether or not it is unique. Using the symmetries in Kneser graphs, we can show that any perfect dominating set can be permuted into another perfect dominating set.

Lemma 4 *Let a and b be vertices of a $\binom{n}{k}$ Kneser graph. Then a and b share s elements iff $\pi(a)$ and $\pi(b)$ share s elements, where π is any permutation.*

proof: Clearly π can be composed of transpositions $i \leftrightarrow j$ where $i, j \in \{1, 2, \ldots, n\}$. If a and b share s elements, then they have no more or less numbers in common in interchanging the numbers i and j.

Conversely, suppose $\pi(a)$ and $\pi(b)$ share s elements but a and b share a different number of elements. Then either the common numbers between a and b must be mapped to two different places or the common numbers between $\pi(a)$ and $\pi(b)$ were mapped from two different places. Neither of these can happen in a permutation. Hence, a and b share s elements iff $\pi(a)$ and $\pi(b)$ share s elements, where π is any permutation. □

Theorem 3 *Suppose C is a set of vertices that are a perfect dominating set for the $\binom{n}{k}$ Kneser graph. Let π be a permutation, and let A be the set such that if $a \in C$ then $\pi(a) \in A$. Then A is also a perfect dominating set on the $\binom{n}{k}$ Kneser graph.*

proof: From Lemma 4 we know that A has the same number of elements as C. Lemma 4 also tells us that the relationships between the words in C is preserved in A. Hence, A contains $\frac{\binom{n}{k}}{\binom{n-k}{k}+1}$ elements, all of which are of distance 3 or greater from each other. Therefore, A is also a perfect dominating set on the $\binom{n}{k}$ Kneser graph. □

8 Decoding to the Dominating Vertex

Let D be our dominating set. Suppose we are given a vertex a, and we want to know to which element of D a is adjacent (itself, if $a \in D$). One type of decoding technique involves a matrix of all the elements of D in the form of binary strings. The length of a binary string for a vertex for the $\binom{2k+1}{k}$ Kneser graph is $2k+1$. A '1' is placed in the ith slot of the binary string if the vertex has an i in it, and a '0' if not. Each string representing an element of D is a row in the matrix, so that the size of the matrix is $\binom{2k+1}{k}\frac{1}{k+2} \times 2k+1$. The vertex to be decoded is also written as a binary string in the same way, and can be considered as a binary vector. The matrix is multiplied by the vertex to be decoded, and a vector will result. There will be one element of the resulting vector that is congruent to 0 (mod$k$). If that element is in the jth row, then our vertex is decoded to the vertex represented by the jth row of our matrix.

This is not an easy decoding technique, as it requires that we have a list of all the elements of D. One would hope that this would not be necessary. Unfortunately, a better decoding technique has not been found.

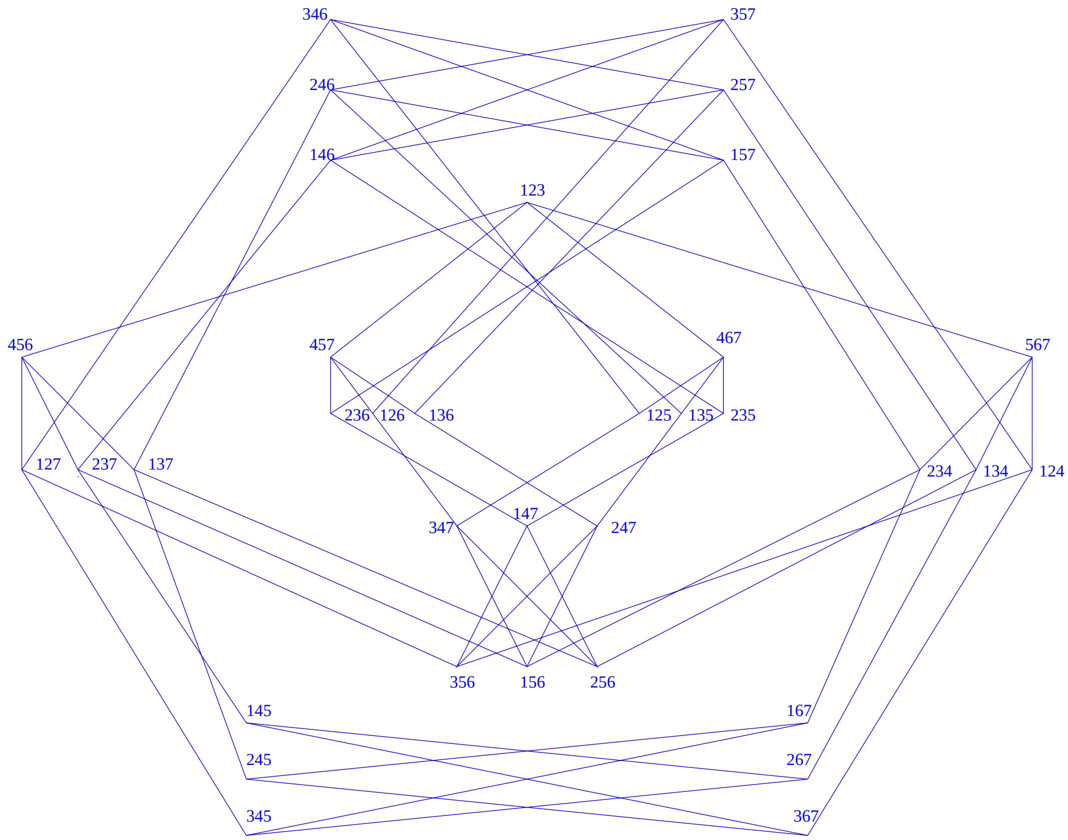
9 Conclusions and Future Work

In order for a Kneser graph to have a perfect dominating set, it must be of the form $\binom{2k+1}{k}$ where $k+2$ is prime. Also, any perfect dominating set on a Kneser graph is a perfect set, and any perfect set on a Kneser graph is a perfect dominating set. We know that the $\binom{n}{1}$, the $\binom{7}{3}$, and the $\binom{11}{5}$ Kneser graphs have perfect dominating sets. Any permutation of a perfect dominating set results in another perfect dominating set. We have given an algorithm with which a perfect dominating set can be found, as well as a method for decoding a vertex to the vertex which dominates it.

Future work could include finding a sufficient condition for which a Kneser graph has a perfect dominating set. We believe that having the graph be of the form $\binom{2k+1}{k}$ where $k+2$ is prime is sufficient, but as yet have been unable to prove so. The algorithm we have created to find a perfect dominating set involves a little bit of guess and check. It would be nice to have a direct algorithm, with no guess involved. Also, the decoding technique described in this paper requires having a list of the elements of the perfect dominating set. This list can be very large, depending on the size of the Kneser graph. A decoding technique that didn't require that list would be much more useful.

Other topics that can be examined are perfect q dominating sets, $q > 1$. A perfect q dominating set on a graph $G = (V, E)$ is a subset $D_q \subset V$ such that, for every $v \in V$, either $v \in D_q$ or $\exists$ exactly one $w \in D_q$ such that the distance between v and w is less than or equal to q. Also, are Kneser graphs Cayley graphs? A Cayley graph is a representation of a group. Each vertex corresponds to a group member, and the edges correspond to a set of generators. Two vertices a and b have an edge if $\exists$ a generator g such that $ga = b$. Perhaps knowing something about a group structure on a Kneser graph would help in knowing whether or not it has a perfect dominating set. Another question for further investigation is whether or not the perfect domination problem for cayley graphs is an NP-complete problem.

A The $\binom{7}{3}$ Kneser graph



B A Perfect Dominating Set for the $\binom{11}{5}$ Kneser Graph

12345	12367	12389	123AB	1246A	12479
1248B	12568	1257B	1259A	1269B	1278A
13469	1347B	1348A	1356A	13578	1359B
1368B	1379A	1456B	1457A	14589	14678
149AB	15679	158AB	167AB	1689A	1789B
23468	2347A	2349B	2356B	23579	2358A
2369A	2378B	24569	24578	245AB	2467B
2489A	2567A	2589B	26789	268AB	279AB
34567	3458B	3459A	346AB	34789	35689
357AB	3678A	3679B	389AB	4568A	4579B
4679A	4689B	478AB	5678B	569AB	5789A

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