Families of Partially Reducible Plane Curves

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Abstract

While plane curves have been classified up to five crossings there has yet been an effective method found that specifies how to create all of the plane curves corresponding to a specified number of crossings. In this paper we give an inductive method for finding partially reducible plane curves with specified properties, using the non-treelike curves of one fewer crossings. These curves, along with the others listed below, will include all non-treelike curves. Also included is a small bit of information on areas of investigation that did not open up as expected.

1 Introduction

By definition, a plane curve is an immersion of a circle into a plane with a derivative that vanishes nowhere. A curve is generic if at none of its self intersections (crossings, double points) are there three paths of the curve involved. We will use the term curve throughout the paper in reference to a generic curve, unless otherwise noted. We will also use the term strand to denote the length of curve between two double points. The definitions from [5] of a tree-like curve and of a Gauss diagram will also be utilized.

If we graph the unit tangent vector of this curve in a plane the number of times the vector rotates is known as the index of the curve. As the index distinguishes between curves that are homeomorphic with respect to change of orientation the direction in which the tangent rotates is of little importance; therefore the absolute value of the index is generally used.

2 Graphs and Plane Curves

In order to discern more about plane curves it is desirable to associate with each one a graph. From this graph a polynomial may be associated, utilizing the method outlined in [1] to find $Z_G(q, v)$, the dichromatic polynomial. Ideally, this polynomial would relate to the basic invariants (St, $J^{\pm}$) and possibly give a more simple way to calculate them. Unfortunately, all the methods of associating a graph that were attempted in this project failed in one form or another.

One method of associating a graph is to utilize the method in [2]–color the curve in an alternating fashion and place a vertex in each colored area. If this method is utilized on the two curves shown below in Figure 1 with their associated graphs we find that each curve gives a sperate polynomial. It may be observed from a table of curves, however, that both curves have the same invariants, indices, and number of crossings (Index=1, St=0, J^+=0, J^-=-2, n=2). Thus this method of graphing will serve no purpose in finding invariants as it is not constant over curves with equivalent invariants.

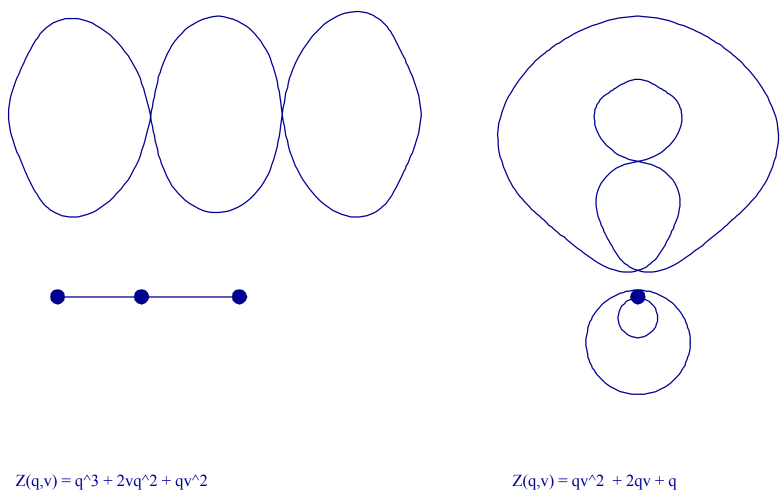

Figure 1

The second method of graphing fails also. If each crossing within the curves in Figure 2 are made into a vertex and each strand an edge we obtain the two graphs shown. Again utilizing the method in [1] to associate a polynomial with the graph we obtain a polynomial for both curves. Unfortunately, though these curves have invariants that are not equal in any case the associated polynomials are equal. Thus utilizing this method of graphing the polynomial clearly does not vary with the invariants either.

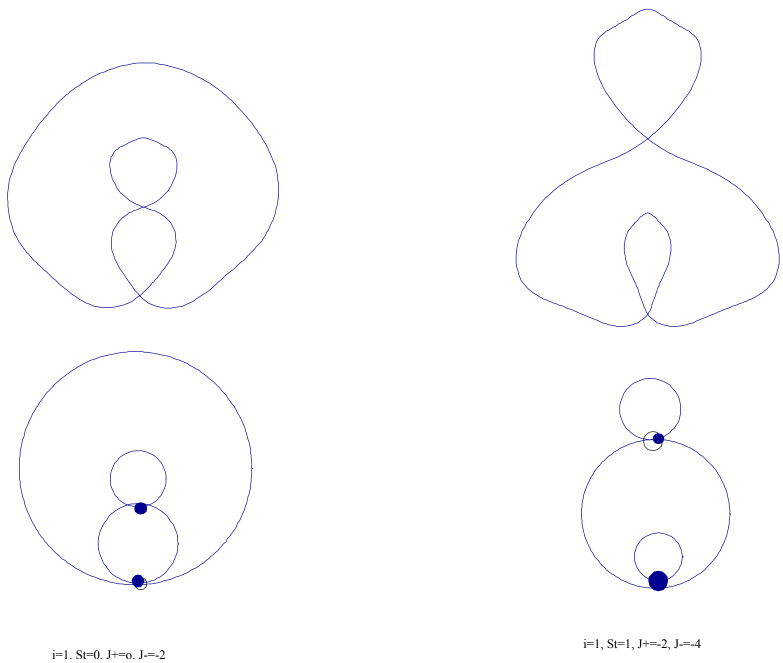

Figure 2: $Z(v, q) = (1 + v)^2(q^2 + 2vq + qv^2)$

3 Plane Curves and Alternating Knots

As knot theory is such a growing field it was decided to attempt to relate plane curves to knots, in hopes of sharing invariants, etc. In order to associate with every plane curve a knot each crossing in the plane must be made into either an over or under crossing in three dimensions. The most simple method of doing this is to make each curve an alternating knot. From [2], any projection of a knot may be made into an alternating knot by adhering to the following steps: select a starting point and an orientation. As the curve is traveled make the strand through the first crossing met an over/under strand, while making the strand traversed in the second crossing an under/over strand. Continue in an alternating fashion until the starting point is reached. The result will be an alternating knot. If a generalization of this procedure is applied to a generic plane curve we will again obtain an alternating knot.

3.1 Curves and Reidemeister Moves

It is of interest to note how the three Reidemeister moves utilized to switch between projections of knots affect the corresponding plane curve. As the second two moves apply to curves with two subsequent crossings which are both over or under we will look at the first move.

Definition 1 *A self resolving crossing is a crossing that is exited and entered by the same strand (see Figure 3).*

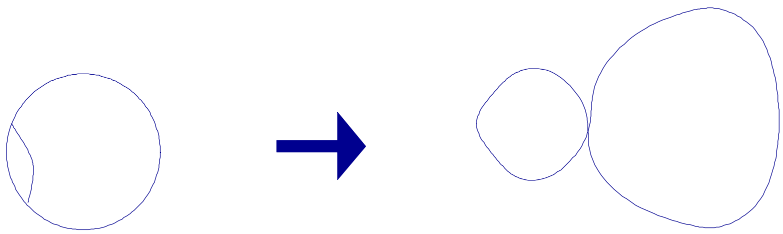

Figure 3: A self-resolving crossing

As may be observed from the illustration in Figure 3 the corresponding chord on a Gauss diagram must be a non-intersected chord which is not separated from the edge of the circle used by any other chords.

Lemma 2 *The reverse of a Reidemeister One (see Figure 4) move corresponds to the removal of a self resolving crossing in the corresponding curve.*

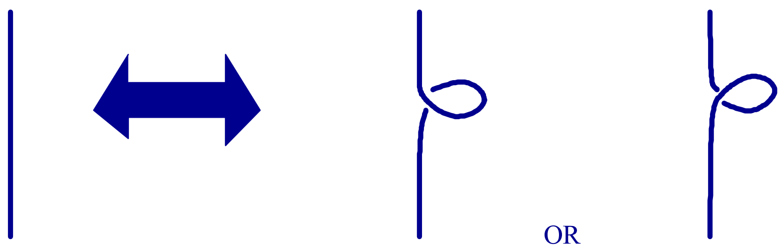

Figure 4

Proof. Evident ■

Theorem 3 *If the outlined method is used to associate a curve with an alternating knot, all treelike curves will correspond to alternating projections of the unknot.*

Proof. *By definition, all chords on the Gauss diagram of a tree-like curve are non-intersecting and thus there must be one with no other chords separating it from the edge of the base circle. This chord will correspond, by definition, to a self resolving crossing and is removed by a Reidemeister One move. After this chord is removed there must be another chord that is not separated from the edge by any others and again this one may be removed by a Reidemeister One move. This process may be continued until all chords are removed and the corresponding knot is the unknot. However, as the original knot corresponding to the curve with no crossings removed is only a sequence of Reidemeister One moves away from the unknot, as shown, the two knots must be equivalent.* ■

The connection between irreducible curves and alternating knots is of more interest. As may be observed from the tables of plane curves the only three crossing irreducible curves, if made into knots, are both projections of the only three crossing knot. Moving to four crossings, the irreducible four crossing curves all correspond to projections of the only four crossing knot. This correlation holds also for irreducible curves with five crossings; all of the five crossing irreducible curves correspond to two Gauss diagrams and there are only two five crossing knots.

3.2 Related Alternating Knots and the Bracket Polynomial

If we associate an alternating knot with every curve it may be asked if we may utilize the invariants of knots, specifically polynomial invariants as they may be calculated more readily, to distinguish between curves or give a more simple method to derive the basic invariants. Derived by Jones in 1984, the polynomial that shares his name is unfortunately related through a composition to the dichromatic polynomial described above and is therefore of no value for the reasons listed in Section 1.

4 Families of Curves

As defined in [3] a treelike curve is one such that every crossing divides it into two loops that have no other points in common. The set of non-treelike curves may, however, be further divided into curves that have some crossings where they may be divided and curves that can be divided at no point.

Definition 4 *A reduction point is a crossing on the curve where the curve may be cut into two disjoint curves.*

4.1 Partially Reducible Curves

Utilizing the definition above we see that a treelike curves has as many reduction points as it has crossings while an irreducible curve has zero reduction points. We would like to focus on the n crossing curves with k reduction points, where $1 \leq k \leq n$. These curves, known as partially reducible curves, may be further divided, as will be seen.

Lemma 5 *A crossing is a reduction point of a curve if and only if it corresponds to a non-intersecting chord on the curve's corresponding Gauss diagram.*

Proof. Suppose we have a reduction point α whose corresponding chord does intersect another chord, say β. The Gauss diagram will then have the form shown in the figure below. As α is a reduction point we may cut along the corresponding crossing and separate the curve into two separate curves. However, the two pieces of the curve that are joined by α are also joined by β (see Figure 5). Thus cutting along α will fail to separate the original curve and a contradiction is reached.

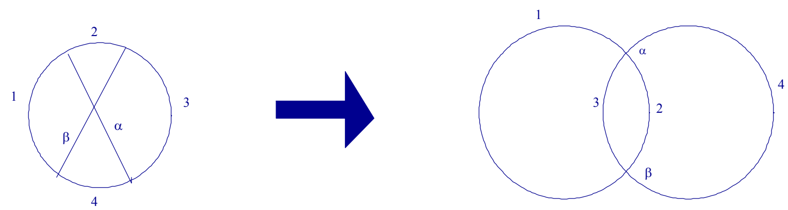

Figure 5: An intersected chord

Suppose, conversely, that we have a non-intersecting chord on a Gauss diagram. If there are no other chords separating it from the edge then the chord corresponds to a self resolving crossing which is clearly a reduction point (see Figure 3). Other chords on both sides of α will lead to a Gauss diagram of the form in Figure 6, which must resolve to a curve of the type shown. As none of the other chords may cross α all other crossings must be resolved within either A or B, thus when the original curve is cut along α two separate curves will be obtained.

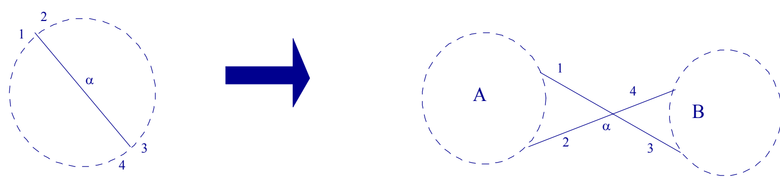

Figure 6: A non-intersected chord

■

4.2 Families of Partially Reducible Curves

As promised above, we now divide the non-treelike curves into groups, which we will call families.

Definition 6 *A family of curves, F_i^n is a group of curves whose Gauss diagrams vary by only non-intersecting chords, and thus vary by only the number and placement of reduction points. The bases of a family are the curves with n crossings that correspond to the i^{th} common Gauss diagram (see Figure 7).*

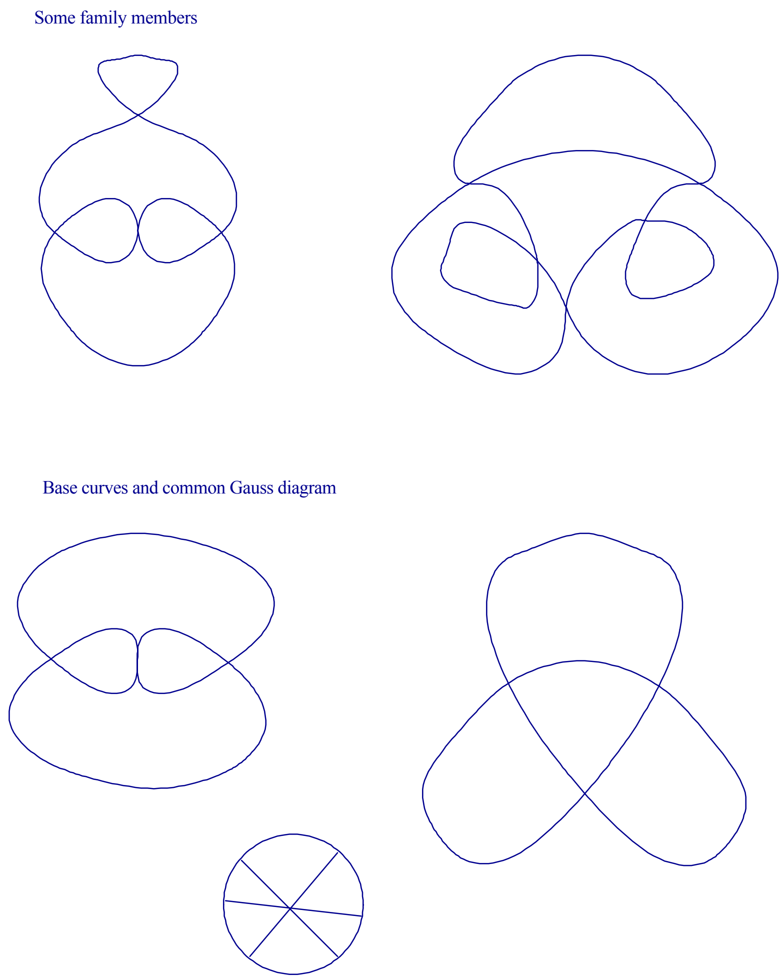

Figure 7

The bases of a family are necessarily irreducible curves, as the only Gauss diagram common in the group of diagrams that vary by only non-intersecting chords must have no non-intersecting chords.

Theorem 7 *If the set of all curves that are members of any family is intersected with the set of all n crossing curves all n crossing non-treelike curves will then be accounted for. Succinctly, if C_n is the set of all curves, and Nt_n is the set of non-treelike curves each with n crossings then* $Nt_n \subseteq (\bigcup_{n=3}^{\infty}(\bigcup_{k=1}^{i} F_k^n)) \cap C_n$

Proof. Suppose not. Then $\exists$ a curve κ with n crossings that is non-treelike, but not a member of any family. As κ is non-treelike it must have k reduction points, where $0 \leq k \leq n-1$. If $k = 0$ then the curve is irreducible and must be a basis and thus a member of F_i^n. If $k \neq 0$ then κ must have a finite number of reduction points. By definition of reduction points, the curve may be cut along these points to form some disjoint curves. If the connected sum of these curves is then formed, taking care to ensure that the union is made along the same strand that was formed when the curve was cut along the reduction point and with the same orientation as they were cut apart it is clear that an irreducible curve with $n-k$ crossings is formed. This curve must be a base for some F_i^{n-k} family of curves and thus the original curve κ must be a member of the same family as it only varies from the base in reduction points. ■

4.3 Special Families

If all the chords in the base Gauss diagram all intersect each other (e.g. the Gauss diagram may not be split into two separate diagrams of irreducible curves by a chord) then an inductive method for finding all family members exists (see Figure 8).

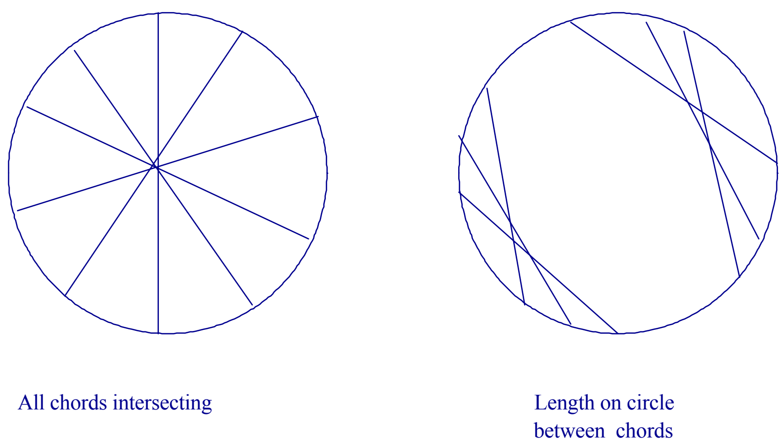

Figure 8

Select a member of F_i^n. With a copy of this curve select a strand and, if the strand is exterior (no other strands separate it from the immersion plane) deform it in one of the three ways shown below in Figure 9. If the strand is not an exterior strand pull it in either way one or way two. When there are copies of the curve such that each strand has been mutated in every way possible move on to the next curve of the family. Continue in this fashion until all of the curves in F_i^n have been altered. The family F_i^{n+1} will be a subset of the union of all the curves produced.

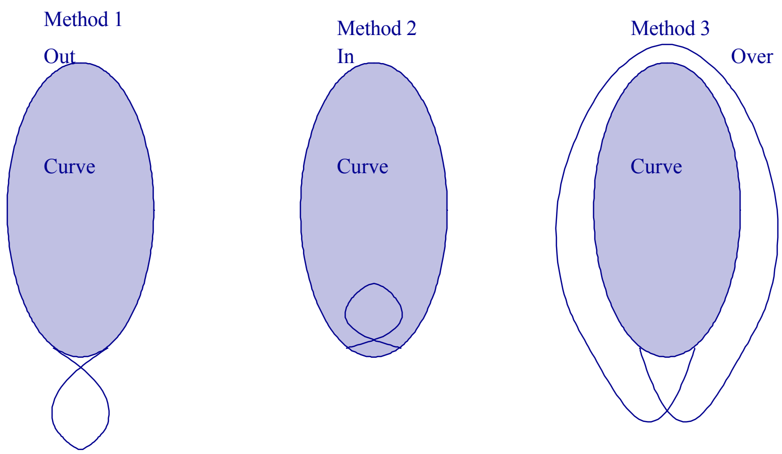

Figure 9

4.3.1 Bound for Cardinality of these Families

Families that are formed in the fashion above will have a bounded number of $n+1$ crossing curves.

Lemma 8 *The number of strands in a curve with n crossings is equal to* $2n$.

Proof. From [4], each curve is created by $\mathrm{J}^{\pm}$ and St moves on the standard curves K_i. Thus if the number of strands is $2n$ for each standard curve and varies appropriately under the moves our lemma will hold.

First note that the number of strands add when a connected sum is preformed, while the number of crossings of the new curve is equal to the sum of the number of crossings of the two factor curves.

Upon inspection the theory holds for K_0 and $K_{\pm 1}$. Assume it holds for $K_{\pm n}$. Again from [4] we know that $K_{\pm(n+1)}$ is the connected sum of $K_{\pm 1}$ and $K_{\pm n}$ which we assume have 2 and $2n$ strands, respectively. Thus we have a $n+1$ crossing curve which must have $2+2n = 2(n+1)$ strands. Thus the lemma is verified for all standard curves by principle of mathematical induction.

The lemma also holds under the basic moves, as illustrated in Figure 10.

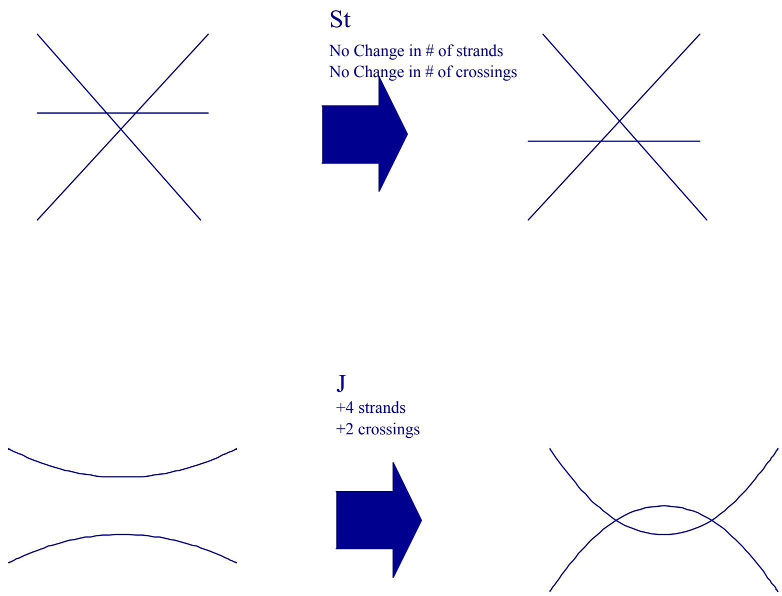

Figure 10

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Thus the number of curves with $n+1$ crossings is bounded by the number of strands, the number of curves in the family with n crossings, and the number of ways to mutate these curves. If $\#(F_i^n \cap C_n)$ is the number of n crossing curves in a family then $\#(F_i^{n+1} \cap C_{n+1}) \leq \#(F_i^n \cap C_n) * 2n * 3$, by simple combinatorics.

4.3.2 Variation of the Index in these Families

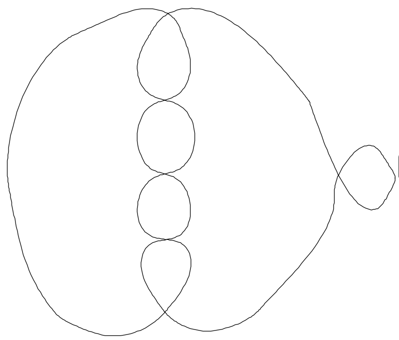
As noted above, the index of a curve is simply the rotation number of the tangent vector. Therefor, as we are only adding self resolving crossing in each case, the variation of the index must reflect this. From [5] we obtain the following theorem of Whitney for the index of a curve.

Theorem 9 *The index of an immersed circle is equal to* $a + \sum$*, where* $a = \pm 1$ *and* $\sum$ *is the sum of* ± 1 *over all double points of the curve.*

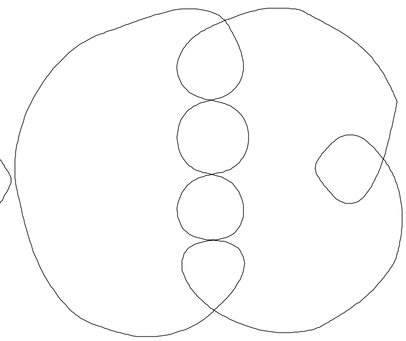
Thus if a certain curve κ has index $= i$ and we insert a crossing we may only alter $\sum$ by one and thus alter the index by one.

4.4 Table of Curves Belonging to F_2^6

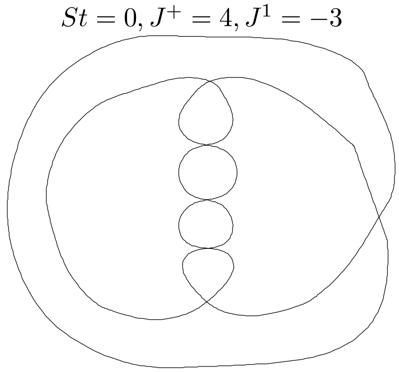
Utilizing the method described above to derive the curves, we present a table of the curves belonging to this family, along with their invariants.



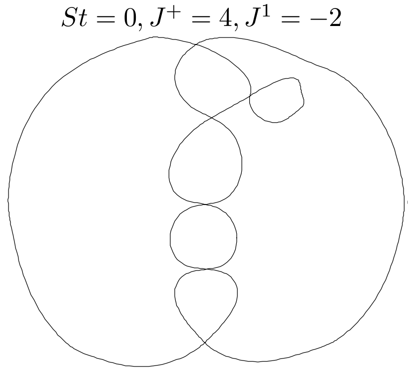
$St = 0, J^+ = 4, J^1 = -3$



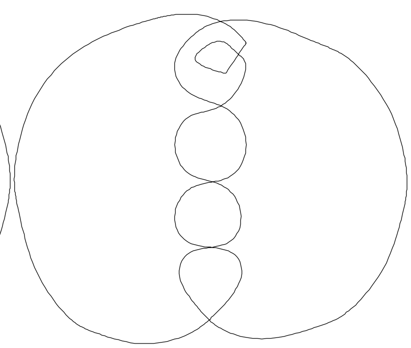
$St = 1, J^+ = 2, J^1 = -4$



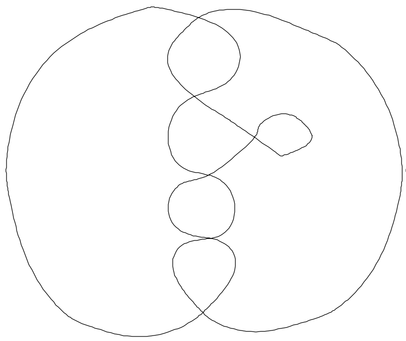
$St = 0, J^+ = 4, J^1 = -2$



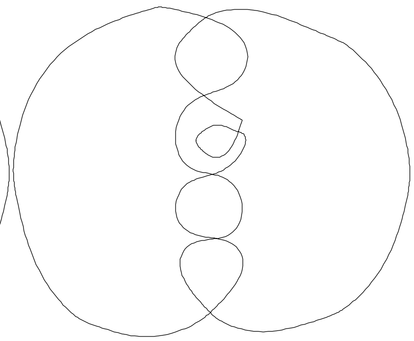
$St = 1, J^+ = 4, J^1 = -2$



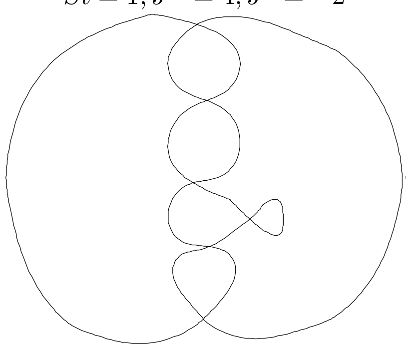
$St = 0, J^+ = 4, J^1 = -2$



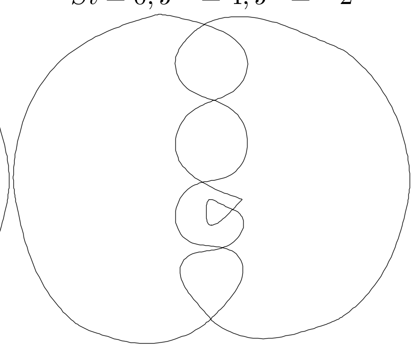
$St = 1, J^+ = 4, J^1 = -2$



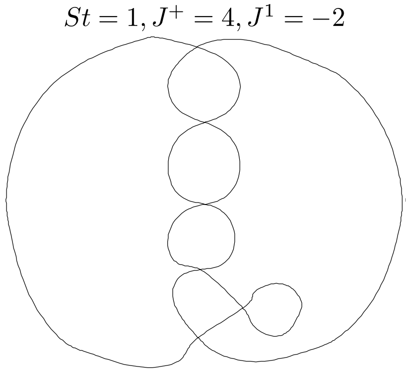
$St = 0, J^+ = 4, J^1 = -2$



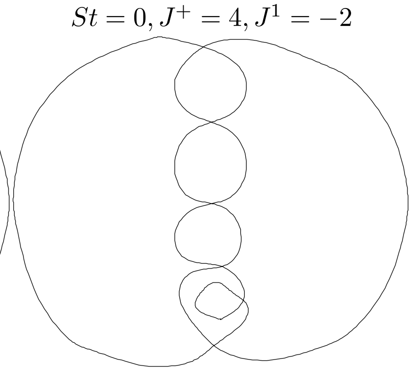
$St = 1, J^+ = 4, J^1 = -2$



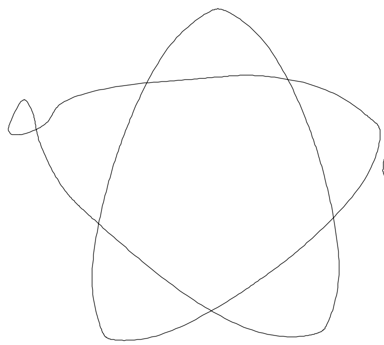
$St = 0, J^+ = 4, J^1 = -2$



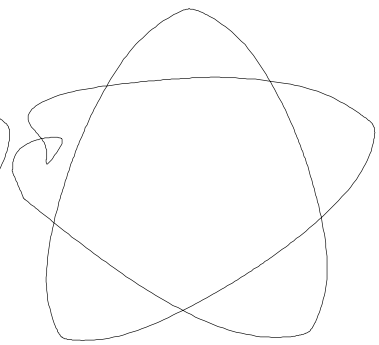
$St = 1, J^+ = 4, J^1 = -2$



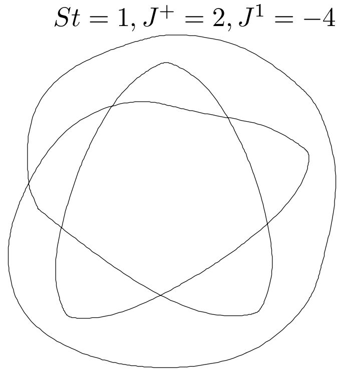
$St = 0, J^+ = 4, J^1 = -2$



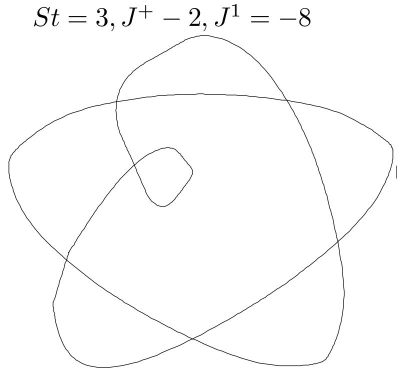
$St = 1, J^+ = 2, J^1 = -4$



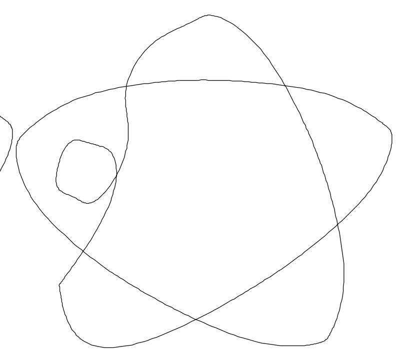
$St = 2, J^+ = 0, J^1 = -6$



$St = 3, J^+ - 2, J^1 = -8$



$St = 3, J^+ = -2, J^1 = -8$



$St = 0, J^+ = 4, J^1 = -2$

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