Connected Sums and Decompositions of Plane Curves

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Abstract

Just as the connected sum is used to combine plane curves, it can be used to define decompositions of composite plane curves. We do just that, applying this idea to equivalent compositions of the A-structures of tree-like curves as well. Following this trail yields a proof of uniqueness of prime decomposition for tree-like curves, as well as some interesting results on spherical tree-like curves (with corresponding S-structures). We then examine the restrictions on index through the connected sum and conclude with a discussion of well-defined connected sums, including related symmetry conditions.

0 Introduction

The theory of plane curves dates back at least to Gauss, as far back as the beginnings of its cousin, knot theory. Recent fundamental work by Arnold [4, 5] attracted new attention to this field, introducing tools for analysis via the local invariants $J^{\pm}$ and St. However, the question of full classification up to ambient diffeomorphisms of the plane and the curve remains open.

The first modern treatment of plane curves was given by Whitney in [7], who examined the question of classification up to isotopy (i.e. deformations of the curve without vanishing tangent vector on intermediate curves). This problem was completely solved by his introduction of the *index* of a curve, the rotation number of the tangent vector. He introduced local as well as integral formulas for this value.

More recent progress has been made by Aicardi [3], who presented a full classification of curves with the property that every double point divides it into two disjoint loops (tree-like curves). This was achieved by way of A-structures, combinatorial structures associated with tree-like curves that prove useful in

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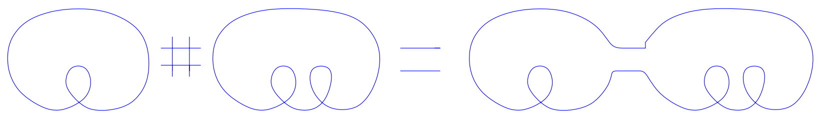


Figure 1: The connected sum of two curves

many applications. Similar structures (S-structures) were introduced for immersions of the circle into the sphere. Much of this was rooted in a treatment of plane curves in terms of their associated Gauss diagrams.

While plane curves have been the study of many other investigations as well, none thus far has focused on the composition (and, naturally, decomposition) of curves. From Arnold, we have at least one method of composition — the connected sum. The connected sum is especially interesting as Arnold's invariants are additive through it. Thus, the following is a preliminary survey of the connected sum, working under the scope of the work described above.

Within the paper itself, the terms *plane curve* and *curve* are generally used to denote a diffeomorphism class of immersions of the circle into the plane, with *spherical curve* used for immersions onto the sphere. By *immersion*, we mean generic immersions, those with only natural double points (i.e. no triple points or self-tangencies). *Diffeomorphism* means ambient diffeomorphisms of the plane and curve, while *classification* refers to classification up to these diffeomorphisms. The use of these terms and others will be made clear by the context.

1 Definitions

We begin by laying the necessary groundwork, starting with the following definition from [5]:

Definition 1 *The* connected sum *of two immersions, the first* (Γ_1) *into the left half-plane, the second* (Γ_2) *into the right half-plane, is defined as in Figure 1 by an embedding of a connecting bridge into the complement to the images of the two original immersions. We write* $\Gamma_1\#\Gamma_2$.

This naturally leads to:

Definition 2 *A* decomposition *is the separation of a curve into two disjoint, well-defined curves such that there exists a connected sum between the two curves which yields the original curve.*

A few of the implications of these two definitions arise immediately. Notice that the connected sum is not an operation on the classes of immersions, since the bridges might be different (though $\Gamma_1\#\Gamma_2$ and $\Gamma_2\#\Gamma_1$ denote the same collection). However, the basic invariants are additive under any choice of the bridge. Also, the number of double points is additive under the connected sum.

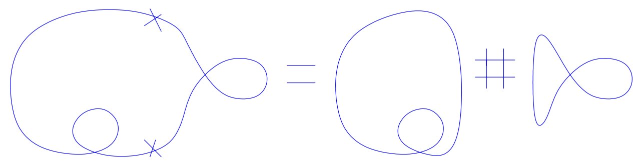


Figure 2: Curve decomposition

This guarantees that the trivial curve (i.e. the circle) cannot be expressed as the connected sum of two non-trivial curves, and leads towards seeing that the connected sum of a trivial curve and a non-trivial curve yields the non-trivial curve. Furthermore, the connected sum of two tree-like curves yields a tree-like curve, whereas the connected sum of a non-tree-like curve with any other curve is non-tree-like. Finally, the embedded connecting bridge of the connected sum demands that each of the discs joined by the bridge must be an exterior disc.

Defining decomposition in terms of the connected sum holds some interesting implications as well. A decomposition amounts to a detachment of the curve at two points lying on the boundary of the same external disc, attaching the new endpoints in such a manner that two curves are created. Neither of these two detachment points may lie at a double point. No discs other than the disc in question are affected by a decomposition – for instance, external discs remain external, and adjacent discs remain adjacent. See Figure 2 for an example of a decomposition.

Of course, if curves can be composed and decomposed in some manner, it makes sense to consider such ideas as prime curves, composite curves, and factor curves. We define these in the expected manner:

Definition 3 *A* prime curve *is a well-defined curve that cannot be expressed as the connected sum of two well-defined curves.*

Definition 4 *A* composite curve *is a well-defined curve that can be expressed as the connected sum of two or more well-defined curves.*

Definition 5 *A* factor curve *of a composite curve is a well-defined curve that can be combined with some other well-defined curve or curves by means of the connected sum to yield the composite curve.*

Thus, a curve with n double points is composed of, at most, n factor curves, due to the additive nature of double points. Also, every factor curve of a composite curve with n double points must have less than n double points. And, of course, every curve is either composite or prime.

More importantly, these definitions beg the question of how to determine whether a curve is prime or composite. We begin with a definition:

Definition 6 *A* decomposition line *of a generic curve is a line, up to diffeomorphism, that can be drawn such that it intersects the curve at exactly two points, neither of them double points, with one or more double points lying on either side of the line.*

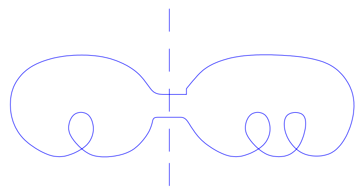


Figure 3: A decomposition line on a curve

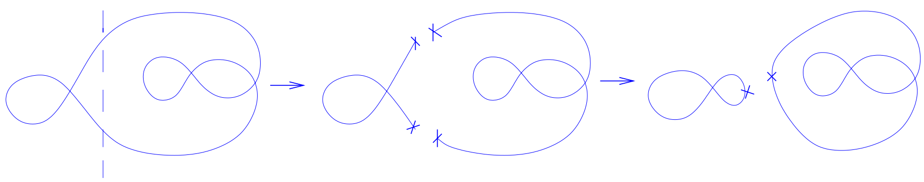


Figure 4: A decomposition line guarantees a composite curve

Proposition 1.1 *A curve is composite iff it has a decomposition line.*

proof $\Rightarrow$ We show that the act of a connected sum creates a decomposition line. Perform a connected sum, and assume that there does not exist a decomposition line. Clearly, there exist many lines intersecting the embedded connecting bridge twice, once through each of the two arcs. If every one of these lines intersects the curve at least one other time, then removing the connecting bridge does not result in two curves – these curves are connected at some other point, and therefore actually comprise only one curve. Thus, there exists at least one line intersecting the embedded connecting bridge twice, and intersecting the curve nowhere else. We know at least one double point lies on either side of the connecting bridge in a well-defined connected sum, and thus, at least one double point lies on either side of the line in question. Thus, the act of a connected sum creates a decomposition line, as in Figure 3. Since this line bisects the plane, it remains a decomposition line under ambient diffeomorphisms.

$\Leftarrow$ If a curve has a decomposition line, it intersects the curve in exactly two points. Disconnect the curve at these two points. This leaves two disjoint immersions of the line segment into the plane, each with at least one double point. Attach the four endpoints in pairs such that the two resultant nontrivial immersions of the circle remain disjoint, with every disc (and only those discs) external on the original curve external on the resultant curves, including the two discs created by this endpoint attachment. Then, an embedded bridge connecting these two discs is a connected sum yielding the original curve (see Figure 4). □

Thus, the prime or composite nature of a curve is a global invariant. Adding this consideration to the curve characteristics previously discussed yields a more robust classification scheme. What's more, including this characteristic creates a number of relatively minor implications. We prove one of the more interesting ones:

Proposition 1.2 *Every curve with* $St > \frac{n(n-1)}{2} + 1$ *is prime.*

proof Assume this curve is composite. Then, it can be expressed as $\Gamma_1 \# \Gamma_2$, where the number of double points in Γ_1 is k and the number in Γ_2 is $n-k$. We know that Arnold's invariants are additive through the connected sum. From [4], we know that the maximum value for St on a curve is $\frac{n(n+1)}{2}$, where n is the number of double points. Thus, the maximum value for St on $\Gamma_1 \# \Gamma_2$ is $\frac{k(k+1)}{2} + \frac{(n-k)(n-k+1)}{2}$. This function has a minimum at $k = \frac{n}{2}$ and a maximum on the boundaries (i.e. $k = 1$ or $n - k = 1$). For these boundary values, $\frac{k(k+1)}{2} + \frac{(n-k)(n-k+1)}{2} = \frac{n(n-1)}{2} + 1$. This is a contradiction, and therefore the curve in question must be prime. □

2 Decomposition and the connected sum on tree-like curves

Having considered the preliminary questions of connected sums and decompositions, we turn to tree-like curves, and consider the same questions on their A-structures, as defined in [3]. Much of this requires the use of Gauss diagrams, also found in [3]. Thus, more specific treatments of these ideas will clarify the subsequent examination:

Definition 7 *The* Gauss diagram *of a curve with* n *ordinary double points is the class of chord diagrams formed by* n *chords connecting the preimages of each self-intersection point of the immersed curve in the standard disc bounded by the standard circle.*

Definition 8 *The* planar tree *of a tree-like curve is the tree of its Gauss diagram formed by* n *nonintersecting chords in the standard oriented disc, consisting of* $n+1$ *vertices (one point inside each of the* $n+1$ *domains into which the disc is cut by the chords) and of* n *edges (straight segments connecting the points into neighboring domains across the chords).*

Definition 9 *The* associated disc *of a vertex of the tree of a tree-like curve is the disc bounded by the images of the boundary arcs of the convex Gauss diagram domain containing this vertex.*

Definition 10 *The last vertex on the path from the subtree* F *to a vertex* v *not belonging to* F *is called the* father *of* v *and is denoted by* $f(v)$.

Definition 11 *The* A-structure *of a tree-like curve is the following collection:*

- T *— the tree of the curve*

- *F — the subtree of the exterior associated discs*
- *c — the character function, defined as follows:*
  1. *$c(v) = -1$ if $v \in F$ is an exterior vertex*
  2. *$c(v) = +1$ if the associated disc of v lies inside the associated disc of its father*
  3. *$c(v) = -1$ if the associated disc of v lies outside the associated disc of its father*

Naturally, we begin the examination itself by defining a connected sum between A-structures:

Proposition 2.1 *A connected sum between two tree-like curves is equivalent to combining their two A-structures by means of an identification of the two vertices, one from each tree, whose associated discs are joined by the embedded connecting bridge of the connected sum, in such a way that the resultant A-structure is the A-structure of the resultant curve.*

proof First, examine the tree T. We draw the associated Gauss diagrams of the two curves joined by the connected sum. From [3], we know each arc of the curve is associated with an arc on the Gauss diagram. The connected sum joins two of these arcs, one from each curve, by the connecting bridge. Notice this is a local action (i.e. no other sections of the curve are affected). Thus, we can do the same for the Gauss diagrams (join associated arcs by a connecting bridge), yielding the Gauss diagram of the resultant curve.

From the Gauss diagrams, we find the trees directly, associating a vertex to every region and connecting edges between points associated to neighboring domains. Since the Gauss diagrams can be connected as described above, the trees can be added in the same way, identifying the vertices associated with the two regions of the Gauss diagram joined by the connected sum.

Next, examine the subtree F. Again, we note that a connected sum is a local action connecting two exterior discs. Thus, the two discs connected by the connecting bridge form an exterior disc, since the bridge itself is exterior. Also, all other discs remain unaffected by this local action, and, therefore, all exterior discs remain exterior, and no others are newly exterior. Thus, those vertices in the subtree F (i.e. those vertices with exterior associated discs) remain the only members of F.

Finally, examine the character function. The value of the character function at a vertex is dependent on two things:

1. the father of the vertex in question
2. whether the associated disc of the vertex in question is interior or exterior to the associated disc of its father

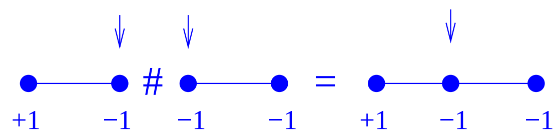


Figure 5: A connected sum of two A-structures. Reading the equation from right to left expresses a decomposition.

Since the tree and subtree F hold as shown above, the father of a vertex remains unchanged through a connected sum. Also, as above, since the connected sum is local, the interior/exterior value of a disc in relation to another disc remains unchanged. Thus, the character function holds through the connected sum. □

Corollary *A decomposition of a tree-like curve is equivalent to the separation of the A-structure into two well-defined A-structures such that a connected sum between the two yields the orginal A-structure.*

We define prime, composite, and factor A-structures in the expected manner.

Just as a connected sum is the identification of two vertices, a decomposition is the opposite — the splitting of one vertex into two, as seen in Figure 5. Other properties of the connected sum have implications for decomposition as well. For example, combining Lemma 2.2 in [3] with the proposition just proved demands that the vertices involved in a connected sum be vertices of the subtree F (i.e. vertices with exterior associated discs). Thus, a decomposition can only be performed at a vertex in the subtree F. To ensure that the factor A-structures are both well-defined, a decomposition can only be performed at a vertex v if there are at least two vertices adjacent to v. Finally, in order that the connected sum of two factor A-structures yields the original, we have:

Note 1 *A decomposition must retain the character function of the original A-structure.*

Even while taking into account all of the above properties, compositions and decompositions can sometimes be done in different ways, even at the same vertex, as in Figure 6. Thus, as before, these are not operations on A-structures.

Applying these properties tells us much about the A-structures of prime and composite tree-like curves. The A-structure of a prime tree-like curve cannot have a vertex in the subtree F with two adjacent vertices. Thus, the only prime tree-like curve with two vertices in the subtree F is shown in Figure 7 and is denoted K_0. Its A-structure is comprised only of those two vertices. Every other prime tree-like curve has exactly one vertex in the subtree F, with exactly one adjacent vertex, also shown in Figure 7. All tree-like curves not having these structures are composite.

Using A-structures to examine the constructions found in [3] of curves with extremal values of Arnold's invariants creates yet another set of minor implications. For example, it is easy to see, from examination of the A-structure, that

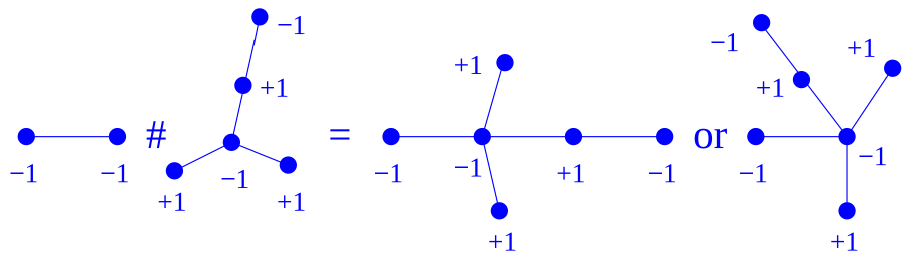


Figure 6: The different connected sums of two A-structures

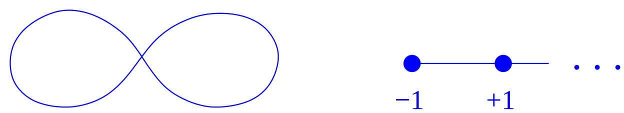


Figure 7: The prime curve K_0, and the A-structure of all other prime curves.

every tree-like curve with n double points on which St is a maximum is prime. More examples of implications of this type abound.

3 Prime decomposition of tree-like curves

It is quite natural to next pursue the question of prime decomposition. The structure lent by tree-like curves yields a quite useful environment for this investigation:

Proposition 3.1 *Let a vertex $v \in F$. Let x_v be the number of vertices adjacent to v. Then, a decomposition can be performed $x_v - 1$ times at v, resulting in x_v A-structures.*

proof By examination, if $x_v = 2$, a decomposition can be performed once at v, resulting in two A-structures (see Figure 8).

We proceed inductively. Let $x_v = n$, and assume the proposition holds for all values of x_v up to $n - 1$. Perform one decomposition at v. The result is two A-structures, one with y vertices adjacent to v, the other with z adjacent vertices, with $y + z = n$. Since y and z are less than n, we know we can perform decompositions at v on our new A-structures $y - 1$ and $z - 1$ times, resulting in y and z A-structures, respectively. Therefore, the total number of decompositions we can perform at v is $1 + (y - 1) + (z - 1) = n - 1$, resulting in $y + z = n$ A-structures. □

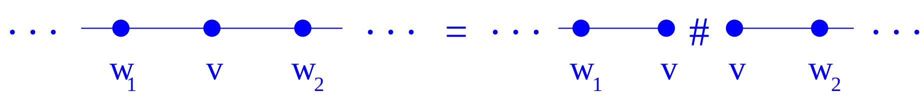


Figure 8: Decomposition for $x_v = 2$

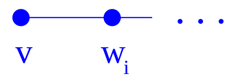


Figure 9: A resultant A-structure after $x_v - 1$ decompositions

Proposition 3.2 *The total decomposition at a vertex v (i.e. the end result of performing $x_v - 1$ decompositions) is unique.*

proof From Proposition 3.1, we know the total decomposition at v consist of x_v well-defined A-structures. Thus, each of these has the form shown in Figure 9, where w_i is one of the x_v vertices adjacent to v. Since the collection of A-structures of this form is unique, the total decomposition is unique. □

Proposition 3.3 *A decomposition performed at one vertex does not affect possible decompositions at other vertices.*

proof Vertices at which decompositions can be performed must be in F, and by Note 1, decomposition does not affect which vertices are in F. By Proposition 3.1, decomposition at a vertex depends only on the vertices adjacent to the vertex in question. Decomposition at any other vertex leaves this unaffected, as the decomposition action is performed only at that vertex. □

Note 2 *The total number of decompositions that can be performed on an A-structure is* $\sum_F (x_v - 1)$, *resulting in* $\sum_F (x_v - 1) + 1$ *A-structures.*

Follows from Propositions 3.1 and 3.3.

Lemma 3.1 *There exists a natural bijection between the sets of nonplanar A-structures having n edges and of classes of nonoriented tree-like curves in the nonoriented plane having n double points.*

proof From [3] Theorem 2.1, we have a bijection between planar A-structures and oriented tree-like curves in the nonoriented plane. Also, from [3] Remark 2.3, we know that the reversal of the orientation of the curve acts on the planar structures as a reflection (reversing the cyclic order at every vertex). Thus, ignoring orientation on the curve is equivalent to ignoring planar orientation on the A-structure. □

Defining prime decomposition in the usual manner, we have the following:

Theorem 3.1 *The prime decomposition of a tree-like curve is unique.*

proof Combining Propositions 3.2 and 3.3 above yields the conclusion that prime decomposition of an A-structure is unique. Combining this result with Lemma 3.1 yields the same for the associated tree-like curve. □

A unique prime decomposition on the tree-like curve yields yet another method of classification, as well as a useful method for determining Arnold's invariants and other properties. Furthermore, it allows tabulation by primes alone, similar to the standard practice for knots.

Since the connected sum is not an operation, we already know that two tree-like curves that share the same prime decomposition need not be identical. However, we can say that these two curves share the same values for Arnold's invariants, their Gauss diagrams share the same number of chords, and their A-structures have the same number of vertices, the same number of vertices for each character value, and the same number of vertices in the subtree F.

4 Spherical curves

Since the S-structures on spherical tree-like curves are closely related to A-structures on plane curves, it seems appropriate to examine these in the same light. We first need to understand these S-structures:

Definition 12 *If a vertex v is the father of the father of another vertex w, then it is called the* grandfather *of w and is denoted $v = ff(w)$. If v and w have a common father, then they are called* brothers *and we write $v = b(w)$ or $w = b(v)$.*

Definition 13 *The* S-structure *of a tree-like spherical curve is the following collection:*

- *T — the tree of the curve*
- *ρ — the function "rebro" defined by the following:*
  1. *$\rho(g(v,w)) = 0$ if v and w are brothers with the same value of the character function or if $v = ff(w)$ and $c(w) = -1$*
  2. *$\rho(g(v,w)) = 1$ if v and w are brothers with different values of the character function or if $v = ff(w)$ and $c(w) = +1$*

Since the sphere does not lend itself towards an easy definition of a connected sum, we examine instead the associated plane curves of a spherical curve (i.e. those plane curves which result from placing the point at infinity in one of the regions defined by the spherical curve). A quick examination of the tables of spherical curves gives rise to the following:

Theorem 4.1 *Every tree-like spherical curve ($n \geq 3$) can be associated with at least one composite plane curve.*

proof Since every spherical curve can be associated to at least one plane curve, it suffices to show that the A-structure of a prime plane curve yields an S-structure which can also be associated to the A-structure of a composite plane curve.

From the discussion of tree-like curves, the A-structure of a prime plane curve has the following properties (recall Figure 7):

1. the number of vertices in the subtree F is one
2. the number of vertices connected to the subtree F at distance one is one

We construct the A-structure of a composite plane curve having the same S-structure. Let v_0 denote the vertex comprising the subtree F. Then the vertex v_1 denotes the unique vertex connected to v_0 at distance one. Thus, v_1 has no brothers. Therefore, all vertices w for which $\rho(g(v_1, w))$ is defined must be a grandfather or grandson of v_1. Since v_1 has no grandfather, $v_1 = ff(w)$. Then, from [3], the value of $\rho(g(v_1, w))$ is dependent only on the value of the character function at w. Thus, the value of the character function at v_1 has no bearing on the function ρ.

Since v_1 is connected to the subtree F at distance one, the value of the character function at v_1 must be $+1$. However, as shown above, the value of the character function at v_1 has no bearing on the function ρ. Thus, utilizing the same planar tree, we let $c(v_1) = -1$, and leave the value of the character function unchanged at every other vertex. The resultant A-structure denotes a composite plane curve for $n \geq 3$, and has the same S-structure as the original A-structure. □

The method used in this proof lends suspicions that it could be altered to prove the apparent opposite, a corrolation with prime plane curves, as well. Indeed, this turns out to be the case:

Theorem 4.2 *Every tree-like spherical curve ($n \geq 2$) can be associated with at least one prime plane curve.*

proof As before, it suffices to show that the A-structure of a composite plane curve yields an S-structure which can also be associated to the A-structure of a prime plane curve.

Given the A-structure of a composite plane curve, we construct the A-structure of a prime plane curve having the same S-structure. Let v_0 be a terminal vertex of the original tree. Let $v_0 \in F$ in our new A-structure. Then, $c(v_0) = -1$. Since v_0 was a terminal vertex, only one vertex is connected to v_0 at distance one. Let v_1 denote this vertex, and let $c(v_1) = +1$. Let $v_2, ..., v_n$ be the remaining vertices of the tree. Then, for $i \geq 2$, $ff(v_i)$ is defined. For all vertices v_k such that $v_0 = ff(v_k)$, define $c(v_k)$ so that $\rho(g(v_k, v_0)$ has the same value as on the S-structure of the original A-structure. Repeat the same operation for all grandsons of $v_1, ..., v_n$ so that the character function is defined for all vertices. Then, for all vertices v and w such

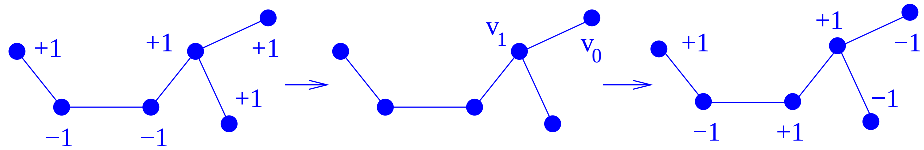


Figure 10: Producing the A-structure of a prime curve from that of a composite curve while preserving the S-structure.

that $v = ff(w)$, the function ρ is defined and is equal to ρ on the S-structure of the original A-structure. All that remains is to consider all vertices y and z such that $y = b(z)$. Since y and z are brothers, $f(y) = f(z)$ and, thus, $ff(y) = ff(z)$. From [3], we know $\rho(g(y,a)) + \rho(g(z,a)) + \rho(g(y,z)) = 0 \bmod 2$. Letting $a = ff(y) = ff(z)$, $\rho(g(y,a))$ and $\rho(g(z,a))$ are already defined, which uniquely determines $\rho(g(y,z))$. Thus, the function ρ (and, therefore, the S-structure) of our new A-structure is identical to that of the original A-structure. Furthermore, the properties of v_0 and v_1 as defined above demand that the plane curve associated with this new A-structure be prime (see Figure 10). □

While the combination of these two theorems seems to erase any possible results of interest, they do, in fact, yield some noteworthy points. Combining them with a result from [6], that the Gauss diagram is a complete invariant for spherical curves, produces the following:

Note 3 *Every tree-like Gauss diagram ($n \geq 3$) can be associated with at least one composite curve and at least one prime curve.*

Note 4 *The number of spherical plane curves with n double points is less than the number of composite plane curves with n double points as well as the number of prime plane curves with n double points.*

Examination of the tables of curves with n double points for $n \leq 5$ confirms this second note in a convincing manner (e.g. the number of spherical curves with $n = 5$ is 76; the composite and prime curves each number over 200). In any event, these provide some clues toward relating spherical curves with their associated plane curves without appealing to the direct method of placing the point at infinity inside one of the regions the curve defines on the sphere.

Turning to non-tree-like curves on the sphere, we would hope to find a similar correlation. Some non-tree-like curves can indeed be associated with both a composite and a prime plane curve, but in general, this is not true, and there does not seem to be a simple method for determining which can or can't (see Figure 11).

5 Index and the connected sum

Unlike Arnold's invariants, the index is not necessarily additive through the connected sum. However, we know from other cases that the index is restricted.

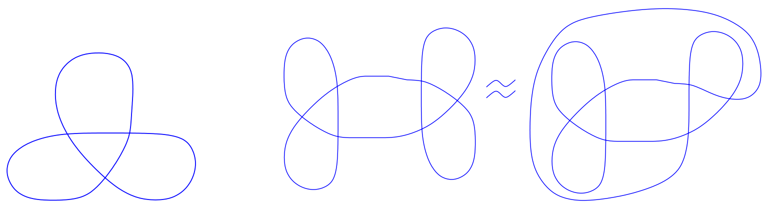


Figure 11: A spherical non-tree-like curve with only prime associated plane curves and one with both prime and composite associated plane curves.

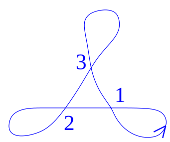
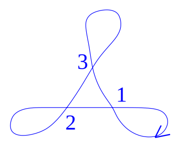
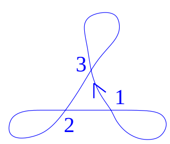
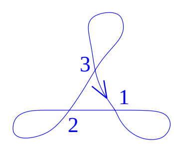
	$orient. = +1$ $\varepsilon_1 = -1$ $\varepsilon_2 = +1$ $\varepsilon_3 = +1$ $\lvert\sum\rvert = 2$		$orient. = -1$ $\varepsilon_1 = +1$ $\varepsilon_2 = +1$ $\varepsilon_3 = -1$ $\lvert\sum\rvert = 2$
	$orient. = +1$ $\varepsilon_1 = -1$ $\varepsilon_2 = -1$ $\varepsilon_3 = -1$ $\lvert\sum\rvert = 2$		$orient. = -1$ $\varepsilon_1 = +1$ $\varepsilon_2 = +1$ $\varepsilon_3 = +1$ $\lvert\sum\rvert = 2$

Figure 12: While index is constant, the values of ε_i vary with choice of base point and orientation

For example, a curve with n double points must have $ind \leq n+1$, with $ind - n$ an odd integer. Likewise, the index is restricted by the connected sum. From [7], we know:

$$ind = \sum \varepsilon_i \pm 1 \tag{1}$$

where the summation is over the set of n double points, ε_i is a sign associated to each double point, and the term ± 1 depends on the orientations of the curve and of the plane.

Since we are considering the nonoriented curves immersed in the nonoriented plane, we instead use:

$$ind = \left|\sum \varepsilon_i \pm 1\right| \tag{2}$$

This expression is independent of choice of orientation of the curve and of the plane. However, as seen in Figure 12, the values of ε_i depend on the orientation on the curve and plane as well as the choice of base point.

Applying this equation to the curves in a connected sum yields the restrictions on the index:

Theorem 5.1 $ind(\Gamma_1 \# \Gamma_2) = |ind(\Gamma_1) \pm ind(\Gamma_2) \pm 1|$

proof Pick a base point on $\Gamma_1 \# \Gamma_2$ previously on Γ_1, and orient the curve positively with respect to the plane. Then, from Equation 2:

$$ind(\Gamma_1 \# \Gamma_2) = \left| \sum \varepsilon_i + 1 \right|$$

We can examine the double points formerly in Γ_1 and Γ_2 separately. Thus:

$$ind(\Gamma_1 \# \Gamma_2) = \left| \sum_{\Gamma_1} \varepsilon_i + \sum_{\Gamma_2} \varepsilon_i + 1 \right|$$

Because our base point was on Γ_1 and the curve is oriented positively, we know:

$$\begin{aligned} ind(\Gamma_1) &= \left| \sum_{\Gamma_1} \varepsilon_i + 1 \right| \\ \pm ind(\Gamma_1) - 1 &= \sum_{\Gamma_1} \varepsilon_i \end{aligned}$$

Since the connected sum induces a base point and orientation on Γ_2 dependent on which connecting bridge is chosen, all we can say is:

$$\begin{aligned} ind(\Gamma_2) &= \left| \sum_{\Gamma_2} \varepsilon_i \pm 1 \right| \\ \pm ind(\Gamma_2) \pm 1 &= \sum_{\Gamma_2} \varepsilon_i \end{aligned}$$

Thus:

$$\begin{aligned} ind(\Gamma_1 \# \Gamma_2) &= |(\pm ind(\Gamma_1) - 1) + (\pm ind(\Gamma_2) \pm 1) + 1| \\ &= |\pm ind(\Gamma_1) \pm ind(\Gamma_2) \pm 1| \\ &= |ind(\Gamma_1) \pm ind(\Gamma_2) \pm 1| \end{aligned}$$

□

An alternate expression, for the index of tree-like curves (with corresponding A-structures), can be found in [3]:

$$ind = \left| \sum s(v_i) \right|$$

where v_i are the A-structure vertices, $s(v_0) = \pm 1$, and $s(v_i) = c(v_i)s(f(v_i))$. Thus, we can substitute this expression into Theorem 5.1:

Corollary *Let Γ_1 and Γ_2 be tree-like curves, and let the vertices of their corresponding A-structures be denoted v_i and w_i, respectively. Then:*

$$ind(\Gamma_1 \# \Gamma_2) = \left| \sum s(v_i) \pm \sum s(w_i) \pm 1 \right|$$

Notice that Theorem 5.1 implies that, for some curves Γ_1 and Γ_2, $\Gamma_1 \# \Gamma_2$ will yield at least four distinct curves. Intuitively, this cannot be true for all curves. It is to this question that we next turn.

6 Some 'nice' compositions

As it turns out, determining the possible products of two curves is, in general, a very difficult problem. We take the specific case of determining which products are well-defined, and restrict ourselves to tree-like curves. The reader may already know that, in the case of knots, the idea of 'invertibility' on a knot (i.e. the invariance of the knot through changes of orientation) is integral to determining whether or not the product of two knots is well-defined (see [1, pp. 10–12]). Similarly, symmetry conditions are necessary in solving the same problem for tree-like curves. We begin with a look at those conditions.

6.1 Symmetry conditions

It would be quite useful to be able to relate symmetry on the curve to symmetry on its A-structure. We begin with this task. Let Σ and σ denote the standard orientation reversal of the plane and circle, respectively, and let T and τ denote the standard antipodal involution of the same. Then, the three symmetric involutions are (T, σ), (Σ, τ), and (Σ, σ).

Note that in all of the explicit curve constructions found in proofs below, the method found in [3] may be substituted without affecting the results. Recall that all curves discussed in this section are tree-like.

Definition 14 *A* symmetric representative *is a curve which is exactly invariant under one of the symmetry involutions. Every symmetric representative has a* fixed point.

Proposition 6.1 *A tree-like curve has symmetry of type* $[\Gamma\sigma] = [\Gamma]$ *iff its A-structure can be drawn such that there exists a line, not containing a vertex, across which the A-structure has perfect reflective symmetry.*

proof $\Rightarrow$ From [4], given a curve with symmetry of type $[\Gamma\sigma] = [\Gamma]$, it has a symmetric representative. Also from [4], we know a curve with this symmetry type must have $ind = 0$. Thus, the curve has an odd number of double points, and the fixed point must be one of these double points.

We draw the Gauss diagram, using our fixed double point as a base point. We scale the length of the curve linearly such that the entire length of the curve scales to the circumference of the Gauss diagram. Thus, the symmetry of the curve about the fixed double point implies that the chord corresponding to that double point bisects the Gauss diagram. Beginning at the fixed double point, we follow the curve until we reach the next double point, and denote it on the Gauss diagram as a, at the appropriately scaled distance along the circumference. Because the curve is symmetric under σ, change of orientation, following the curve in the opposite direction (i.e. reversing the orientation) must yield a double point at the same distance, which we denote a'. We continue in this manner, noting all double points and their symmetric counterparts, until we return to the fixed double point. Constructing the chords of the diagram from these double points, then, yields a Gauss diagram perfectly symmetric across

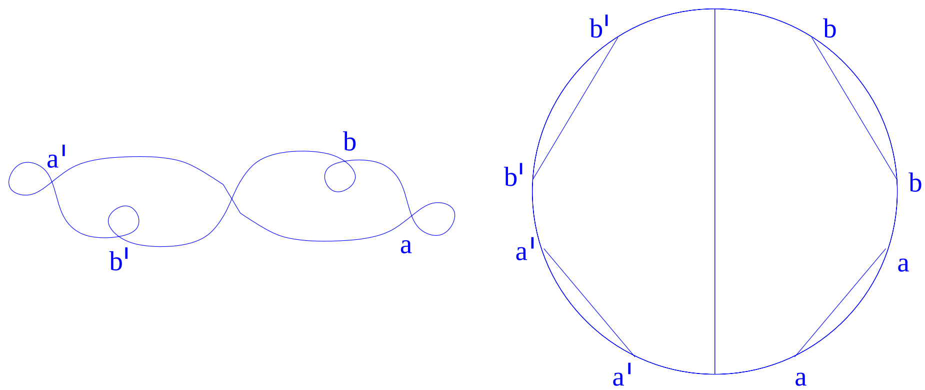


Figure 13: A curve with symmetry of type $[\Gamma\sigma] = [\Gamma]$ and its Gauss diagram.

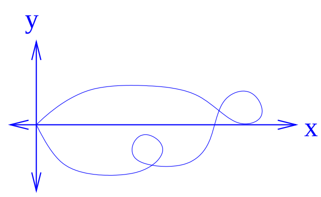


Figure 14: Redrawing half of the curve shown in Figure 13 or 15.

the line corresponding to the chord associated with the fixed double point, as in Figure 13. Since the Gauss diagram is symmetric, we can easily construct a perfectly symmetric tree corresponding to this Gauss diagram, with the line of symmetry not passing through a vertex.

We then turn to investigate the subtree F and the character function on the tree. We need only examine the interior/exterior value of the associated discs of vertices in relation to their neighbors. Returning to the Gauss diagram and starting again at the fixed double point, the orientation symmetry on the curve demands that this interior/exterior value be symmetric. Thus, the A-structure is perfectly symmetric across a line not passing through a vertex.

$\Leftarrow$ From an A-structure that is perfectly symmetric across a line not passing through a vertex, we can easily build the associated Gauss diagram, perfectly symmetric about a line corresponding to one chord. Being careful to keep track of the character function values, we construct the associated curve. Let the double point corresponding to the chord about which the diagram is symmetric lie at $(0, 0)$ on the plane. Follow the circumference of the Gauss diagram in one direction and construct half the curve, with interior/exterior values from the character function, utilizing only quadrants one and four and stopping on return to $(0, 0)$, as in Figure 14. Because of the symmetry on the Gauss diagram, we can complete our construction with another copy of this half-curve, constructing a point $(-x, -y)$ for every (x, y) already constructed. The resultant curve has

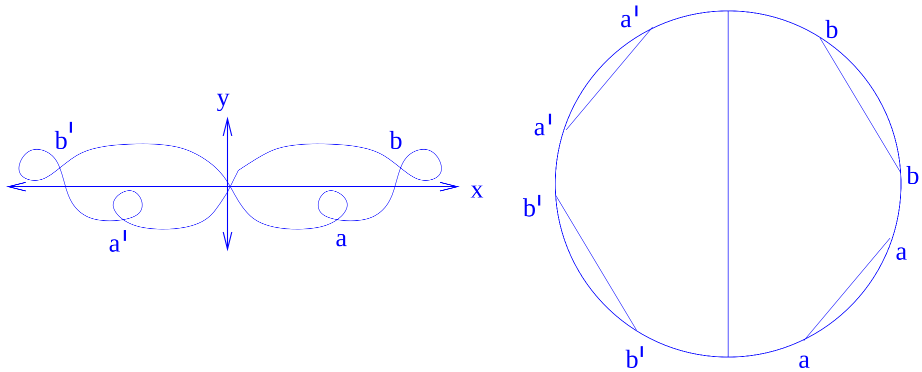


Figure 15: A curve with symmetry of type $[\Sigma\Gamma] = [\Gamma]$ and its Gauss diagram.

symmetry of type $[\Gamma\sigma] = [\Gamma]$ and corresponds to the A-structure and Gauss diagram in question. □

Proposition 6.2 *A tree-like curve has symmetry of type* $[\Sigma\Gamma] = [\Gamma]$ *iff its A-structure can be drawn such that there exists a point, not at a vertex, about which the A-structure has perfect 180° rotational symmetry.*

proof ⇒ As before, given a curve with symmetry of type $[\Sigma\Gamma] = [\Gamma]$, there exists a symmetric representative, with fixed point at a double point.

The action $\Sigma\Gamma\tau$ is equivalent to a reflection. Thus, a curve symmetric under this involution has a line of reflective symmetry passing through the fixed double point.

Let the fixed double point be at the origin, and let the line of symmetry be the y-axis. As in the proof of Proposition 6.1, we draw the Gauss diagram. We scale the length of the curve to the circumference of the Gauss diagram as before. Again, the chord corresponding to the fixed double point bisects the Gauss diagram. Using this point as a base point, we draw half of the Gauss diagram, stopping upon return to our base point.

We know that, for every point (x, y) on the curve, there is a point $(-x, y)$. Thus, following the second half of the curve and completing the Gauss diagram, we find an exact replica of the first half, yielding perfect 180° rotational symmetry on the Gauss diagram, even while keeping track of the interior/exterior value of discs in relation to their neighbors (see Figure 15. Thus, the A-structure associated to this Gauss diagram can be constructed with 180° symmetry. Since the point about which the A-structure is symmetric must be associated with the bisecting chord on the Gauss diagram, it cannot be at a vertex of the A-structure.

⇐ Given a 180° symmetric A-structure with fixed point not at a vertex, we can easily build the associated 180° symmetric Gauss diagram with bisecting chord. Keeping careful track of the character function values as before, we construct a curve, starting with one half. Let the double point corresponding to the bisecting chord lie at $(0,0)$. Follow the circumference of the Gauss diagram in one direction and, just as before, construct one half of the curve, utilizing quadrants one and four and stopping on return to the origin, again as in Figure 14. Because of the symmetry on the Gauss diagram, we can complete our construction with another copy of this half-curve, constructing a point $(-x,y)$ for every point (x,y) already constructed. The resultant curve has symmetry of type $[\Sigma\Gamma]=[\Gamma]$ and corresponds to the A-structure and Gauss diagram in question. □

Proposition 6.3 *A tree-like curve has symmetry of type* $[\Sigma\Gamma\sigma]=[\Gamma]$ *iff its A-structure can be drawn such that there exists a line not passing through a vertex and a point not at a vertex, with reflection about the line identical to rotation by 180° about the point.*

proof Propositions 6.1 and 6.2 give us the correspondence between curves of symmetric type $[\Gamma\sigma]=[\Gamma]$ and $[\Sigma\Gamma]=[\Gamma]$, respectively, and their A-structures. We know that the trivial involution (i.e. Γ alone) corresponds to no action on the A-structures. Thus, Propositions 6.1 and 6.2 yield corresponding actions on A-structures for the involutions $\Sigma\Gamma$ and $\Gamma\sigma$. Since $\Sigma\Gamma\sigma$ is their composition, it follows that the corresponding action on A-structures is the composition of the actions corresponding to $\Sigma\Gamma$ and $\Gamma\sigma$. □

Combining Propositions 6.1, 6.2, and 6.3 allows us to relate all types of symmetry on curves. If two symmetry conditions hold, then all do, and the curve is supersymmetric. If none of them do, then the curve is asymmetric. This puts restrictions on the symmetry of prime curves — K_0 is the only supersymmetric prime curve, and all others are either asymmetric or have symmetry of type $[\Sigma\Gamma\sigma]=[\Gamma]$.

An interesting byproduct of these propositions is that they imply that there exist five supersymmetric tree-like curves with the number of double points $n=5$. Examination of the chart for $n=5$ found in [2] reveals only one tree-like curve designated as supersymmetric. Personal communication with F. Aicardi has confirmed that the other four curves are indeed supersymmetric. These curves are shown in Figure 16.

6.2 Well-defined compositions

Utilizing the symmetry conditions on prime tree-like curves, we find a set of well-defined compositions. While the list appears to be exhaustive, there is no proof of this. Furthermore, none of the compositions listed below can be directly generalized to non-tree-like curves. Each of the compositions has an index equal to $ind(\Gamma_1)+ind(\Gamma_2)-1$.

We prove the first composition is well-defined, simply listing the rest. We begin with a lemma:

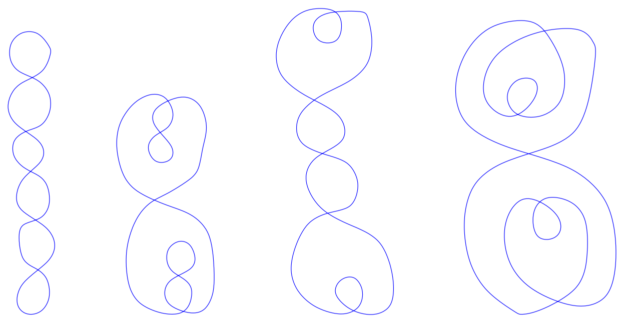


Figure 16: The four supersymmetric curves with $n = 5$ not marked so in [2].

Lemma 6.1 *Let Γ be a prime curve that is not asymmetric. Let A_1 and A_2 denote two immersions of the A-structure of Γ into the plane such that all vertices of A_1 and A_2 not in F lie in an open disc. Let their F subtrees be identified and lie in the closure of the open disc. Then, there exists a plane orientation-preserving ambient diffeomorphism (i.e. not involving a reflection) of the open disc taking A_1 to A_2.*

proof Since this curve must be supersymmetric or have symmetry of type $[\Sigma\Gamma\sigma] = [\Gamma]$, there exists a plane orientation-preserving ambient diffeomorphism of the plane taking A_1 to A_2. All we need to show is that this diffeomorphism can be restricted to the open disc in question. For K_0, this is evident. For the rest, we construct this diffeomorphism vertex by vertex. The vertex in the subtree F is already in place, so start with the one adjacent vertex (with character function value of $+1$), and denote it v_1. Since we are operating in the open disc, there exists a path taking this vertex to its corresponding vertex on A_2, taking the edge connecting v_1 and $f(v_1)$ to its corresponding edge as well, as in Figure 17. Next, examine the vertices whose father is v_1. We repeat the action as with v_1. The symmetry on the curve combined with openness on the set guarantees the correct planar orientation on the A-structure. Continue in this manner, examining successive son-vertices, until all vertices have been considered. The composition of these diffeomorphisms, then, takes all of A_1 to A_2 in the manner in which we desired. □

Composition 1 *$\Gamma_1 \# \Gamma_2$ is well-defined if Γ_1 and Γ_2 are prime, and Γ_1 is not asymmetric.*

proof Assume $\Gamma_1 \# \Gamma_2$ yields two distinct curves, and denote them Γ_α and Γ_β. We show there exists an ambient diffeomorphism between the A-structures of Γ_α and Γ_β, denoted A_α and A_β, respectively. Since any two immersions of the same A-structure are identical, there exists an ambient diffeomorphism taking all those vertices of A_α formerly in A_2 to their counterparts in A_β. Thus, all we need to show is that, from this position, their exists a diffeormophism taking

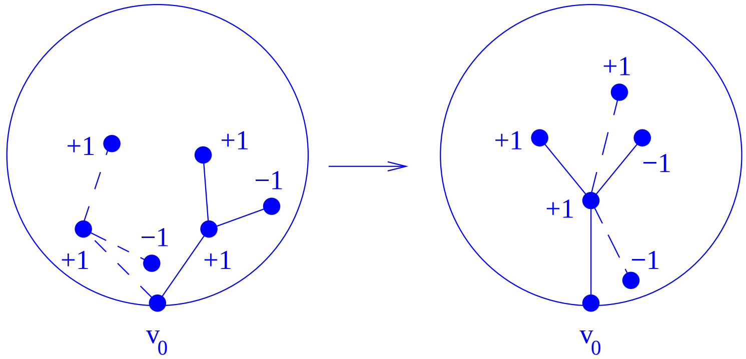


Figure 17: The first diffeomorphism described in Lemma 6.1. A_1 is represented by the solid lines, A_2 by the dashed lines.

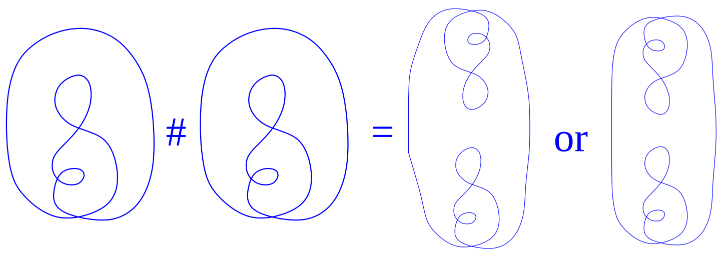


Figure 18: Two different compositions of the same prime asymmetric curve.

the remaining vertices of A_α to those in A_β while preserving the position of all vertices formerly in A_2. Because A_1 has only one vertex adjacent to F, we can construct a closed curve passing through the vertex of the subtree F such that the interior of the curve is contained in $\mathbb{R}^2 \setminus A_2$ and those vertices of A_α and A_β formerly in A_2 are found in this interior. There is a diffeomorphism taking this region to an open disc, and thus we can apply Lemma 6.1 to the remaining vertices (all formerly in A_1), completing the proof. □

It is necessary that Γ_1 or Γ_2 not be asymmetric, as seen in Figure 18. This condition is analogous to invertability on knots.

Composition 2 *$\Gamma_1 \# \Gamma_2$ is well-defined if:*

$$\Gamma_1 = \Gamma_\alpha \# \Gamma_\alpha \# \cdots \# \Gamma_\alpha$$

with Γ_2 and Γ_α prime and Γ_α not asymmetric or supersymmetric.

Composition 3 *$\Gamma_1 \# \Gamma_2$ is well-defined if:*

$$\Gamma_1 = \Gamma_\alpha \# \Gamma_\alpha \# \cdots \# \Gamma_\alpha \quad \textit{and} \quad \Gamma_2 = \Gamma_\beta \# \Gamma_\beta \# \cdots \# \Gamma_\beta$$

with Γ_α and Γ_β prime and not asymmetric or supersymmetric.

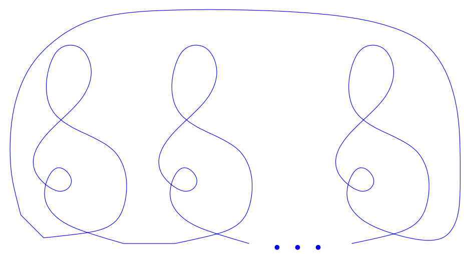


Figure 19: A curve $\Gamma\#\Gamma\#\cdots\#\Gamma$ with an orientation on the curve inducing the same orientation on every representative of Γ.

Composition 4 $\Gamma_1\#\Gamma_2$ *is well-defined if:*

$$\Gamma_1 = \Gamma_\alpha\#\Gamma_\beta$$

with Γ_2*,* Γ_α*, and* Γ_β *prime, none of them asymmetric, and* Γ_α *and* Γ_β *not supersymmetric.*

Composition 5 $\Gamma_1\#\Gamma_2$ *is well-defined if:*

$$\Gamma_1 = \Gamma_\alpha\#\Gamma_\alpha\#\cdots\#\Gamma_\alpha\#\Gamma_\beta \quad \textit{and} \quad \Gamma_2 = \Gamma_\alpha\#\Gamma_\alpha\#\cdots\#\Gamma_\alpha$$

with Γ_α *prime and not asymmetric or supersymmetric.*

Composition 6 $\Gamma_1\#\Gamma_2$ *is well-defined if:*

$$\Gamma_1 = \Gamma_\alpha\#\Gamma_\alpha\#\cdots\#\Gamma_\alpha \quad \textit{or} \quad \Gamma_1 = K_0 \quad \textit{and} \quad \Gamma_2 = \Gamma_\beta\#\Gamma_\beta\#\cdots\#\Gamma_\beta$$

with Γ_α *and* Γ_β *both prime,* Γ_α *not asymetric or supersymmetric, but with* Γ_β *asymmetric and an orientation on* Γ_2 *inducing the same orientation on every representative of* Γ_β (see Figure 19).

The next three compositions involve K_0, and each requires supersymmetry. Thus, the index of each reduces to $ind(\Gamma_2) - 1$. Recall the restrictions on the A-structure of supersymmetric curves.

Composition 7 $\Gamma_1\#\Gamma_2$ *is well-defined if:*

$$\Gamma_1 = \Gamma_\alpha\#K_0\#\Gamma_\alpha \quad \textit{and} \quad \Gamma_2 = \Gamma_\beta\#\Gamma_\beta\#\cdots\#\Gamma_\beta$$

with Γ_α *and* Γ_β *both prime, not asymmetric or supersymmetric, and with* Γ_1 *supersymmetric.*

Composition 8 $\Gamma_1\#\Gamma_2$ *is well-defined if:*

$$\Gamma_1 = \underbrace{\Gamma_\alpha\#\cdots\#\Gamma_\alpha}_{n}\#K_0\#\underbrace{\Gamma_\alpha\#\cdots\#\Gamma_\alpha}_{n} \quad \textit{and} \quad \Gamma_2 = \Gamma_\alpha\#\Gamma_\alpha\#\cdots\#\Gamma_\alpha$$

with Γ_α *prime and not asymmetric or supersymmetric, and with* Γ_1 *supersymmetric.*

Composition 9 $\Gamma_1 \# \Gamma_2$ *is well-defined if:*

$$\Gamma_1 = \underbrace{\Gamma_\alpha \# \cdots \# \Gamma_\alpha}_{2n} \# \Gamma_\beta \# K_0 \# \Gamma_\beta \# \underbrace{\Gamma_\alpha \# \cdots \# \Gamma_\alpha}_{2n} \quad \textit{and} \quad \Gamma_2 = \Gamma_\alpha \# \Gamma_\alpha \# \cdots \# \Gamma_\alpha$$

with Γ_α *and* Γ_β *prime, not asymmetric or supersymmetric, and with* Γ_1 *supersymmetric.*

7 Conclusion

While the results described above provide a solid introduction to the connected sum and decomposition of plane curves, it raise a number of questions and leaves much open for further research. Most conspicuously, the conjecture that all plane curves (not just tree-like curves) have a unique prime decomposition is left unproven. This would provide a key tool for a more complete classification of curves. Another topic worthy of examination is the results of composition — what curves result, how many are there, and how can one determine whether two curves have the same factor curves. Along with this comes the problem of proving that the list of well-defined compositions above is complete (or, if not, to complete it). Finally, a further investigation into the relation between spherical curves and their associated plane curves is warranted, as well as curves on other surfaces. In any event, this topic is wide open for further research.

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