# Classification of Extremal Curves with 6 Double Points and of Tree-like Curves with 6 Double Points and I = 5

Tim Ritter Bethel College Mishawaka, IN timritter@juno.com

Advisor: Juha Pohjanpelto Oregon State University Corvallis, OR

### Abstract

Plane curves considered as generic immersions of the circle into the plane can be classified by the Arnold invariants St, J+, and J- as well as by their index. Such classifications have already been done for all plane curves with at most five double points by a combination of Arnold and Aicardi. The goal in this paper is to begin the classification process for curves with six double points, specifically by classifying all such extremal curves as well as all tree-like curves with an index of five. In order to show that all possible curves belonging to these two subsets have been obtained, a proof is included in addition to the curves.

### Section 1. Introduction

Before beginning any other type of discussion, it is necessary to define some of the terms that will be frequently used. As stated in the abstract, a *plane curve* is defined as the generic immersion of a circle into the plane. A *double point in a plane curve* is a self-intersection of the curve. An *extremal curve* is a plane curve with n double points that has an index of n + 1 (index will be defined later in the introduction), and finally, a plane curve is *tree-like* if, when cut at any double point, it is divided into two disjoint curves. With these basic definitions in hand, we can now move on to some deeper information regarding plane curves that will be used in the proof mentioned in the abstract.

This information deals primarily with the A-structures devised by Aicardi and is simply a review of that process for uniquely determining curves. (Unless otherwise denoted, all cited

work comes from Aicardi.) The first definition that must be mentioned is that of *a Gauss diagram of a curve with n double points*. Definition 2.2 defines this as the class of chord diagrams formed by n chords connecting the preimages of each self-intersection point of the immersed curve in the standard disc bounded by the standard circle. In other words, a Gauss diagram consists of a circle with chords, where each chord corresponds to a double point in the curve. It is important to notice that a tree-like curve must correspond to a Gauss diagram in which the chords do not intersect.

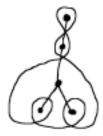
From the Gauss diagram, one can then construct the *planar tree*, assuming that the Gauss diagram is tree-like. Definition 2.4 describes this construction as placing a vertex inside each of the n + 1 regions of the diagram and then connecting them with n edges where only vertices of neighboring domains are connected. This planar tree is the basic skeleton from which Aicardi builds the A-structures. It is also important to note Definition 2.5 which states that the planar tree of a tree-like curve is the tree of its Gauss diagram. Thus, each tree-like curve can be described by a particular planar tree and Gauss diagram.

With this background in mind, we can move on to the actual construction of Aicardi's A-structures. Each of these A-structures consists of three distinct parts. The first part is simply the tree of the curve which has already been discussed. The second part is what Aicardi calls the *subtree F of the exterior associated discs*. An *associated disc of a vertex* is the region of the curve (or Gauss diagram) in which that vertex of the planar tree lies. The *exterior* associated discs comprising F are those regions which do not lie inside any other region of the curve. (Conversely, an *interior* associated disc is a region that lies within another region of the curve.) Finally, the A-structure consists of a *character function*. This function assigns a value to each vertex in the planar tree according to the following rules by Proposition 3.1. Choose an arbitrary vertex  $v_0 \in F$  to be the root of the A-structure, and label all other vertices  $v_1, v_2, \ldots, v_n$ . Also, let the last vertex on the way from  $v_0$  to  $v_i$  be called the *father of*  $v_i$  and be denoted  $f(v_i)$ . Now, the character function defines  $c(v_0) = -1$ ,  $c(v_i) = 1$  if the associated disc of  $v_i$  lies outside the associated disc of its father. Thus, an A-structure will consist of a planar tree, a subtree F, and the described character function.

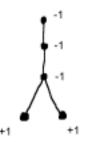
In order to help make this process a little more clear, consider the following example on how to obtain an A-structure from a curve. Start with the given curve with four double points.



Now, the first step in forming the A-structure is to form the planar tree of the curve. To do this, place a vertex in each region of the curve and then connect the neighboring regions by straight lines. The resulting planar tree should have four edges and five vertices since the curve has four double points.

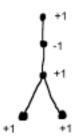


The next step in finding the A-structure of this particular curve is to determine the subtree F which consists of the exterior associated discs. Obviously, by looking at the picture above, the top three vertices all correspond to exterior associated discs so they must comprise the subtree F. The final step in determining the A-structure is to apply the character function to each of the vertices. It is evident that the given values must be the following:



Thus, this last image is the actual A-structure for our original curve because it has the planar tree, the subtree, and the character function applied to it.

The final major concept from Aicardi that needs to be discussed is that of computing the index of a curve from its A-structure. The *index* is defined as the angle measured in  $2\pi$  units, which is swept out by the tangent vector to the curve at a point running along the entire curve (Definition 3.1). In other words, the index tells how many counterclockwise rotations the tangent vector makes as it travels along the curve. Aicardi defines the index in terms of the A-structure in Definition 3.4 as  $I = \sum s(v_i)$  where *s* is the following function on each vertex. First, choose any vertex from the subtree F and label it  $v_0$ . If the associated disc of  $v_0$  is positively oriented (oriented in a counterclockwise direction), then  $s(v_0) = 1$ . If the associated disc of  $v_0$  is negatively oriented (oriented in a clockwise direction), then  $s(v_0) = -1$ . Then, for all other vertices  $v_i$ ,  $s(v_i) = c(v_i)s(f(v_i))$ . So for our prior example, the *s* function at each vertex would have the values given below if we assume that the associated disc of the top vertex (our  $v_0$  for this case) is positively oriented.



That means that  $I = \sum s(v_i) = 1 - 1 + 1 + 1 + 1 = 3$  so the index of this curve is 3.

There is one other minor note to cover before heading into the proof that all curves in the afore-mentioned subsets are listed in the curve charts at the end of the paper. This note deals with the fact that A-structures must deal with oriented curves in the nonoriented plane while the listed curves in this paper are all nonoriented. Obviously, for each of the nonoriented curves, there are two possible orientations. The only real difference in them from the point of view of the A-structures is that the indices of the two oriented curves will be opposites, one positive and one negative. In order to avoid this type of confusion, the nonoriented curves will be assigned a positive orientation with the negative orientation following similar arguments. This also helps because only one oriented curve from each nonoriented curve needs to be examined.

That finishes the background information that is needed in order to understand the proof that follows. For further study on these briefly discussed concepts, a thorough reading of Aicardi is recommended. This may be necessary because the character function and the index formula are especially important in the arguments of this proof.

### Section 2. Proof that all extremal curves with 6 double points are listed.

The first step in the general proof to establish that I have all possible extremal curves with 6 double points and all tree-like curves with 6 double points and an index of 5 is obviously to show that I have all of the extremal curves. To begin with, there must be a finite number of Gauss diagrams which correspond to curves that are tree-like and have 6 double points, regardless of the index. This is because each diagram must contain 6 non-intersecting chords. So by simple combinatorics, one can obtain each of these Gauss diagrams. Next, form the planar tree for each of these diagrams. As previously stated in this paper and by Aicardi, this is done by placing a vertex in each region of the Gauss diagram and by connecting the neighboring regions. This process results in a planar tree corresponding to each Gauss diagram.

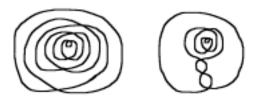
Here, note a theorem from Arnold which says that there exists a bijection between the set (of classes) of extremal curves with n double points and the set (of classes) of plane rooted trees with n edges. Since all possible planar trees with 6 edges can be constructed (as shown previously), it follows that all possible rooted trees with 6 edges may be constructed from these. To do this, simply consider each vertex of each planar tree as the root of that particular tree. This must necessarily result in the existence of all possible

rooted trees with 6 edges. Because Arnold established that a bijection exists between these rooted trees and the extremal curves with 6 double points, then it follows that all possible extremal curves with 6 double points have been obtained if one draws the actual curve corresponding to each rooted tree. Since the preceding is the exact method that I used to come up with the extremal curves with 6 double points, I know that they must all be listed.  $\Box$ 

# Section 3. Proof that all tree-like curves with 6 double points and an index of 5 are listed.

Now that it has been established that I have all of the extremal curves, I move on to the proof that I also have obtained all of the tree-like curves with 6 double points and an index of 5. In order to do this, I must first explain a method that I will call the *orientation reversal of an associated disc in a curve*. (From now on, this method will be referred to as an orientation reversal.) An orientation reversal is defined as causing the desired associated disc to lie outside its father as well as causing each of its sons to lie outside of it (provided that the sons exist). The exterior or interior contact of any other associated disc in the curve to its respective father will not be altered. The reason that I refer to this method as an orientation reversal is that the orientation of the specified disc in the newly formed curve is now the opposite of its orientation in the original curve, and the orientation of all other discs in the curve remains the same.

In order to illustrate this process, the following example is included. This example shows an orientation reversal of the second associated disc from the root of the extremal curve on the left. Notice that, in the resulting curve on the right, the specified disc lies outside of its father and that its son lies outside of it (by definition). Also, notice that its new orientation within the curve is now the opposite of what it was in the original curve on the left.



Now that the orientation reversal method has been explained, examine the following lemma

**Lemma:** An orientation reversal of one and only one associated disc in an extremal curve with n double points results in a tree-like curve with n double points and I = n - 1.

As has already been mentioned in Section 2, Arnold has established the bijection between the extremal curves with n double points and the rooted trees with n edges. Thus, consider the plane rooted trees with n edges, and let the root of each of these trees be labeled vertex  $v_0$ . Each of these rooted trees can also be considered as one of Aicardi's A-structures. It obviously contains the planar tree structure because that is how we were able to come up with the rooted tree. It is also evident that there is only one element of the subtree F, namely  $v_0$ . The reason is that its associated disc is the root of the rooted tree which is the only exterior associated disc for any extremal curve.

Now the final step in making these rooted trees into A-structures is to apply the character function to each of the vertices. Obviously, we know that  $c(v_0) = -1$  because  $v_0$  is the only element of F in an extremal curve, and the character function dictates that we must begin with  $c(v_0) = -1$  for some  $v_0 \in F$ . In order to determine the value of the character function for the rest of the vertices, consider the function *s* that is used to determine the index. For extremal curves, we know that  $I = \sum s(v_i) = n + 1$ . In order for this to happen, each value of  $s(v_i)$  must be equal to 1 because there are n + 1 vertices. (They are equal to 1 and not -1 because this is assuming that  $v_0$  is positively oriented, as mentioned in the introduction.) Since each vertex has a value of *s* equal to 1, it is evident that  $s(f(v_i)) = 1$  for all  $v_i$  because  $s(v_i) = c(v_i)s(f(v_i)) = c(v_i)(1) = c(v_i)$ . Thus, the character function on any rooted tree corresponding to an extremal curve assigns  $c(v_0) = -1$  (assuming that  $v_0$  is the root of the tree) and  $c(v_i) = 1$  for any other  $v_i$ .

Since we now have an A-structure defined on each rooted tree, we can move on to the two possible cases of an orientation reversal. The first case considers the orientation reversal of the associated disc of a vertex  $v_j$  where  $j \neq 0$ . The second case considers the orientation reversal of the associated disc of  $v_0$ . In both cases, keep in mind that the original extremal curves are all such that the associated disc of  $v_0$  is positively oriented.

#### Case 1: The orientation reversal of the associated disc of $v_i$ where $j \neq 0$

According to the definition of an orientation reversal, the associated disc of  $v_j$  now lies outside the associated disc of its father, and the associated discs of any sons of  $v_j$  now lie outside the associated disc of  $v_j$ . Because of this, the character function of the original extremal curve will need to be adjusted in order to form an A-structure that fits the newly formed curve. Obviously, the only adjustments that may need to be made will be to  $v_j$ , its father, and its sons since their associated discs are the only ones that were altered. First of all, consider  $f(v_j)$ . As we travel from the associated disc of  $v_0$  to the associated disc of  $f(v_j)$ , we will find no changes in the exterior or interior contact of the associated discs with respect to their fathers. That means that  $c(f(v_j))$  is still equal to 1. Now consider  $v_j$ . Since the orientation reversal defines its associated disc to lie outside the associated disc of its father, the character function mandates that  $c(v_j) = -1$ . Finally, consider the sons of  $v_j$ . The orientation reversal also dictates that the associated discs of these sons must lie outside the associated disc of their father  $(v_j)$ . Therefore, the character function assigns a value of -1 to each son.

Now that the character function has been established, we have the A-structure for the generic curve formed by the previously described orientation reversal. The next step is to use this A-structure to compute the index of the curve by using Aicardi's formula for the

index. To do this, we must apply the function *s* to each vertex in the A-structure. As previously stated, we begin by assigning  $s(v_0) = 1$  since the root of the original extremal curve is positively oriented. Next, note that  $s(v_k) = 1$  for all  $v_k$  such that k < j because, in each of these cases,  $c(v_k) = 1$  and  $s(f(v_k)) = 1$ . However, the next vertex is  $v_j$  which has  $s(f(v_j)) = 1$  but also has  $c(v_j) = -1$ . That means that  $s(v_j) = -1$ . The next set of vertices that need to be considered are the sons of  $v_j$  (label them  $v_s$ ). Each son has the values  $c(v_s) = -1$  and  $s(f(v_s)) = s(v_j) = -1$  which means that  $s(v_s) = 1$ . Finally, consider all of the vertices  $v_l$  such that l > s. As in the case of all  $v_k$ , all  $v_l$  have the values  $c(v_l) = 1$  and  $s(f(v_l)) = 1$  which results in  $s(v_l) = 1$  for all  $v_l$ .

Since the function *s* has been applied to all vertices in the A-structure, we can compute the index of the curve by using Aicardi's formula  $I = \sum s(v_i)$ . It is obvious that the value of *s* at each vertex in the newly formed curve is equal to the value of *s* at the same vertex in the original extremal curve except at  $v_j$ . Here, the value of *s* has dropped from 1 to -1, a difference of 2. Thus, the value of the index will also decrease by 2, from n + 1 in the extremal curve to n - 1 in the new curve. We can also note that the curve is still tree-like since it has a specific A-structure and that it still has n double points because no edges were removed from the planar tree at any time. That means that the orientation reversal in case 1 does indeed change an extremal curve with n double points into a tree-like curve with n double points and an index of n - 1.

### Case 2: The orientation reversal of the associated disc of $v_0$

We again begin this process by noting the definition of the orientation reversal. However, in this case,  $v_0$  does not have a father since its associated disc is the root of the extremal curve. That means that the orientation reversal only causes the associated discs of the sons of  $v_0$  to lie outside the associated disc of  $v_0$ . Again, notice that this mandates that the A-structure of the extremal curve be altered in order to become the A-structure for the newly formed curve. Also, as in case 1, the possible changes will occur with  $v_0$  and its sons because their associated discs are the ones that were altered. Obviously, the planar tree of the extremal curve will still be the planar tree of the newly formed curve. However, the subtree F has changed. Since the associated discs of the sons of  $v_0$  now lie outside the associated disc of  $v_0$ , these discs all have exterior contact and must then be part of F. Finally, the character function needs to be adjusted. In this case,  $c(v_0)$  is still equal to -1 because it is still part of F, but because all of the sons of  $v_0$  are also part of F and have the exterior contact to  $v_0$ , their character function is now  $c(v_s) = -1$  (where  $v_s$  is a son of  $v_0$ ).

Now that the new A-structure has been formed, we can begin to look at the index of this new curve. The first step in doing that is to establish the value of the function *s* for each vertex in the A-structure so we begin with  $s(v_0)$ . Recall that we are assuming that the associated disc of  $v_0$  is positively oriented in the original extremal curve. However, in this case, the orientation reversal is acting on this particular associated disc, and in the explanation of the orientation reversal, it was noted that this process changes the orientation of that one disc with respect to the rest of the curve. That means that the associated disc of  $v_0$  is negatively oriented in the new curve and so  $s(v_0) = -1$ . Now, consider the function *s* when it is applied to each  $v_s$ . Obviously, for each of these vertices,  $c(v_s) = -1$  and  $s(f(v_s)) = s(v_0) = -1$  which means that  $s(v_s) = 1$ . Finally, consider what happens when *s* is applied to all vertices  $v_t$  where t > s. In each of these cases, it can easily

be noted that  $c(v_t) = 1$  and that  $s(f(v_t)) = 1$  which again results in  $s(v_t) = 1$ .

Since we now have the function *s* defined on each vertex in the A-structure of the newly formed curve, consider the value of the index according to Aicardi's formula  $I = \sum s(v_i)$ . As in case 1, notice that the value of *s* at each vertex in the newly formed curve is equal to the value of *s* at that vertex in the original extremal curve except at  $v_0$ . Here, the value of  $s(v_0)$  drops by 2 which results in the value of the index dropping from n + 1 to n - 1. We again note that the new curve must be tree-like since it corresponds to a specific A-structure and that it still contains n double points since no edges were removed from the planar tree during the process. Therefore, case 2 also describes an orientation reversal that changes an extremal curve with n double points into a tree-like curve with n double points and an index of n - 1.

Since both of these 2 possible cases hold, the lemma must be true in general. Thus, we know that an orientation reversal of any one disc in an extremal curve with n double points will result in a new curve that is tree-like, has n double points, and has an index equal to n-1.  $\Box$ 

Now, remember that the original goal is to show that I do, in fact, have all of the possible curves with 6 double points that are tree-like and have an index of 5. In section 2, I have already mentioned that simple combinatorics will give all of the possible Gauss diagrams for tree-like curves with 6 double points and that it is obviously possible to obtain the planar tree corresponding to each diagram. That means that each curve with 6 double points that is tree-like and has an index of 5 must correspond to one of these planar trees. In section 2, I also showed that I was able to obtain all of the possible extremal curves by designating each vertex of each planar tree as a root of the tree and forming the rooted tree which corresponds to one and only one extremal curve. Since these extremal curves cover every possibility from the planar trees, I can use them to come up with the curves with 6 double points that are tree-like and have an index of 5. Finally, by the lemma that was just proved, I can perform an orientation reversal on the associated disc corresponding to each vertex on these extremal curves and get my desired curves. Since I have reversed the orientation of each possible disc in every extremal curve with 6 double points and because these extremal curves can be traced back directly to the planar trees corresponding to all possible Gauss diagrams that are tree-like with 6 double points, there are no possible tree-like curves with 6 double points and an index of 5 that would not be covered by this method. Therefore, I must have all of the possible curves in this specific subset.  $\Box$ 

### Section 4: The curve charts

Since I have already shown that I must have all of the possible curves in these two subsets, the only thing that remains is to present the charts of these curves. I have already explained the methods that I have used in order to come up with the actual curves. In order to compute the Arnold invariants for each curve, I was able to use the method devised by Polyak. This method is very convenient because it allows the invariants to be computed from the Gauss diagrams. This is done by assigning positive or negative signs to each

chord and then using formulas to compute each invariant. For more information, I suggest reading Polyak's paper. With that being said, the following pages present the extremal curves with 6 double points, the tree-like curves with 6 double points and an index of 5, and the invariants of each curve.

## References

1. F. Aicardi: Tree-like Curves, Advances in Soviet Mathematics 21 (1994), 1-31.

2. V. I. Arnold: Plane Curves, Their Invariants, Perestroikas, and Classifications, *Advances in Soviet Mathematics* **21** (1994), 33-91.

3. V. I. Arnold: Topological Invariants of Plane Curves and Caustics, AMS, Providence, 1994.

4. M. Polyak: Invariants of Curves and Fronts Via Gauss Diagrams, *Topology* **37**(1998), 989-1009.

