Determining a Triangle from Two X-ray Sources

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June 17, 2002

Abstract

Here, we attempt to determine a triangle for all other convex bodies from directed X-rays from two point sources.

1 Introduction.

The two notions of determination and verification are important concepts when trying to identify convex polygons through computed tomography. The results of Gardner [3] proved that, save for the class of parallel wedges, convex polygons are uniquely determined within the set of all other convex polygons from one directed X-ray. We will focus on determining a triangle from all other convex bodies, a result that hopefully will extend to determining convex polygons from all other convex bodies. Also, the concept of determination is a rather strong notion – focusing on verification is a better starting point. Now, the results from Falconer [2] and Gardner [3] proved that a convex body is uniquely determined from all other convex bodies by its directed X-rays from two sources, provided the line through the sources meets the convex body. Therefore, our interest is with two sources where the line passing through the sources does not intersect the convex body. Thus, throughout the remainder of this paper, we will assume without loss of generality, that our convex body lies in the open upper half plane, and our two sources lie on the x-axis.

Definition 1 *We say that a convex body K_0 from a family of convex bodies K is* ***determined from a family of convex bodies*** P ***by*** n ***sources*** *if we randomly choose n X-ray sources, $\{s_1, s_2, ..., s_n\}$, and if some body $P_0 \in P$ has the same X-ray data from the s_i's as K_0, then $K_0 = P_0$.*

Definition 2 *We say that a convex body K_0 from a family of convex bodies K is* ***verified from a family of convex bodies*** P ***by*** n ***sources*** *if we can choose n X-ray sources, $\{s_1, s_2, ..., s_n\}$, such that if some body $P_0 \in P$ has the same X-ray data from the s_i's as K_0, then $K_0 = P_0$.*

And so, starting by verifying a triangle from all other convex bodies from two directed X-ray sources, we attempt to address the problem of determination.

2 Notation and Definitions.

Definition 3 *A* ***convex body*** *is a compact body $K \subset \mathbb{R}^2$ with nonempty interior, where for all points x, $y \in K$, the segment $(1-t)x + ty$, where $0 \leq t \leq 1$, lies in K.*

Definition 4 *The characteristic function of a convex body K is defined as:*

$$\chi_K(x) = \begin{cases} 1 & \text{if} \quad x \in K \\ 0 & \text{if} \quad x \notin K \end{cases}$$

for all $x \in \mathbb{R}^2$.

Now, throughout our paper, we will make it standard notation for the symbols ϕ and θ to refer to points on the unit circle S^1, while the symbols α, β, φ, and ω refer to angles.

Definition 5 *A* ***directed X-ray transform****, also known as a point source X-ray or a fan-beam X-ray, gives the chord length of the convex body along a particular direction ϕ from a given point s, which is called the* ***point source****. Thus, the directed X-ray of K is given by the function $\mathcal{D}_{K_o} : [0, \pi) \longrightarrow \mathbb{R}$, where:*

$$D_{K_o}(\omega) = \int_0^\infty \chi_K(s + t\phi)dt,$$

with $\phi = (\cos\omega, \sin\omega)$.

Throughout the remainder of the paper, we will denote the ray $r = r_0 + t\phi$, where $r_0 \in \mathbb{R}^2$, $t \in [0, +\infty)$, and $\phi = (\cos\omega, \sin\omega)$, by the ordered pair (r_0, ω).

Definition 6 *A* ***supporting cone*** *$C_s[\alpha, \beta]$ of a convex body K is the largest cone such that $D_{K_o}(\omega) > 0$, $\alpha < \omega < \beta$. The rays $\omega = \alpha$ and $\omega = \beta$ are called* ***supporting rays****.*

3 Verification of a triangle.

Let us first introduce a trivial lemma, which allows us to verify the triangle.

Lemma 7 *If K, a triangle with vertices p_0, p_1, and p_2, and K', a convex body with three nonsmooth points at p_0, p_1, and p_2, have the same directed X-ray data from two sources, then $K = K'$.*

Proof. Clearly, $K \subseteq K'$. Thus, we can express $K' = K \cup A_0 \cup A_1 \cup A_2$. Now, choose one of the sources to be on the line that contains the edge $e_1 = \overline{p_1 p_2}$ of K, and translate the picture so that this source is at the origin, denoted o (Figure 1).

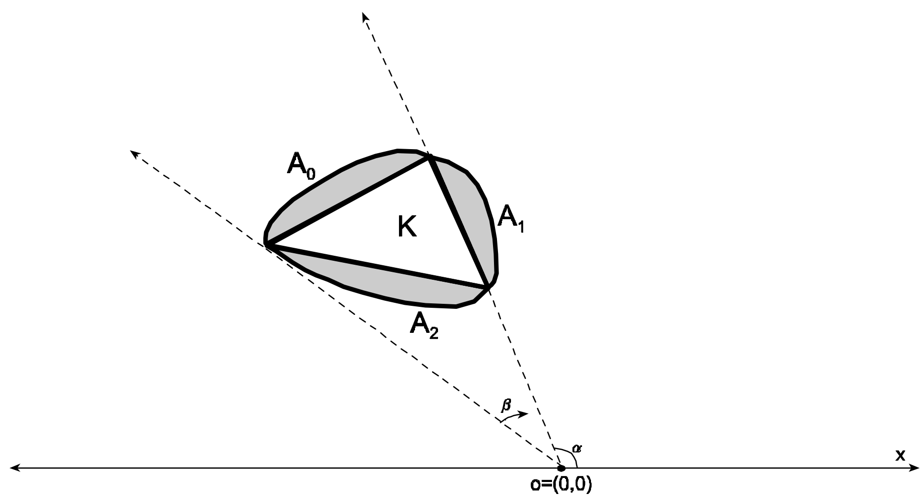

Figure 1.

Now, assume that K and K' lie in the supporting cone $C_o[\alpha, \beta]$, where the ray (o, α) contains the edge e_1. Thus, we must have $A_1 = \varnothing$, else K' would lie outside this supporting cone. Now, K and K' have equal X-ray data from o, so $\mathcal{D}_{K_o}(\varphi) = \mathcal{D}_{K'_o}(\varphi)$, for all φ. Therefore, from our definition of directed X-rays, we obtain:

$$\int_0^\infty \chi_K(t\varphi)\,dt = \int_0^\infty \chi_{K'}(t\varphi)\,dt = \int_0^\infty \chi_K(t\varphi)\,dt + \int_0^\infty \chi_{A_0}(t\varphi)\,dt + \int_0^\infty \chi_{A_2}(t\varphi)\,dt, \tag{2.1}$$

for all φ. However, since the characteristic function is a positive function, we must have:

$$\int_0^\infty \chi_{A_0}(t\varphi)\,dt = \int_0^\infty \chi_{A_2}(t\varphi)\,dt = 0. \tag{2.2}$$

Therefore, we must have $A_0 = A_2 = \varnothing$, and therefore $K' = K \cup A_0 \cup A_1 \cup A_2 = K \cup \varnothing = K$, which is as desired. ∎

In order to verify a triangle from all other convex bodies from two sources, we must first partition the x-axis so that we can choose sources that will be sufficient for verifying a triangle. Let us denote the vertices of our triangle K by p_0, p_1, and p_2. Also, let $e_0 = \overline{p_0 p_1}$, $e_1 = \overline{p_1 p_2}$, and $e_2 = \overline{p_2 p_0}$. Now, let γ_i be the angle e_i makes with the positive x-axis. Thus, $\gamma_i \in [0, \pi)$. Now, we can locate three (two if one of the edges is horizontal) points on the x-axis which are the roots of the lines ℓ_i, where ℓ_i contains e_i. Let us denote these points $a_1 = (a_1, 0)$, $a_2 = (a_2, 0)$, and $a_3 = (a_3, 0)$, where a_1 is the root of ℓ_i, a_2 is the root of ℓ_j, and a_3 is the root of ℓ_k, where i, j, k is an ordering of 0, 1, 2 such that $\gamma_i < \gamma_j < \gamma_k$. Therefore, we have $a_1 < a_2 < a_3$. Using the above methods, we can find points a_1, a_2, and a_3 for any triangle in the open upper half plane.

Theorem 8 *We can verify any triangle K in the open upper half plane from two sources on the x-axis, namely any two of the points a_1, a_2, and a_3.*

Proof. For our triangle K in the open upper half plane, choose, without loss of generality, the points a_1 and a_2 as our X-ray sources. The proof follows similarly for any other two sources from $\{a_1, a_2, a_3\}$. Assume that K lies in the supporting cones $C_{a_1}[\alpha, \beta]$ and $C_{a_2}[\alpha', \beta']$, and that edges of K lie on the rays (a_1, β) and $\big(a_2, \beta'\big)$. Thus, there are two vertices of K on the rays (a_1, β) and $\big(a_2, \beta'\big)$, and one vertex of K on the rays (a_1, α) and (a_2, α'). But since two vertices lie on each of the rays (a_1, β) and $\big(a_2, \beta'\big)$, they cannot be distinct, else we would have 4 vertices – too many for a triangle. Thus, one vertex must be the intersection of the rays (a_1, β) and $\big(a_2, \beta'\big)$, call this point $r = (r_1, r_2)$. Thus, we still can detect two other distinct vertices of K, one each on the rays (a_1, β) and $\big(a_2, \beta'\big)$. Therefore, if the intersection of the rays (a_1, α) and (a_2, α') was a vertex of K, we would have a contradiction, as we would have at least 4 vertices – too many for a triangle. So, we must have the intersection of the rays (a_1, β) and (a_2, α') as a vertex – call it $s = (s_1, s_2)$. Similarly, the intersection of (a_1, α) and $\big(a_2, \beta'\big)$ is a vertex, call it $p = (p_1, p_2)$. Thus, from our two sources, we have uniquely determined our triangle K as having vertices r, s, and p.

Now, assume there is another convex body K' with the same directed X-ray data from a_1 and a_2 as K. From Theorem 2.4 in the paper of D. Lam and D. Solmon [4], we can conclude that $\partial K'$ has exactly 3 non-smooth points – exactly two on the ends of the segment which lies on the ray (a_1, β), and only one on the ray (a_1, α), since $\mathcal{D}_{K'_{a_1}}(\alpha) = 0$, and none on the rays (a_1, φ), for all $\varphi \in (\alpha, \beta)$. Then, using a method which is exactly the same as the one used above to identify vertices of K, we reach the conclusion that the nonsmooth points of $\partial K'$ are p, r, and s, also. Then, from lemma 7, we must obtain $K = K'$, as desired. ■

4 Verification of a triangle from other points.

Given a triangle K in the upper half plane, choose for a source the point a_1 which is the intersection of the line passing through e_1 of K and the x-axis. Then K lies in the supporting cone $C_{a_1}[\alpha', \beta']$, where e_1 lies on the ray $\big(a_1, \beta'\big)$. Now, choose for a second source some point $o \in (a_1, a_2)$. For simplicity, we will hereafter denote our source a_1 by a. Also, let us now translate our picture so that $o = (0, 0)$ is the origin of our coordinate system. Now, if K lies in the supporting cone $C_o[\alpha,\beta]$, we will find the vertices of K on the rays (o,β), (o,φ), (o,α), for some φ where $0<\alpha<\varphi<\beta<\pi$. It is not hard to show, from the results of [4], that the vertices of K are given by the intersection of the rays: (o,α) and (a,α'); (o,φ) and $\left(a,\beta'\right)$; (o,β) and $\left(a,\beta'\right)$. Let $p=(p_1,p_2)$, $s=(s_1,s_2)$, and $r=(r_1,r_2)$ denote these vertices, respectively. Now, assume there exists a convex body K' with the same directed X-ray data from the sources o and a. Then, we know K' has three nonsmooth points, two on the ray $\left(a,\beta'\right)$ and only one on (a,α'), and none on rays (a,φ'), for all $\varphi'\in\left(\alpha',\beta'\right)$ (Theorem 2.4 of [4]). Now, since the three nonsmooth points are also located on the rays (o,α), (o,φ), and (o,β), at least one of the nonsmooth points of K' is also a vertex of K on the ray $\left(a,\beta'\right)$, as it is not possible to have 4 distinct nonsmooth points on the ray $\left(a,\beta'\right)$. So, let us now consider two cases.

4.1 The vertex r of K is also a nonsmooth point of $\partial K'$.

We know from above that $\overline{rs}=e_1$ is an edge of K, and the X-ray data from a gives us the length of $\overline{rs}$, let $|\overline{rs}|=x$. Now, the X-ray data from a indicates that K' has a straight line for an edge on the ray $\left(a,\beta'\right)$, and this edge has one end at the point r. It cannot extend 'down' $\left(a,\beta'\right)$ towards a, since $K'\subset C_o[\alpha,\beta]$. Thus, it must extend 'up' $\left(a,\beta'\right)$ a distance x, and so s is the other end of this edge of K', and thus the intersection of rays $\left(a,\beta'\right)$ and (o,φ) is a nonsmooth point of $\partial K'$. Therefore, we must have the intersection of (a,α') and (o,α) as the third nonsmooth point of $\partial K'$, and therefore the vertices of K are exactly the nonsmooth points of K', from lemma 7, $K=K'$.

4.2 The vertex r is not a nonsmooth point of $\partial K'$.

First, let us denote the intersection of the rays $\left(a,\beta'\right)$ and (o,α) by t. Now, we must have s and t as a nonsmooth points of $\partial K'$, since the only possible nonsmooth points of $\partial K'$ on the ray $\left(a,\beta'\right)$ are r, s, and t, and, assuming r to not be a nonsmooth point, the X-ray data from a tells us we must have two nonsmooth points on this ray – namely, s and t. Note, also, that the X-ray data from a tells us that $\left|\overline{st}\right|=x$. Now, if we go 'out' from s on the ray $\left(a,\beta'\right)$ a distance x to the point t, and draw a line through t and the vertex p, where this line intersects the x-axis is the only such point in the interval (a_1,a_2) that will provide us with this case – thus, we have only one such 'bad point' to worry about. Then, it follows that the remaining nonsmooth point on $\partial K'$ must be the intersection of the rays (a,α') and (o,β) – let us denote this point $q=(q_1,q_2)$. Note, however, that the ray (o,φ) intersects both ∂K and $\partial K'$ at the point s. Let $\mathcal{D}_{K_o}(\varphi)=\mathcal{D}_{K'_o}(\varphi)=w$. Thus, if we move 'in' towards o on the ray (o,φ), we will arrive at another point, call it $g=(g_1,g_2)$, which lies on both ∂K and $\partial K'$. Therefore, we have found 4 points on $\partial K'$, namely s, t, g, and q, giving us a quadrilateral Q with these vertices such that $Q\subset K'$. Figure 2 gives us a nice picture of what's going on here.

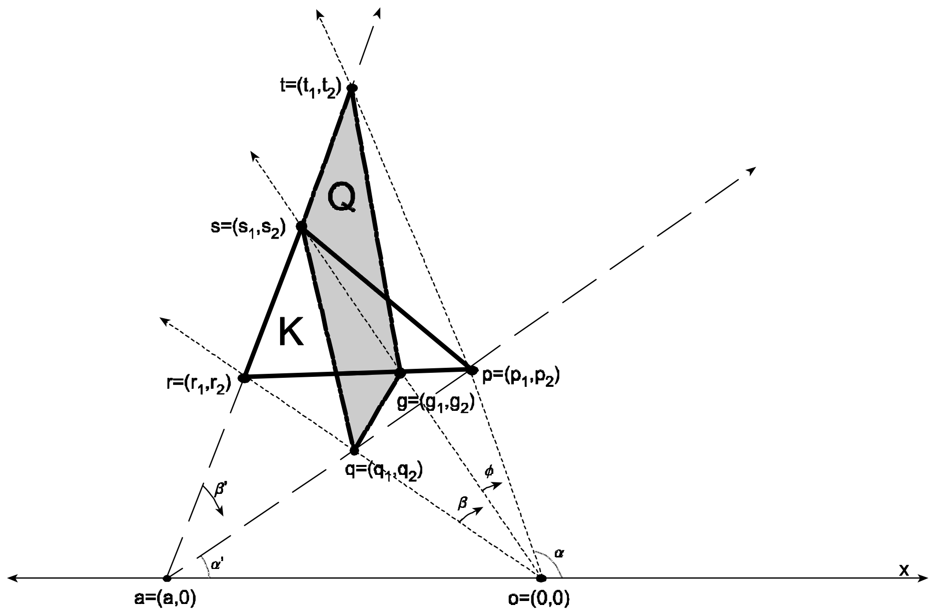

Figure 2.

Now, a brief disgression as we spell out some wonderful facts about directed X-ray data.

Now, recall the consistency condition from Volčič [5] that states:

$$\int_{S^1} \mathcal{D}_{K_o}(\theta)\cdot|\theta\cdot\phi|^{-1}\,d\theta = \iint_{\mathbb{R}^2} \chi_K(x)\cdot|(x-o)\cdot\phi|^{-1}\,dA = \iint_K |(x\cdot\phi)-(o\cdot\phi)|^{-1}\,dA, \tag{3.1}$$

where ϕ is allowed to be any unit vector. Now, remember that we assumed that another convex body K' has the same X-ray data as our triangle K from our source o. Since $K \neq K'$, certainly $\chi_K \neq \chi_{K'}$, although $\mathcal{D}_{K_o}(\theta) = \mathcal{D}_{K'_o}(\theta)$. But, the left-hand side of the above equation exploits this equality, allowing us to write an equality between K and K' that relies only on the characteristic functions of both convex bodies, ignoring completely their directed X-ray data. It is:

$$\iint_K |(x\cdot\phi)-(o\cdot\phi)|^{-1}\,dA = \iint_{K'} |(x\cdot\phi)-(o\cdot\phi)|^{-1}\,dA. \tag{3.2}$$

But, recall that we translated things so that $o = (0,0)$. This being done, the above equation simplifies greatly to:

$$\iint_K |x\cdot\phi|^{-1}\,dA = \iint_{K'} |x\cdot\phi|^{-1}\,dA. \tag{3.3}$$

Being able to select any unit vector ϕ, will select one that simplifies my calculations the most. The two most promising choices, then, were one of the unit vectors $\phi = (1,0)$ or $\phi = (0,1)$, as $x \cdot \phi$ would simplify to either x or y before integrating. Notice that x is just lazy notation for the vector $\widehat{x} = (x,y)$. I will abuse this notation hereafter and let x simply denote the x-coordinate of this vector. Anyhow, the best choice for ϕ is the vector $\phi = (1,0)$. So, the new version of equation 3.3 equality reads:

$$\int\int_{K} \frac{1}{x} dA = \int\int_{K'} \frac{1}{x} dA. \tag{3.4}$$

To clean up our calculations a bit more, we will rotate our picture by some angle θ where $\beta + \theta = \frac{\pi}{2}$. In doing this, we can now write the points q and r as $q = (0, q_2)$ and $r = (0, r_2)$. Furthermore, scale our entire picture by some scalar λ so that $s_1 = 1$. Doing this, we simplify calculations, and also note that since $\overline{rs} = \overline{st} = x$, and the segments lie on the same line, with $r_1 = 0$, and $s_1 = 1$, then we must obtain $t_1 = 2$ (Figure 3).

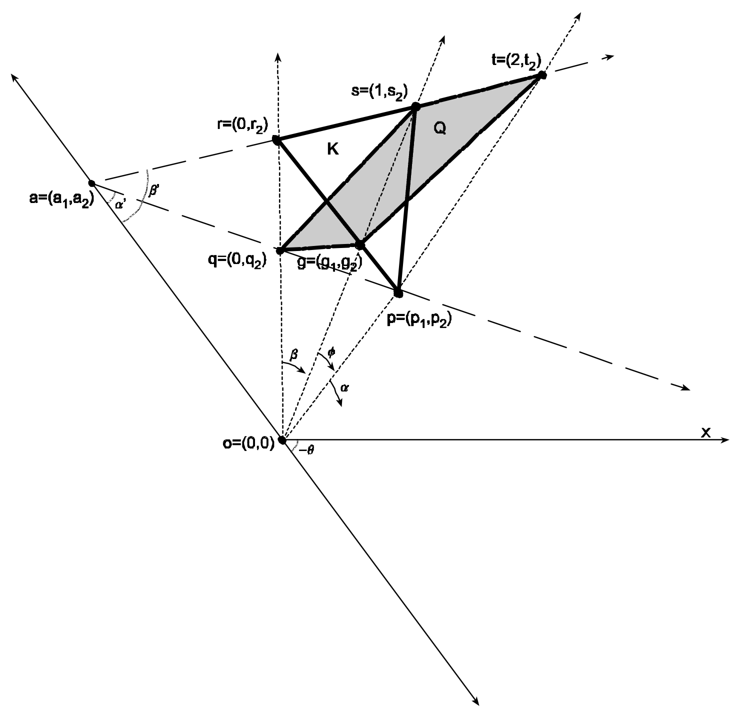

Figure 3.

After our rotation, looking at the picture, one can notice that we will have the relations $0 < g_1 < p_1 < t_1$ and $0 < g_1 < s_1 < t_1$ – although we do not know how p_1 and s_1 relate to one another. Now, we must take the integral of $\frac{1}{x}$ on both K and K', although we will do it several steps, partitioning the x-axis a bit, to make things a bit more feasible. However, we must do things in three cases, each one specifying exactly how p_1 relates to s_1.

4.2.1 Case where $p_1 < s_1$.

Since $p_1 < s_1$, we will take the integral on the intervals $x \in [0, g_1]$, $x \in [g_1, p_1]$, $x \in [p_1, 1]$, and $x \in [1, 2]$. Remember, our goal is to verify our triangle K from all other convex bodies, and, thus, show that $K = K'$. We are trying to prove by contradiction, as we are assuming that $K \neq K'$. But, with this new convex body K', if we can show that the equality in 3.5 fails, we will obtain our desired contradiciton. Now, recall the quadrilateral Q that we identified is inscribed in K', and so $Q \subset K'$. Since after we rotated the picture, both $int(Q)$ and $int(K')$ (here int denote the interior of both bodies) lie in the open half-plane with positive x coordinates, and since the function $f(x) = \frac{1}{x}$ is positive in this region, we must have:

$$\int\int_{Q} \frac{1}{x} dA < \int\int_{K'} \frac{1}{x} dA. \tag{3.5}$$

Then, since this inequality must hold, if we can obtain our desired contradiction by showing that:

$$\int\int_{Q} \frac{1}{x} dA \geq \int\int_{K} \frac{1}{x} dA. \tag{3.6}$$

Now, then, we are ready to integrate. First, we integrate $\frac{1}{x}$ on K for the interval $x \in [0, g_1]$. But the y-values are bounded on top by the line passing through the points r and s, with intercept r_2. Below, they are bounded by the line passing through r and g, also with intercept r_2. So, our double integral will look like:

$$\int_{0}^{g_1} \int_{\frac{g_2 - r_2}{g_1} x + r_2}^{\frac{s_2 - r_2}{s_1} x + r_2} \frac{1}{x} dy dx = \frac{g_1(s_2 - r_2) + s_1(r_2 - g_2)}{s_1} \tag{3.7}$$

Now, let's integrate over Q on the same interval, $x \in [0, g_1]$. Now, Q is bounded above by the line passing through the points q and s, with intercept q_2. On the bottom, the line through points q and g bounds our region. Our integral is:

$$\int_{0}^{g_1} \int_{\frac{g_2 - q_2}{g_1} x + q_2}^{\frac{s_2 - q_2}{s_1} x + q_2} \frac{1}{x} dy dx = \frac{g_1(s_2 - q_2) + s_1(q_2 - g_2)}{s_1} \tag{3.8}$$

Now, in order to obtain (hopefully) our contradiction, we need to integrate over both K and Q in the four intervals. So, to keep better track of what all is going on here, we denote the integrals of $\frac{1}{x}$ over K and Q by $I(K)$ and $I(Q)$ respectively. Once again, we want to show that $I(Q) \geq I(K)$, or that $I(Q)-I(K) \geq 0$. And, since we are calculating these integrals over four smaller intervals, let us denote the integral of $\frac{1}{x}$ over K on the interval $x \in [0, g_1]$ by K_1, the integral on the interval $x \in [g_1, p_1]$ by K_2, and so on, continuing in a similar manner for $I(Q)$. Thus, we just computed above the values for K_1 and Q_1. Also, we can rewrite:

$$I(Q) - I(K) = (Q_1 - K_1) + (Q_2 - K_2) + (Q_3 - K_3) + (Q_4 - K_4), \qquad (3.9)$$

with our goal being to show that this difference is greater than or equal to 0. At this point, we will turn over all the caculations to Maple. Performing the integrals on Maple, and then plugging the values into equation 3.9, we obtain:

$$s_2 - r_2 - p_2 - g_2 + \frac{r_2 - t_2}{r_1 + t_1} + q_2 \ln(g_1) + r_2 \ln(2)(\frac{g_1}{g_1 - 2}) + \frac{\ln(\frac{2}{g_1})(t_1 g_2 - g_1 t_2)}{(g_1 - 2)} + \frac{\ln(p_1)(s_1 p_2 - p_1 s_2)}{(p_1 - 1)}. \qquad (3.10)$$

Notice that the term $\frac{r_2 - t_2}{r_1 + t_1}$ is simply the negative slope of the line containing $\overline{rs}$ and $\overline{st}$. And, since $s_1 = 1$ and $r_1 = 0$, notice that $s_2 - r_2$ is the slope of this line. Therefore, these terms cancel, and we can simplify equation 3.10 so it reads:

$$-(p_2 + g_2) + q_2 \ln(g_1) + r_2 \ln(2)(\frac{g_1}{g_1 - 2}) + \frac{\ln(\frac{2}{g_1})(t_1 g_2 - g_1 t_2)}{(g_1 - 2)} + \frac{\ln(p_1)(s_1 p_2 - p_1 s_2)}{(p_1 - 1)}. \qquad (3.11)$$

Now, since the ray (o, φ) containing the segment $\overline{os}$ is oriented at a greater angle than the ray (o, α) containing the segment $\overline{op}$, and thus we must have the relationship $\frac{s_2}{s_1} > \frac{p_2}{p_1}$. Since p_1 and s_1 are both positive, we get the relationship $s_1 p_2 - p_1 s_2 < 0$. And, since we assumed $p_1 < 1$, we must have $\ln(p_1) < 0$, and certianly $p_1 - 1 < 0$ also. Thus, the final term in equation 3.11 is negative. And, by similar arguments, we can show that the second to last term in 3.12 is negative also. Since $g_1 < 1$, $\ln(g_1) < 0$, and certainly $g_1 - 2 < 0$, all the terms in 3.12 are negative. And, $I(Q) - I(K) < 0$, although we wanted to show that this value was greater than or equal to 0. Thus, our method for obtaining contradiction was ineffective.

Notice, however, that we did not confirm the existence of a second convex body with the same X-rays as our triangle K. Rather, we simply failed to prove conclusively that a second such body does not exist.

4.2.2 Case where $p_1 = s_1 = 1$.

Since $p_1 = s_1$, we need only to integrate on the intervals $x \in [0, g_1]$, $x \in [g_1, 1]$, $x \in [1, 2]$. And, as in the previous case, we will integrate the function $\frac{1}{x}$ on these intervals over K and Q and calculate $I(Q)-I(K)$ from 3.9, although here $K_4 = Q_4 = 0$. Using Maple to calculate the integrals, we obtain:

$$t_2 - s_2 + p_2 - r_2 + r_2\ln(2) - \ln(g_1)q_2 + (\ln(2) - \ln(g_1))(\frac{t_1g_2 - g_1t_2}{g_1 - t_1}). \quad (3.12)$$

Notice now, that the term $\frac{t_1g_2-g_1t_2}{g_1-t_1}$ is simply the negative value of the y-intercept of the line through the points t and g. Now, the magnitude of this term must be less than q_2, otherwise violating the convexity of Q. And, we can write the last two terms of 3.12 as:

$$q_2((r_2 - q_2)\ln(2) - \ln(g_1)). \quad (3.13)$$

Notice that $((r_2-q_2)\ln(2)-\ln(g_1)) > (\ln(2)-\ln(g_1))$. Therefore, the sum of the last three terms of equation 3.12 is positive. Thus, equation 3.12 is greater than or equal to 0 if:

$$t_2 - s_2 \geq r_2 - p_2. \quad (3.14)$$

However, since $r_1 = 0$, $s_1 = p_1 = 1$, and $t_1 = 2$, that we simply need to show that the slope of the line through the points s and t is greater than or equal to the negative slope of the line through the points p and r. Now, we can rewrite the slope of the line through the points s and t as $s_2 - r_2$. Thus, substituting into equation 3.14, we can obtain the inequality:

$$\frac{s_2 + p_2}{2} \geq r_2. \quad (3.15)$$

Now, if this inequality is satisfied, the value of $I(Q) - I(K) \geq 0$, and we obtain our desired contradiction, thus verifying our triangle K from a point a_i and any other point in the interval (a_i, a_{i+1}).

4.2.3 Case where $p_1 > s_1$.

Since $p_1 > s_1$, we need to integrate on the intervals $x \in [0, g_1]$, $x \in [g_1, 1]$, $x \in [1, p_1]$, and $x \in [p_1, 2]$. Again, we will integrate the function $\frac{1}{x}$ on these intervals over K and Q and calculate $I(Q) - I(K)$ from 3.10. Maple provides us with the values of the integrals:

$$\frac{r_2}{2}(p_1 - 3) - \frac{t_2}{2}(1 + p_1) + s_2 + p_2 + r_2\ln(2) - \quad (3.16)$$
$$q_2\ln(g_1) + (\ln(2) - \ln(g_1))(\frac{t_1g_2 - g_1t_2}{g_1 - t_1}).$$

Using the same arugments that we did for the previous case, we know that the last three terms of 3.17 are altogether positive. Now, let us closely examine the first 4 terms. Now, since t_2 lies on the line through the points r and s, and the $\overline{rs} = \overline{st} = x$, we can express t_2 in terms of r_2 and s_2. Namely:

$$t_2 = 2s_2 - r_2. \quad (3.17)$$

Substituting into the first 4 terms of 3.17, and manipulating, we obtain:

$$(p_1 - s_1)(r_2 + \frac{s_1 p_2 - p_1 s_2}{p_1 - s_1}). \tag{3.18}$$

But, looking at 3.19, we realize that we are in trouble. $\frac{s_1 p_2 - p_1 s_2}{p_1 - s_1}$ can be verified to be the negative y-intercept of the line through the points p and s. This value must be greater in magnitude than r_2, and so the entire second term is negative. But, since we assumed $p_1 > s_1$, the first term is positive, and 3.19 is negative. Thus, 3.17 consists of a large positive chunk, and a large negative chunk, and the two are so complex that it is unfeasible to compare the two values. Thus, for this case, our results are inconclusive.

5 Conclusions.

Other than the one success for our second case, we had great difficulty verifying our triangle from two sources, where one is a point a_i, and the other is a point in the interval (a_i, a_{i+1}). There are other methods that we could look at, however, that might produce a positive result. When rotating the picture, we can consider different angles of rotation. Or, when scaling the picture, we could scale so that a different coordinate is equal to 1. We could also make easy changes to allow our integrand to be $\frac{1}{y}$. These changes could have a profound impact in how the final calculations work out. And, most importantly, we only focused on the integrals derived from the X-ray data from only one of our sources. Our key piece of information is that the two convex bodies have the same X-ray data from two sources. However, I focused on the similarities in data from only one source, due to the simpler integrals. Although introducing the second source might give us messier integrals, in the long run, it might also prove to be more effective in achieving a positive result.

A natural follow-up is the question of whether this result carries over to two sources where one is a point a_i, and the other is a point in the interval (a_i, a_{i-1}). If these questions can be answered positively, then by considering other cases, where our points lie in other intervals on the x-axis, we can address the problem of determining a triangle from two directed X-ray sources.

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