Functions that are Directed X-rays of Planar Convex Bodies

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Abstract

We develop conditions that ensure that a convex function X is the directed x-ray from a point source at the origin of a convex body K in the upper half plane ($y > 0$).

1 Introduction

This paper was motivated by the previous work of Black and Koop [1] in the field of geometric tomography. Geometric tomography is the area of mathematics that developed in response to Hammer's question (posed in 1961) of whether a convex body could be uniquely determined by a finite set of x-rays [3]. This paper focuses on conditions that guarantee that a function is the directed x-ray of a convex body. The main results presented in this paper are essentially that X is the directed x-ray of a convex body in the upper half plane ($y > 0$), if X is a convex function and the set of points with zero curvature on the boundary of the graph of X (if any exist) is connected. For more information on geometric tomography consult Gardner [2].

2 Notation and Basic Definitions

Definition 2.1 *Throughout this paper, a* ***convex body*** *is a compact, convex subset of the plane with non-empty interior.*

Definition 2.2 *The* ***characteristic function*** *of a convex body K is*

$$x_K(p) = \begin{cases} 0 & if \quad p \notin K \\ 1 & if \quad p \in K \end{cases} \; for\, p \in E^2.$$

Definition 2.3 *A* ***directed x-ray transform****, also known as a point source x-ray or fan-beam x-ray, gives the chord length of the convex body along a particular direction θ from a given point O, which is called the* ***point source****. Thus, the directed x-ray of K, $\mathcal{D}_{K_O}$ is*

$$\mathcal{D}_{K_O}(\varphi) = \int_0^{\infty} x_K(O + t\theta)dt, \;\; \theta =< \cos\varphi, \sin\varphi > .$$

Throughout this paper, we will assume that K is a convex body and that the point source O is at the origin. Unless otherwise specified, points will be in polar coordinates (r,φ) where $r \geq 0$ and φ is measured counterclockwise from the positive x-axis.

As did Black and Koop [1], we can define the convex body K from a single point source by a unique pair of functions $r(\varphi)$ and $R(\varphi)$ satisfying

$$K = \{(s, \varphi) : 0 \leq r(\varphi) \leq s \leq R(\varphi)\}.$$

We refer to the points with polar coordinates $(r(\varphi), \varphi)$ as points on the near boundary of K and the function $r(\varphi)$ as the near boundary function of K. In a similar manner, the terms far boundary and far boundary function refer to the points $(R(\varphi), \varphi)$ and the function $R(\varphi)$. Throughout this paper, we use lower case letters to refer to the near boundary and uppercase letters to refer to the far boundary. Using this notation, the directed x-ray of a convex body K in the direction $\theta =< \cos\varphi, \sin\varphi >$ becomes $X(\varphi) = R(\varphi) - r(\varphi)$. Moreover, we will simply refer to these functions as $X, R,$ and r.

Definition 2.4 *Let K be a convex body with an associated directed x-ray from source O. Then the* ***supporting cone*** $\mathcal{C}[\alpha,\beta]$ *of K is the largest cone with vertex at O such that for all φ, $\alpha < \varphi < \beta$, $\mathcal{D}_{K_O}(\varphi) > 0$. The* ***supporting rays*** *of K are $\varphi = \alpha$ and $\varphi = \beta$.*

Throughout this paper, the supporting rays are labeled α and β where $\alpha < \beta$.

3 The Quadratic Form and the Differential Operator $\mathcal{K}$

We will use many tools that have been previously developed in studying the functions that are directed x-rays. One such tool is the quadratic form defined by Lam and Solmon [4].

Definition 3.1 *Given three points on the curve x, $\{x_1, x_2, x_3\}$, with associated angles $\varphi_1, \varphi_2, \varphi_3$ such that $\varphi_1 < \varphi_2 < \varphi_3 < \varphi_1 + \pi$, we define the* ***quadratic form****, $Q(x)$, as*

$$Q(x) = Q(x_1, x_2, x_3) = x_1x_2\sin(\varphi_2 - \varphi_1) + x_2x_3\sin(\varphi_3 - \varphi_2) - x_1x_3\sin(\varphi_3 - \varphi_1)$$

where $x_i = x_i(\varphi)$ is the distance of φ_i from the origin.

This form can also be written $\mathcal{Q}(x) = xAx^T$, where

$$A = \frac{1}{2}\begin{pmatrix} 0 & \sin(\varphi_2 - \varphi_1) & -\sin(\varphi_3 - \varphi_1) \\ \sin(\varphi_2 - \varphi_1) & 0 & \sin(\varphi_3 - \varphi_2) \\ -\sin(\varphi_3 - \varphi_1) & \sin(\varphi_3 - \varphi_2) & 0 \end{pmatrix}$$

and x is the vector $< x_1, x_2, x_3 >$. Both forms appear in [4], but the latter expression will be particularly useful to us since it can be used to express the middle term when expanding $\mathcal{Q}(r + s) = \mathcal{Q}(r) + 2rAs^T + \mathcal{Q}(s)$. The actual expansion of the previous equation, derived by Black & Koop [1], will be particularly useful to us later in the paper and is as follows

$$\mathcal{Q}(r + s) = \mathcal{Q}(r) + \mathcal{Q}(s) + (r_1s_2 + r_2s_1)\sin(\varphi_2 - \varphi_1) + (r_2s_3 + r_3s_2)\sin(\varphi_3 - \varphi_2) - (r_1s_3 + r_3s_1)\sin(\varphi_3 - \varphi_1).$$

We also know that if $\mathcal{Q}(X) \leq 0$, then

$$x_2 \leq \frac{x_1 x_2 \sin(\varphi_3 - \varphi_1)}{x_1 \sin(\varphi_2 - \varphi_1) + x_3 \sin(\varphi_3 - \varphi_2)}.$$

The quadratic form is useful because it can be used to compute the concavity of a curve. The following Lemma from [4] describes the details of this concavity.

Lemma 3.2 *Suppose that* $\varphi_1 < \varphi_2 < \varphi_3 \leq \varphi_1 + \pi$ *and* $r_j \geq 0$, $j = 1, 2, 3$.

(a) $\mathcal{Q}(r) = 0$ *if and only if the points* (r_j, φ_j) *are collinear.*

(b) If $\mathcal{Q}(r) > 0$, *then the line passing through* (r_1, φ_1) *and* (r_3, φ_3) *separates* (r_2, φ_2) *from the origin. If* $\mathcal{Q}(r) < 0$, *then the origin and* (r_2, φ_2) *lie on the same side of the line passing through* (r_1, φ_1) *and* (r_3, φ_3).

(c) If for $j = 1, 2, 3$ *the rays making angle* φ_j *with the positive x-axis meet the boundary of the convex body K in points* (r_j, φ_j) *and* (R_j, φ_j) *where* $0 \leq r_j \leq R_j$, *then* $\mathcal{Q}(R) \geq 0$ *and* $\mathcal{Q}(r) \leq 0$.

One of the problems with the quadratic form is that it requires three points to compute concavity. Black and Koop [1] realized this problem and derived the differential operator $\mathcal{K}$ by taking appropriate limits involving $\mathcal{Q}$ as $\varphi_1, \varphi_3 \to \varphi_2$. Therefore $\mathcal{K}$ only requires one point to compute concavity.

The appropriate limits for computing the differential operator are as follows:

First compute

$$\mathcal{Q}^*(x) = \lim_{\varphi_1 \to \varphi_2} \frac{\mathcal{Q}(x)}{\varphi_2 - \varphi_1}$$

and then

$$\lim_{\varphi_3 \to \varphi_2} \frac{\mathcal{Q}^*(x)}{(\varphi_3 - \varphi_2)^2}.$$

The result will give the differential operator $\mathcal{K}$ which is defined below.

Definition 3.3 *If X is a curve that is twice differentiable, the* ***differential operator,*** $\mathcal{K}(X)$, *is defined as follows*

$$\mathcal{K}(X) = X^2 + 2(X')^2 - X''X.$$

Remark 3.4 *An interesting property of the differential operator is its relationship to curvature κ. The formula for κ in polar coordinates is*

$$\kappa(x) = \frac{x^2 + 2(x')^2 - x''x}{(x^2 + (x')^2)^{\frac{3}{2}}}.$$

Since the denominator of κ is always positive, the numerator determines if the curve has positive or negative curvature, and this numerator is exactly the differential operator.

Remark 3.5 *It will not be a problem that $\mathcal{K}$ requires the function X to be twice differentiable since the functions that we will be examining will have at most a small number of points where this is an issue. In particular, the curvature is zero at a point $(\varphi_0, X(\varphi_0))$ on the graph of X if and only if $\mathcal{K}(X)(\varphi_0) = 0$.*

Since it is useful to apply the differential operator to sums, Black and Koop [1] also derived $\mathcal{K}(X+Y)$,

$$\mathcal{K}(X+Y) = \mathcal{K}(X) + (2XY + 4X'Y' - XY'' - X''Y) + \mathcal{K}(Y).$$

Black and Koop [1] also established conditions for concavity similar to those of $\mathcal{Q}$. These conditions are stated in the following theorem from [1].

Theorem 3.6 *Let $g(\varphi)$ be a C^2 polar function on (α, β) and continuous on $[\alpha, \beta]$, $0 < \beta - \alpha < \pi$. Define the parametric curve $\Gamma =< g(\varphi)\cos\varphi, g(\varphi)\sin\varphi >$. Then,*

i) Γ is concave toward the origin on (α, β) if and only if $\mathcal{K}(g)(\varphi) \geq 0$ for all $\varphi \in (\alpha, \beta)$.

ii) Γ is concave away from the origin on (α, β) if and only if $\mathcal{K}(g)(\varphi) \leq 0$ for all $\varphi \in (\alpha, \beta)$.

4 Rotation

At various times we will be rotating the convex body K so that $\alpha = 0$ or $\beta = \pi$. It is vital that we understand this rotation geometrically to fully develop our results.

So suppose we have a convex body K with near side r and far side R as described previously. If we set $r(\alpha) = r_\alpha > 0$, then a rotation of coordinates so that $\alpha = 0$ places the convex body in the half plane $y \geq 0$ and tangent to the x-axis at $(r_\alpha, 0)$. In the neighborhood of $(r_\alpha, 0)$ the boundary of K is the graph of a convex function $y = f(x)$ that satisfies $f(x) > 0$ for $x < \alpha$, $f(\alpha) = f'(\alpha) = 0$, and $f(x) \geq 0$ for $x > \alpha$. (Refer to diagram 1) Similarly we can rotate the convex body K so that $\beta = \pi$.

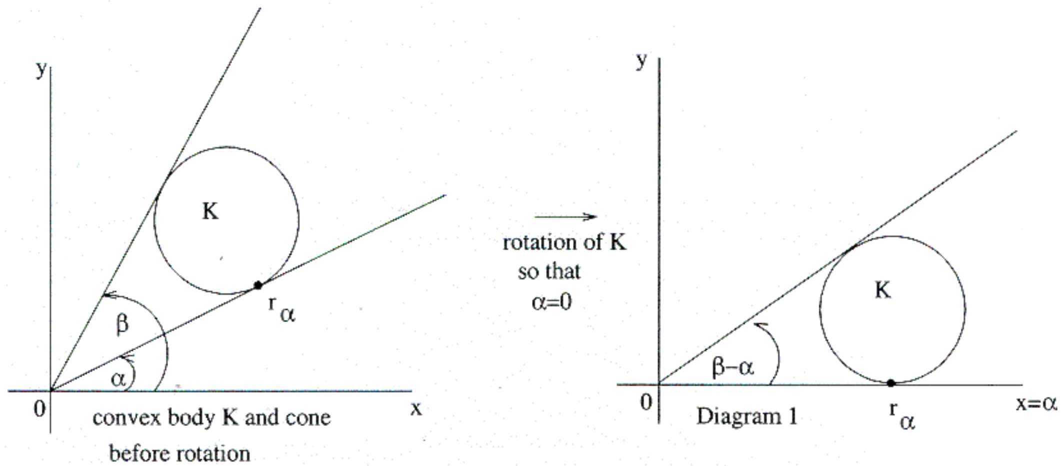
Now consider, after rotation, a point of intersection P on the near side of K with the ray making angle $\sigma > 0$ (measured counterclockwise from the x-axis). P has polar coordinates $(r(\sigma), \sigma)$ and rectangular coordinates $(x, y) = (x, f(x))$, so $x = r(\sigma)\cos\sigma$ and $y = r(\sigma)\sin\sigma$. The slope at this point is given by $m_\sigma = m = f'(x) < 0$. The angle of inclination ψ_σ of the tangent line to the boundary of K at $(r(\sigma), \sigma)$ satisfies $m_\sigma = \tan(\psi_\sigma)$. Clearly as $\sigma \to 0$, then $\psi_\sigma \to \pi$. (Refer to diagram 2)

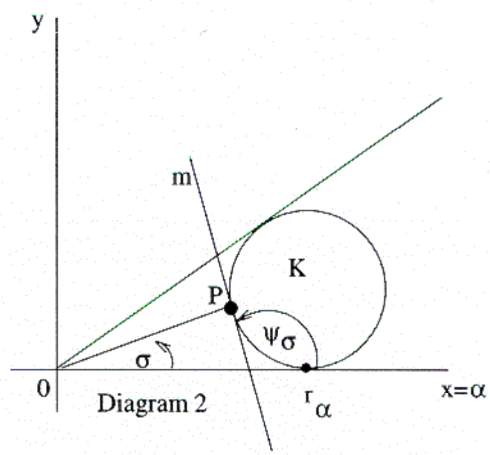
Remark 4.1 *The angle labeled σ in the figures appears as φ throughout the body of the paper.*

5 Functions that are directed x-rays of convex bodies with smooth boundaries

Using the rotation above, we can now prove the following theorem.

Theorem 5.1 *Suppose that K is a convex body in the upper half plane with near side r, far side R and supporting cone $C[\alpha, \beta]$. If the boundary of K is C^2, then as $\varphi \to \alpha$ or $\varphi \to \beta$, $r'(\varphi) \to -\infty$, $R'(\varphi) \to \infty$, $r''(\varphi) \to \infty$, and $R''(\varphi) \to -\infty$. In particular, $X'(\varphi) = R'(\varphi) - r'(\varphi) \to \infty$ and $X''(\varphi) = R''(\varphi) - r''(\varphi) \to -\infty$ as $\varphi \to \alpha$ or $\varphi \to \beta$.*


y
K
β
α
r_α
0
x
convex body K and cone
before rotation
rotation of K
so that
α=0
y
K
β−α
0
Diagram 1
r_α
x=α


y
m
K
P
ψσ
σ
0
Diagram 2
r_α
x=α

Proof. We will begin by allowing $\varphi \to \alpha$. Without harming the computations, we can rotate the convex body K along with its x-ray X such that $\alpha = 0$ (as described previously). So we have $x = r(\varphi)\cos\varphi$ and $y = r(\varphi)\sin\varphi$ and

$$m = \frac{dy}{dx} = \frac{\frac{dy}{d\varphi}}{\frac{dx}{d\varphi}} = \frac{r(\varphi)\cos\varphi + r'(\varphi)\sin\varphi}{-r(\varphi)\sin\varphi + r'(\varphi)\cos\varphi}.$$

Now solving for $r'(\varphi)$, we have

$$r'(\varphi) = \frac{r(\varphi)\cos\varphi - r(\varphi)m\sin\varphi}{m\cos\varphi - \sin\varphi}.$$

And by allowing $\varphi \to \alpha = 0$, we have

$$\lim_{\varphi\to 0} \frac{r(\varphi)\cos\varphi - r(\varphi)m\sin\varphi}{m\cos\varphi - \sin\varphi}.$$

We know that $m < 0$ and that m goes to zero as $\varphi \to \alpha$, hence m goes to zero through negative values. And we know that $\sin\varphi$ goes to zero as $\varphi \to 0$. Therefore the denominator goes to zero through negative values. And clearly, the numerator goes to $r_\alpha > 0$ as $\varphi \to 0$. Hence $r'(\varphi) \to -\infty$.

Now we need to show that $r''(\varphi) \to \infty$. By showing that $r'(\varphi)$ is an increasing function near 0, it follows that $r''(\varphi) > 0$. Then all we need to show is that $r''(\varphi)$ diverges. The slope $m = m_\psi$ can be expressed in terms of the angle of inclination ψ_φ in the following way

$$m = \tan\psi_\varphi = \frac{\sin\psi_\varphi}{\cos\psi_\varphi}.$$

Hence

$$r'(\varphi) = \frac{r(\varphi)\cos\varphi - r(\varphi)\frac{\sin\psi_\varphi}{\cos\psi_\varphi}\sin\varphi}{\frac{\sin\psi_\varphi}{\cos\psi_\varphi}\cos\varphi - \sin\varphi}.$$

Multiplying by $\frac{\cos\psi_\varphi}{\cos\psi_\varphi}$ yields

$$r'(\varphi) = r(\varphi)\frac{\cos\psi_\varphi\cos\varphi - \sin\psi_\varphi\sin\varphi}{\sin\psi_\varphi\cos\varphi - \cos\psi_\varphi\sin\varphi}.$$

Using the trigonometric identities $\cos(x-y) = \sin x \sin y + \cos x \cos y$ and $\sin(x-y) = \sin x \cos y - \sin y \cos x$ we can write the above expression as

$$r'(\varphi) = r(\varphi)\frac{\cos(\psi_\varphi - \varphi)}{\sin(\psi_\varphi - \varphi)} = r(\varphi)\cot(\psi_\varphi - \varphi).$$

Now let $0 < \varphi_1 < \varphi_2 << 1$, then $r(\varphi_1) > r(\varphi_2)$ and $\pi > \psi_{\varphi_1} > \psi_{\varphi_2} > \frac{\pi}{2}$. Hence $(\psi_{\varphi_1} - \varphi_1) > (\psi_{\varphi_2} - \varphi_2)$ and $\cot(\psi_{\varphi_1} - \varphi_1) < \cot(\psi_{\varphi_2} - \varphi_2) < 0$.

So we have

$$r(\varphi_1)\cot(\psi_{\varphi_2} - \varphi_2) < r(\varphi_2)\cot(\psi_{\varphi_2} - \varphi_2)$$

and

$$r(\varphi_1)\cot(\psi_{\varphi_1} - \varphi_1) < r(\varphi_1)\cot(\psi_{\varphi_2} - \varphi_2)$$

hence

$$r(\varphi_1)\cot(\psi_{\varphi_1} - \varphi_1) < r(\varphi_2)\cot(\psi_{\varphi_2} - \varphi_2).$$

Therefore $r'(\varphi_1) < r'(\varphi_2)$ and $r'(\varphi)$ is increasing near 0. So $r''(\varphi) > 0$.

Differentiating $r'(\varphi) = r(\varphi)\cot(\psi_\varphi - \varphi)$, we obtain the following equation for $r''(\varphi)$:

$$r''(\varphi) = r'(\varphi)\cot(\psi_\varphi - \varphi) - r(\varphi)\csc^2(\psi_\varphi - \varphi)\psi'_\varphi.$$

Since $r'(\varphi) \to -\infty$, $\cot(\psi_\varphi - \varphi) \to -\infty$, $\csc^2(\psi_\varphi - \varphi) \to \infty$, and $\psi'_\varphi \leq 0$, then $r''(\varphi) \to \infty$.

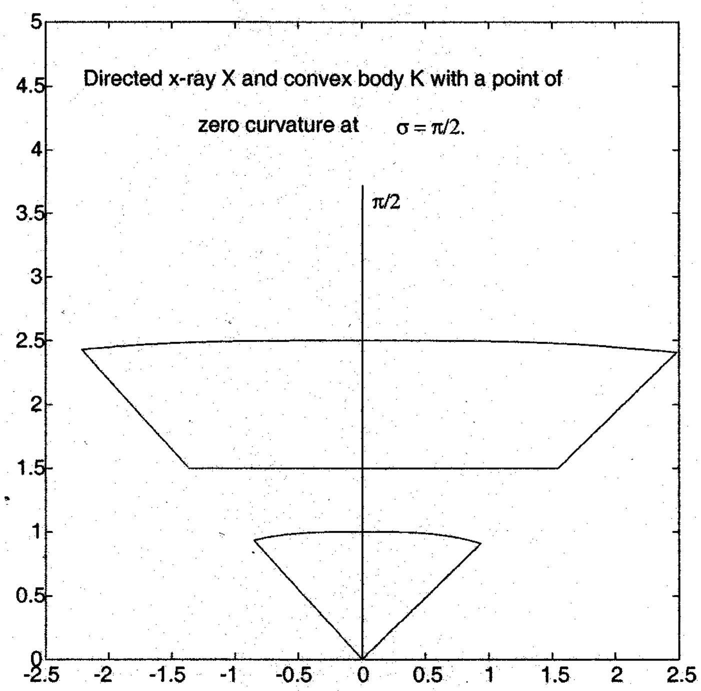
By similar arguments, $R'(\varphi) \to \infty$ and $R''(\varphi) \to -\infty$.

We can also rotate the convex body K so that $\beta = \pi$ and show by similar arguments that as $\varphi \to \beta$, $r'(\varphi) \to -\infty$, $r''(\varphi) \to \infty$, $R'(\varphi) \to \infty$, and $R''(\varphi) \to -\infty$.

Therefore $X' = R' - r' \to \infty - (-\infty) \to \infty$ and $X'' = R'' - r'' \to -\infty - \infty \to -\infty$. ■

Remark 5.2 *Notice that the above proof still holds if the point of tangency of the convex body K is at the origin.*

Black and Koop's Lemma 7.1, [1], is necessary for the following proof. It is as follows.



Lemma 5.3 *If* $K = (r, R)$ *is a convex body bounded away from the origin in* $\mathcal{C}[\alpha, \beta] \cap C^2(\alpha, \beta)$ *and its directed x-ray* X *satisfies* $\mathcal{K}(X)(\varphi_0) = 0$ *for some* $\varphi_0 \in (\alpha, \beta)$,. *then* $\mathcal{K}(r)(\varphi_0) = \mathcal{K}(R)(\varphi_0) = 0$ *and the tangent lines to the graphs of the functions* $X(\varphi), r(\varphi)$, *and* $R(\varphi)$ *are parallel at* $\varphi = \varphi_0$.

Theorem 5.4 *Suppose that* $X \in \mathcal{C}[\alpha, \beta] \cap C^2(\alpha, \beta)$ *satisfies the following:*

i) $X > 0$ *on* (α, β).

ii) $\mathcal{K}(X) \geq 0$ *on* (α, β) *and the set of points where* $\mathcal{K}(X) = 0$ *is connected. (We include* α *or* β *in this set if* $\liminf_{\varphi \to \alpha+} \mathcal{K}(X) = 0$ *or* $\liminf_{\varphi \to \beta-} \mathcal{K}(X) = 0$.)

Then there exists a convex body K *which lies in the upper half plane, has* C^2 *boundary (except possibly at points of intersection of* K *with the rays* $\varphi = \alpha$ *and* $\varphi = \beta$) *and such that* X *is the directed x-ray of* K.

Remark 5.5 *Black and Koop [1] have shown the necessity of hypothesis ii).*

Proof. Let ℓ be a fixed line. We will show that $\mathcal{K}(t\ell + X) \geq 0$. We know that $\mathcal{K}(t\ell + X) = t(2\ell X + 4\ell' X' - \ell'' X - \ell X'') + \mathcal{K}(X)$.

Case 1:

Assume $\mathcal{K}(X)(\varphi) > \delta > 0$ for all $\varphi \in [\alpha, \beta]$.

If $2\ell X + 4\ell' X' - \ell'' X - \ell X''$ is bounded as $\varphi \to \alpha$ and $\varphi \to \beta$ then there exists a $t > 0$ such that $t(2\ell X + 4\ell' X' - \ell'' X - \ell X'') < \delta$. Hence for such t, $\mathcal{K}(t\ell + X) > 0$, and the convex body with near side $t\ell$ and far side $X + t\ell$ has X as its directed x-ray.

However if $2\ell X + 4\ell' X' - \ell'' X - \ell X''$ becomes unbounded as $\varphi \to \alpha$ or as $\varphi \to \beta$ then $\mathcal{K}(t\ell + X)$ will remain positive if we show that $X'(\varphi) \to \infty$ and $X''(\varphi) \to -\infty$.

So suppose that $\mathcal{K}(X) \to +\infty$ as $\varphi \to \alpha$ or $\varphi \to \beta$. And let M be the convex body such that $M = (0, X)$ and rotate M such that $\alpha = 0$.

Then by Theorem 5.1, $X'(\varphi) \to \infty$ and $X''(\varphi) \to -\infty$.

Hence $\mathcal{K}(t\ell + X) > 0$.

Case 2:

Assume $\mathcal{K}(X)(\varphi) = 0$ for a nonempty connected subset $[\varphi_0, \varphi_1]$ of (α, β) and $\mathcal{K}(X) > \delta > 0$ for φ sufficiently close to α and β. We will show that $\mathcal{K}(t\ell + X) > 0$ for $\varphi < \varphi_0$ where φ_0 is the smallest angle such that $\mathcal{K}(X)(\varphi_0) = 0$.

Since $\mathcal{K}(X) = X^2 + 2(X')^2 - X''X$ then $X'' = \frac{-\mathcal{K}(X) + X^2 + 2(X')^2}{X}$. And since $\mathcal{K}(\ell) = 0$ for all $\varphi \in [\alpha, \beta]$ then $\ell'' = \ell + \frac{2(\ell')^2}{\ell}$.

So

$$\mathcal{K}(X + t\ell) = \mathcal{K}(X) + t\left(2\ell X + 4\ell' X' - X(\ell + \frac{2(\ell')^2}{\ell}) - \ell\left(\frac{-\mathcal{K}(X) + X^2 + 2(X')^2}{X}\right)\right)$$

$$\begin{aligned}
&= \mathcal{K}(X) + \frac{t\ell}{X}\mathcal{K}(X) + 4t\ell' X' - \frac{2tX(\ell')^2}{\ell} - \frac{2t\ell(X')^2}{X} \\
&= \mathcal{K}(X) + \frac{t\ell}{X}\mathcal{K}(X) - 2t\ell X\left(\frac{X'}{X} - \frac{\ell'}{\ell}\right)^2 \\
&= \mathcal{K}(X) + t\ell X\left[\frac{\mathcal{K}(X)}{X^2} - 2\left(\frac{X'}{X} - \frac{\ell'}{\ell}\right)^2\right]. \qquad (1)
\end{aligned}$$

By Lemma 5.3, we know that both the near and far side of any convex body with the directed x-ray X must have points of zero curvature at its points of intersection with the ray $\varphi = \varphi_0$. Moreover, the tangent lines to K at these points of intersection must be parallel with a determined slope. By a rotation of coordinates, we may assume that the common slope is 0. The only line that we can add to X and preserve convexity has slope 0, so we assume that

$$\ell = \frac{1}{\sin\varphi}.$$

So

$$\ell' = \frac{-\cos\varphi}{\sin^2\varphi} \tag{2}$$

and

$$\frac{\ell'}{\ell} = \frac{-\cos\varphi}{\sin\varphi} = -\cot\varphi. \tag{3}$$

Again with ψ_φ, the angle of inclination,

$$\tan\psi_\varphi = \frac{X'\sin\varphi + X\cos\varphi}{X'\cos\varphi - X\sin\varphi} = \frac{dy}{dx} = \frac{\frac{dy}{d\varphi}}{\frac{dx}{d\varphi}} = \frac{\frac{d}{d\varphi}(X(\varphi)\sin\varphi)}{\frac{d}{d\varphi}(X(\varphi)\cos\varphi)}$$

Cross-multiplying yields

$$(X'\cos\varphi - X\sin\varphi)\sin\psi_\varphi = (X'\sin\varphi + X\cos\varphi)\cos\psi_\varphi$$

$$\Rightarrow \frac{X'}{X}\cos\varphi\sin\psi_\varphi - \sin\varphi\sin\psi_\varphi = \frac{X'}{X}\sin\varphi\cos\psi_\varphi + \cos\varphi\cos\psi_\varphi$$

$$\Rightarrow \frac{X'}{X}(\cos\varphi\sin\psi_\varphi - \sin\varphi\cos\psi_\varphi) = \cos\varphi\cos\psi_\varphi + \sin\varphi\sin\psi_\varphi$$

Using the trigonometric identities $\cos(x-y) = \sin x\sin y + \cos x\cos y$ and $\sin(x-y) = \sin x\cos y - \sin y\cos x$, we can rewrite the above expression as

$$\frac{X'}{X}\sin(\psi_\varphi - \varphi) = \cos(\psi_\varphi - \varphi).$$

So,

$$\frac{X'}{X} = \cot(\psi_\varphi - \varphi). \tag{4}$$

And

$$\left(\frac{X'}{X}\right)' = -\csc^2(\psi_\varphi - \varphi)\left(\frac{d\psi_\varphi}{d\varphi} - 1\right). \tag{5}$$

We can use the equation for the differential operator to obtain the following equation for $\frac{\mathcal{K}(X)}{X^2}$:

$$\frac{\mathcal{K}(X)}{X^2} = 1 + \left(\frac{X'}{X}\right)^2 - \left(\frac{X'}{X}\right)'.$$

By substituting (4) and (5), we have

$$\begin{aligned}\frac{\mathcal{K}(X)}{X^2} &= 1 + \cot^2(\psi_\varphi - \varphi) + \csc^2(\psi_\varphi - \varphi)\left(\frac{d\psi_\varphi}{d\varphi} - 1\right)\\ &= \csc^2(\psi_\varphi - \varphi)\frac{d\psi_\varphi}{d\varphi}.\end{aligned}$$

And by substituting (3) and (4) into $\left[\frac{X'}{X} - \frac{\ell'}{\ell}\right]^2$, we have

$$\begin{aligned}\left[\frac{X'}{X} - \frac{\ell'}{\ell}\right]^2 &= [\cot(\psi_\varphi - \varphi) + \cot\varphi]^2\\ &= \left[\frac{\cos(\psi_\varphi - \varphi)}{\sin(\psi_\varphi - \varphi)} + \frac{\cos\varphi}{\sin\varphi}\right]^2\\ &= \frac{\sin^2\psi_\varphi}{\sin^2(\psi_\varphi - \varphi)\sin^2\varphi}.\end{aligned}$$

Hence rewriting equation (1) gives,

$$\mathcal{K}(X + t\ell) = \mathcal{K}(X) + t\ell X\csc^2(\psi_\varphi - \varphi)\left[\frac{d\psi_\varphi}{d\varphi} - \frac{2\sin^2\psi_\varphi}{\sin^2\varphi}\right].$$

We know that $\mathcal{K}(X) > 0$, $t\ell X > 0$, and $\csc^2(\psi_\varphi - \varphi) > 0$ for $\varphi < \varphi_0$. So we need to show that $\frac{d\psi_\varphi}{d\varphi} - \frac{2\sin^2\psi_\varphi}{\sin^2\varphi} > 0$ for $\varphi < \varphi_0$.

So assume that $\frac{d\psi_\varphi}{d\varphi} - \frac{2\sin^2\psi_\varphi}{\sin^2\varphi} < 0$ for some φ_1 near φ_0 where $\varphi_1 < \varphi_0$. Then by assumption $0 \le \frac{d\psi_\varphi}{d\varphi} < \frac{2\sin^2\psi_\varphi}{\sin^2\varphi}$.
By the Mean Value Theorem, there exists φ^* such that

$$\psi_\varphi - \psi_{\varphi_0} = \psi'(\varphi^*)(\varphi - \varphi_0) \qquad \text{where } \varphi < \varphi^* < \varphi_0.$$

Let $g(\varphi) = \frac{2\sin^2\psi_\varphi}{\sin^2\varphi}$ then

$$g'(\varphi) = \frac{4\sin\psi_\varphi(\sin\varphi\cos\psi_\varphi\psi'_\varphi - \sin\psi_\varphi\cos\varphi)}{\sin^3\varphi}.$$

Since $0 < \varphi < \frac{\pi}{2}$ and $\frac{\pi}{2} < \psi_\varphi < \pi$, then $g'(\varphi) < 0$. Hence $g(\varphi)$ is a decreasing function and is bounded. So $\psi'(\varphi^*) < \psi'(\varphi) < \frac{2\sin^2\psi_\varphi}{\sin^2\varphi}$.
Since $\psi_{\varphi_0} = \pi$ then,

$$\psi_\varphi - \pi = \psi'(\varphi^*)(\varphi - \varphi_0)$$
$$|\psi_\varphi - \pi| = |\psi'(\varphi^*)|\,|\varphi - \varphi_0|.$$

And since $\varphi < \varphi^* < \varphi_0$ and $\psi'(\varphi^*) < \psi'(\varphi) < \frac{2\sin^2\psi_\varphi}{\sin^2\varphi}$, then

$$|\psi_\varphi - \pi| < \frac{2\sin^2\psi_\varphi}{\sin^2\varphi}|\varphi - \varphi_0|.$$

And dividing by $\sin\psi_\varphi$ gives

$$\frac{|\psi_\varphi - \pi|}{|\sin\psi_\varphi|} < \frac{2\sin\psi_\varphi}{|\sin^2\varphi|}|\varphi - \varphi_0|.$$

and if we are taking the limit as $\varphi \to \varphi_0$, we obtain $1 < 0$, which is a contradiction. Hence there is some interval near φ_o such that $\frac{d\psi_\varphi}{d\varphi} - \frac{2\sin^2\psi_\varphi}{\sin^2\varphi} > 0$ and $\mathcal{K}(t\ell + X) > 0$ for all $t > 0$ and $\varphi < \varphi_0$, for φ sufficiently close to φ_0. By similar arguments we can show that $\mathcal{K}(t\ell + X) > 0$ for all $t > 0$ and $\varphi > \varphi_1$ for φ sufficiently close to φ_1. Since we know $\mathcal{K}(t\ell + X) \ge 0$ for all t sufficiently small when φ is bounded away from φ_0 in $[\alpha, \beta]$.
Case 3) $\liminf_{\varphi\to\alpha}\mathcal{K}(X) = 0$ or $\liminf_{\varphi\to\beta}\mathcal{K}(X) = 0$.
This case can be handled in a manner similar to that of case ii). ■

6 Further Work

Until now we have only considered functions that are the directed x-rays of convex bodies with smooth boundaries. We hope to expand on this work and consider functions that are directed x-rays of convex bodies with nonsmooth boundaries. Working with boundaries that are not $\mathcal{C}^2$ prevents us from using the differential operator $\mathcal{K}$ during our analysis. Instead we need the analogous form of $\mathcal{Q}$ which is derived below.

We know that

$$\mathcal{Q}(X+Y) = \mathcal{Q}(X) + \mathcal{Q}(Y) + 2YAX^T.$$

And Black and Koop's [1] expansion gives

$$\mathcal{Q}(X+Y) = \mathcal{Q}(X) + \mathcal{Q}(Y) + (X_1Y_2 + X_2Y_1)\sin(\varphi_2 - \varphi_1) + (X_2Y_3 + X_3Y_2)\sin(\varphi_3 - \varphi_2) \\ - (X_1Y_3 + X_3Y_1)\sin(\varphi_3 - \varphi_1). \quad (6)$$

We also know that

$$X_2 = \frac{\mathcal{Q}(X) + X_1X_3\sin(\varphi_3 - \varphi_1)}{X_1\sin(\varphi_2 - \varphi_1) + X_3\sin(\varphi_3 - \varphi_2)} \quad (7)$$

and

$$Y_2 = \frac{\mathcal{Q}(Y) + Y_1Y_3\sin(\varphi_3 - \varphi_1)}{Y_1\sin(\varphi_2 - \varphi_1) + Y_3\sin(\varphi_3 - \varphi_2)}. \quad (8)$$

Substituting (7) and (8) into equation (6) gives us the following

$$\mathcal{Q}(X+Y) = \left(1 + \frac{Y_1\sin(\varphi_2 - \varphi_1) + Y_3\sin(\varphi_3 - \varphi_2)}{X_1\sin(\varphi_2 - \varphi_1) + X_3\sin(\varphi_3 - \varphi_2)}\right)\mathcal{Q}(X) \\ + \left(1 + \frac{X_1\sin(\varphi_2 - \varphi_1) + X_3\sin(\varphi_3 - \varphi_2)}{Y_1\sin(\varphi_2 - \varphi_1) + Y_3\sin(\varphi_3 - \varphi_2)}\right)\mathcal{Q}(Y) \\ - \frac{\sin(\varphi_3 - \varphi_2)\sin(\varphi_2 - \varphi_1)\sin(\varphi_3 - \varphi_1)\left[X_1Y_3 - X_3Y_1\right]^2}{\left(X_1\sin(\varphi_2 - \varphi_1) + X_3\sin(\varphi_3 - \varphi_2)\right)\left(Y_1\sin(\varphi_2 - \varphi_1) + Y_3\sin(\varphi_3 - \varphi_2)\right)}.$$

When $Y = \ell$ is a line, $\mathcal{Q}(\ell) = 0$, and this becomes

$$\mathcal{Q}(X+\ell) = \left(1 + \frac{\ell_1\sin(\varphi_2 - \varphi_1) + \ell_3\sin(\varphi_3 - \varphi_2)}{X_1\sin(\varphi_2 - \varphi_1) + X_3\sin(\varphi_3 - \varphi_2)}\right)\mathcal{Q}(X) \\ - \frac{\sin(\varphi_3 - \varphi_2)\sin(\varphi_2 - \varphi_1)\sin(\varphi_3 - \varphi_1)\left[X_1\ell_3 - X_3\ell_1\right]^2}{\left(X_1\sin(\varphi_2 - \varphi_1) + X_3\sin(\varphi_3 - \varphi_2)\right)\left(\ell_1\sin(\varphi_2 - \varphi_1) + \ell_3\sin(\varphi_3 - \varphi_2)\right)}$$

which is the analogue of (1).

Remark 6.1 *We believe that Theorem 5.4 remains valid without the hypothesis $X \in C^2(\alpha, \beta)$, provided (of course) that the conclusion that K has C^2 boundary is dropped. From properties of convex functions $\mathcal{K}(X)(\varphi)$ exists for almost every φ.*

References

[1] W. Black and D. Koop, *Properties of Planar Convex Bodies from One Directed X-Ray*, Proceedings of OSU NSF Sponsored REU, 2000.

[2] R.J. Gardner, *Geometric Tomography*, Cambridge University Press, 1995.

[3] P.C. Hammer, Problem 2. In *Convexity*. Proceedings of Symposia in Pure Mathematics, vol. VII. American Mathematical Society, Providence, RI, 1963, pp. 498-499.

[4] D. Lam and D.C. Solmon, *Reconstructing Convex Polygons in the Plane from One Directed X-Ray*, Discrete and Computational Geometry, 26:105-146, 2001.